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LIPSCHITZ GROUPS AND LIPSCHITZ MAPS

LAURENT POINSOT

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ABSTRACT. This contribution mainly focuses on some aspects of Lipschitz groups, i.e., metrizable groups with Lipschitz multiplication and inversion map. In the main result it is proved that metric groups, with a translation-invariant metric, may be characterized as particular group objects in the category of metric spaces and Lipschitz maps. Moreover, up to an adjustment of the metric, any metrizable abelian group also is shown to be a Lipschitz group. Finally we present a result similar to the fact that any topological nilpotent element x in a Banach algebra gives rise to an invertible element $1 - x$, in the setting of complete Lipschitz groups.

1. Introduction

Historically lots of works have been done in two rather antipodal settings, namely about continuous maps and about C^∞ maps, in order to classify singularities or geometries up to homeomorphisms or diffeomorphisms. With the seminal work of P. Assouad [3], M. Gromov [6], P. Pansu [10] on geometric group theory, the interest into quasi-isometries and bilipschitz maps has grown up, since they provide finer classifications. This contribution does not concern the classification of singularities or geometries up to bilipschitz transformations nor the study of bilipschitz embedding from a metric space to another, but rather the (bi-)Lipschitz maps themselves. More precisely, this contribution concerns the metrizable groups with Lipschitz multiplication and Lipschitz inversion map that are referred to as *Lipschitz groups*. These groups are exactly the group objects in the category of metric spaces and Lipschitz maps, as e.g. topological groups are group objects in the category of topological spaces and continuous maps. We are also interested in subcategories of such groups, namely Lipschitz abelian groups, and Lipschitz groups with 1-Lipschitz multiplication, called *1-Lipschitz groups*. We

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prove that 1-Lipschitz groups are exactly the groups with a two-sided translation-invariant metric (Theorem 3.3), while any metrizable abelian group may be equipped with a topologically equivalent metric that turns the group into a Lipschitz abelian group (Corollary 3.5). Therefore, metrizable abelian groups and Lipschitz abelian groups are essentially the same objects. Finally we mention a result, Theorem 3.6, that presents a relation between any complete 1-Lipschitz group and the group of bilipschitz automorphisms of its underlying complete metric space. This relation is similar to a well-known property of Banach operator algebras on Banach spaces which relates a Banach space and the group of units of its algebra of bounded operators: for any bounded linear operator f of norm < 1 on a Banach space, $id - f$ is an invertible (bounded linear) operator. In the setting of Lipschitz groups, the corresponding result Theorem 3.6 states that given a complete 1-Lipschitz group (say in multiplicative notation), for any K -Lipschitz (endo)function on the group with $K < 1$, $id \cdot f^{-1}: x \mapsto xf(x)^{-1}$ belongs to the group of bilipschitz (endo)functions of the underlying metric space of the group.

Except this introduction, the contribution is organized into two parts. In section 2 are recalled the basic definitions about Lipschitz maps, and also some category-theoretic properties for metric spaces and Lipschitz maps. Section 3 is devoted to Lipschitz-groups. In this section we characterize all Lipschitz abelian groups (up to a choice of a topologically equivalent metric) and also all Lipschitz groups with 1-Lipschitz multiplication as groups with a two-sided translation-invariant metric. In this section we also describe the construction of a bilipschitz map from a K -Lipschitz map, with $K < 1$, on a complete 1-Lipschitz group as described above.

2. Lipschitz maps: basic notions

2.1. A glance at Lipschitz maps. Basic notions of Lipschitz maps may be found for instance in [1, 14]. Let $(E, d), (E', d')$ be two metric spaces. Let $K \in \mathbb{R}_+ = [0; +\infty[$. A set-theoretic map $f: E \rightarrow E'$ is said to be a K -Lipschitz map if for every $x, y \in E$, $d'(f(x), f(y)) \leq Kd(x, y)$. A Lipschitz map is a K -Lipschitz map for some K . For any set-theoretic map $f: E \rightarrow F$, let us define

$$K_f = \sup_{x \neq y} \frac{d'(f(x), f(y))}{d(x, y)}$$

where the supremum is taken in $\overline{\mathbb{R}}_+ = [0; +\infty]$, so that f is a Lipschitz map if, and only if, $K_f < +\infty$. K_f is called the *Lipschitz constant* of f , and it actually depends not only on f but also on the metrics d, d' . In particular, any Lipschitz map f is a K_f -Lipschitz map, and it is also a K -Lipschitz map if, and only if, $K_f \leq K$. Alternatively (see for instance [15]), if f is Lipschitz, then K_f may be defined as

$$K_f = \min\{K \in \mathbb{R}_+ : f \text{ is } K\text{-Lipschitz}\}.$$

Consequently, any constant map is a 0-Lipschitz map (and these are the only 0-Lipschitz maps). When $K_f < 1$, the map f is said to be a *contraction map*, and when $K_f \leq 1$, it may be called a *distance-non-expanding map* (following the terminology of [11]). Of course, a Lipschitz map is also uniformly continuous.

Example 2.1. (1) Let V be a complex or real normed vector space, with norm $\| \cdot \|$. Then, $\| \cdot \|: V \rightarrow \mathbb{R}_+$ is 1-Lipschitz, when V is equipped with the metric associated to $\| \cdot \|$, and \mathbb{R}_+ has its usual metric induced by the absolute value.

(2) Let (E, d) be a metric space. Then, for every $a \in E$, $d(a, \cdot): x \in E \mapsto \mathbb{R}_+$ is 1-Lipschitz.

Let (E, d) and (E', d') be two metric spaces, and $f: E \rightarrow E'$. We say that f is *bilipschitz* if f is an invertible Lipschitz map, and if f^{-1} is also a Lipschitz map. E.g., an isometry is bilipschitz.

2.2. The category of metric spaces with Lipschitz maps. Composing Lipschitz maps yields a Lipschitz map, and it is immediate that $K_{g \circ f} \leq K_g K_f$. In particular, if f and g both are 1-Lipschitz maps, then so is $g \circ f$. The identity function of any metric space being a 1-Lipschitz map, one thus gets a category, denoted by Lip , with objects the metric spaces, and with Lipschitz maps as morphisms (we observe that it is not the same category as in [12]). We denote by $Lip((E, d), (E', d'))$ the set of all Lipschitz maps from (E, d) to (E', d') . The endomorphism monoid (under composition) $Lip((E, d), (E, d))$ is denoted by $EndLip(E, d)$. An isomorphism in the category Lip is exactly a bilipschitz map. The group of automorphisms of (E, d) is denoted by $AutLip(E, d)$. According to [8], $AutLip(E, d)$ may be equipped with a metric in such a way that $AutLip(E, d)$ becomes a Hausdorff topological group. Moreover, if (E, d) is complete, then also is $AutLip(E, d)$.

Remark 2.2. One observes that any map $f: E \rightarrow F$ between sets becomes a 1-Lipschitz map when E and F are both equipped with their own discrete metric. This defines a functor from the category of sets to Lip , which, being full and injective on objects (see [9] for these notions), makes possible to embed the category of sets within Lip as a full subcategory.

The category Lip admits a categorical product (see [9] for the notion of a product in a category). Let (E, d) and (E', d') be two metric spaces. Let us define $d + d'$ to be a distance on $E \times E'$ given by $(d + d')((x, x'), (y, y')) = d(x, y) + d'(x', y')$ for every $x, y \in E$ and $x', y' \in E'$; call it the *product metric*. In what follows we always assume that a product of metric spaces is equipped with the product metric. Let $\pi_E: E \times E' \rightarrow E$ and $\pi_{E'}: E \times E' \rightarrow E'$ be the canonical projections. Both of them are 1-Lipschitz maps. Now, let $f: (F, d_F) \rightarrow (E, d)$ be a K -Lipschitz map and $f': (F, d_F) \rightarrow (E', d')$ be a K' -Lipschitz map. Let us define $(f, f'): F \rightarrow E \times E'$ using the universal property of the product of sets, i.e., $(f, f')(a) = (f(a), f'(a))$ for $a \in F$. Then,

$$(2.1) \quad d(f(a), f(b)) + d'(f'(a), f'(b)) \leq K d_F(a, b) + K' d_F(a, b) \leq 2 \max\{K, K'\} d_F(a, b)$$

for every $a, b \in F$. Therefore, (f, f') is also a Lipschitz map, and it is the only one with the properties that $\pi_E \circ (f, f') = f$ and $\pi_{E'} \circ (f, f') = f'$.

3. Lipschitz groups

3.1. Category-theoretic preliminaries. In any cartesian monoidal category C , i.e., a category with binary categorical products \times and a terminal object 1 (i.e., an object of C with a unique morphism $t_c: c \rightarrow 1$ for each object c of C), we may define *group objects in C* (see [9]). First of all, let us recall

that in such a category, for any two morphisms $f: a \rightarrow b$, and $g: c \rightarrow d$, there is a unique morphism $f \times g: a \times c \rightarrow b \times d$ such that $(f \times g) \circ \pi_a = \pi_b \circ f$, and $(f \times g) \circ \pi_c = \pi_d \circ g$ (where $\pi_a, \pi_b, \pi_c, \pi_d$ are the canonical projections associated to the products). A group object is an object G in C with three morphisms (in C), $m: G \times G \rightarrow G$, $i: G \rightarrow G$ and $e: 1 \rightarrow G$ that satisfy group axioms, which are illustrated by the following commutative diagrams.

(1) The associativity law (where $G \times (G \times G)$ and $(G \times G) \times G$ are canonically identified):

$$(3.1) \quad \begin{array}{ccc} (G \times G) \times G & \xrightarrow{m \times id_G} & G \times G \\ \downarrow \cong & & \searrow m \\ G \times (G \times G) & \xrightarrow{id_G \times m} & G \times G \\ & & \nearrow m \end{array} \quad G$$

(2) e is a two-sided unit of m :

$$(3.2) \quad \begin{array}{ccccc} 1 \times G & \xrightarrow{e \times id_G} & G \times G & \xleftarrow{id_G \times e} & G \times 1 \\ & \searrow \pi_2 & \downarrow m & \swarrow \pi_1 & \\ & & G & & \end{array}$$

(3) i is a two-sided inverse (where $\Delta_G: G \rightarrow G \times G$ is the diagonal map defined by $\pi_1 \circ \Delta_G = id_G = \pi_2 \circ \Delta_G$, with $\pi_1, \pi_2: G \times G \rightarrow G$ the canonical projections):

$$(3.3) \quad \begin{array}{ccccccc} G & \xrightarrow{\Delta_G} & G \times G & \xrightarrow{i \times id_G} & G \times G & \xleftarrow{id_G \times i} & G \times G & \xleftarrow{\Delta_G} & G \\ t_G \downarrow & & & & \downarrow m & & & & \downarrow t_G \\ 1 & \xrightarrow{e} & G & & G & \xleftarrow{e} & 1 & & \end{array}$$

An *abelian group object* in C is a group object (G, m, i, e) in C such that furthermore $m = \sigma \circ m$, where $\sigma: G \times G \rightarrow G \times G$ is the twist isomorphism given by $\pi_1 \circ \sigma = \pi_2$ and $\pi_2 \circ \sigma = \pi_1$ (where $\pi_1, \pi_2: G \times G \rightarrow G$ are the canonical projections). A morphism $f: (G_1, m_1, i_1, e_1) \rightarrow (G_2, m_2, i_2, e_2)$ between group objects is a morphism $f: G_1 \rightarrow G_2$ in C such that $f \circ m_1 = m_2 \circ (f \times f)$, $f \circ i_1 = i_2 \circ f$, and $f \circ e_1 = e_2$. Whence one may form a category of group objects in C and it appears that it is itself a cartesian category with the same categorical product and terminal object as that of C . For instance, a usual group (respectively, topological group, Lie group) is a group object in the category of sets (respectively, topological spaces, smooth manifolds). Moreover, any abelian group object in C is a group object in the category of group objects in C (by the well-known Eckmann-Hilton argument, see [5]).

3.2. Lipschitz groups and Lipschitz-equivalence of metrics. A *Lipschitz group* thus is a group object in the category of Lipschitz spaces (we observe that a terminal object in Lip is given by any one-point set $*$ with its discrete metric). Hence, it is a usual group G together with a metric d such

that the multiplication m and the inversion operation i in the group G are Lipschitz maps. (Note that the identity element 1_G of the group G , seen as a constant map from $*$ to G , automatically is a 0-Lipschitz map for any group G .) This means that there are two constants $K_1, K_2 \geq 0$ such that for every $x, y, g, h \in G$, $d(xy, gh) \leq K_1(d(x, g) + d(y, h))$, and $d(x^{-1}, y^{-1}) \leq K_2d(x, y)$.

Example 3.1. (1) *Let G be a group. A metric on G is said to be a (a two-sided) translation-invariant metric whenever for every $x, y, g \in G$, $d(gx, gy) = d(x, y) = d(xg, yg)$ (see [7]). Then, a group equipped with such a metric is easily seen to be a Lipschitz group. The group law of G and the inversion map are even 1-Lipschitz maps (the latter is actually an isometry).*
 (2) *For any group G , (G, d) is a Lipschitz group with d the discrete metric.*

Let E be a set, and let d, d_1 be two metrics on E . One defines the relation $d \preceq d_1$ if, and only if, $id_E: (E, d) \rightarrow (E, d_1)$ is a Lipschitz map. This turns to be a pre-order relation on the set of metrics of E . Therefore it provides an equivalence relation: $d \sim_L d_1$ if, and only if, $d \preceq d_1$ and $d_1 \preceq d$, and the set of equivalence classes inherits an order relation given by $[d] \leq [d']$ if, and only if, $d \preceq d'$, where by $[d]$ is denoted the equivalent class of $d \bmod \sim_L$. It is rather clear that \sim_L is nothing but Lipschitz-equivalence of metrics on E , since $d \sim_L d_1$ if, and only if, $id_E: (E, d) \rightarrow (E, d_1)$ is a bilipschitz map. If $d \sim_L d_1$, then d and d_1 are also topologically equivalent.

What is important with this notion of Lipschitz-equivalence is the fact that being Lipschitz for a map does not depend on the metrics on its domain and codomain but on their equivalence classes under Lipschitz-equivalence, i.e., if $f: (E, d) \rightarrow (E', d')$ is a Lipschitz map, then $f: (E, d_1) \rightarrow (E', d'_1)$ remains a Lipschitz map for every $d_1 \sim_L d$ and every $d'_1 \sim_L d'$.

Remark 3.2. *It is always possible to turn a Lipschitz map into a 1-Lipschitz map, up to a change of a Lipschitz-equivalent metric: if $f: (E, d) \rightarrow (E', d')$ is a K -Lipschitz map, then with $d_1 = \max\{1, K\}d \sim_L d$, $f: (E, d_1) \rightarrow (E', d')$ is a 1-Lipschitz map. Note however that the new Lipschitz-equivalent metric depends on the Lipschitz constant of f .*

Whence one may define a category \mathcal{LIP} with objects the sets (E, Q) where Q is a Lipschitz-equivalent class of metrics of E , and for morphisms from (E, Q) to (E', Q') those maps $f: E \rightarrow E'$ such that $f: (E, d) \rightarrow (E', d')$ is a Lipschitz map for some, and thus for all, $d \in Q$ and $d' \in Q'$.

The category \mathcal{LIP} also is a monoidal cartesian category with product given by $(E, Q) \times (E', Q') = (E \times E', Q + Q')$, where $Q + Q' = [d + d']$ for $d \in Q$ and $d' \in Q'$ (this is well-defined), and with the usual projections $\pi_E, \pi_{E'}$ (it is easy to see that for every $d \in Q, d' \in Q'$, if $d_0 \sim_L d + d'$, then $\pi_E: (E \times E', d_0) \rightarrow (E, d)$ and $\pi_{E'}: (E \times E', d_0) \rightarrow (E', d')$ are Lipschitz maps), and terminal object given by $(*, [d])$ where $*$ is any one-point set, and d is the discrete metric on $*$.

There is an obvious “projection” functor $\Pi: \mathcal{Lip} \rightarrow \mathcal{LIP}$ given by $\Pi(E, d) = (E, [d])$, and $\Pi(f) = f$ for each Lipschitz map f . It is surjective on objects, full and faithful, so that it provides an equivalence of categories between \mathcal{Lip} and \mathcal{LIP} . Therefore, there exists a left-adjoint-right-inverse for Π , i.e., a “section” functor $s: \mathcal{LIP} \rightarrow \mathcal{Lip}$ such that $\Pi \circ s = id_{\mathcal{LIP}}$ and $s \circ \Pi \simeq id_{\mathcal{Lip}}$ (natural isomorphism), where $id_C: C \rightarrow C$ denotes the identity functor of a category C . The reader should refer to [9, pp. 92–95]

for more details on equivalence of categories. In particular, Π is a right adjoint to s , and right adjoint functors preserve all limits that exist in their domain category (see again [9, Theorem 1, p. 118]). In our case, this implies that Π preserves binary products, and terminal objects.

It then follows easily that Π lifts to a functor between the category $LipGrp$ of group objects in Lip and the category $LIPGrp$ of group objects in LIP . By this is meant that there is a functor $\tilde{\Pi}: LipGrp \rightarrow LIPGrp$ such that the following diagram commutes, where both vertical arrows are the obvious forgetful functors (forgetting the group structure).

$$(3.4) \quad \begin{array}{ccc} LipGrp & \xrightarrow{\tilde{\Pi}} & LIPGrp \\ \downarrow & & \downarrow \\ Lip & \xrightarrow{\Pi} & LIP \end{array}$$

Therefore, if (G, d) is a group object in Lip , then $\tilde{\Pi}(G, d) = (G, [d])$ is a group object in LIP , and if $f: (G, d) \rightarrow (G', d')$ is morphism of groups, then $\tilde{\Pi}(f) = f: (G, [d]) \rightarrow (G', [d'])$ also is a morphism of groups. Moreover $\tilde{\Pi}$ is readily surjective on objects, full and faithful, so the equivalence between Lip and LIP also lifts to an equivalence of categories between $LipGrp$ and $LIPGrp$. In conclusion there is no much advantage to choose one or the other category as that of Lipschitz maps, and for a matter of taste one chooses the former.

3.3. Lipschitz groups and translation-invariant metrics. We may consider a (non-full) subcategory of Lip given as the category of all metric spaces and 1-Lipschitz maps (that is, Lipschitz maps f with $K_f \leq 1$). Whilst the canonical projections of a categorical product are 1-Lipschitz and any terminal object in Lip also is terminal in this subcategory, due to Equation (2.1), one cannot consider the latter as a cartesian monoidal category of its own right, and cannot formally talk about group objects within it. However, one may introduce the notion of a 1-Lipschitz group. It is defined as a group equipped with a metric such that its multiplication is a 1-Lipschitz map (with the product metric on its domain). 1-Lipschitz groups completely characterize groups with a translation-invariant metric, and they appear to be particular group objects in Lip .

Theorem 3.3. *A group G with a metric d is a 1-Lipschitz group if, and only if, d is a two-sided translation-invariant metric.*

Proof. We have $d(gx, gy) \leq d(g, g) + d(x, y) = d(x, y)$. Moreover, $d(x, y) = d(g^{-1}gx, g^{-1}gy) \leq d(g^{-1}, g^{-1}) + d(gx, gy) = d(gx, gy)$, so that $d(gx, gy) = d(x, y)$ for every $x, y, g \in G$. One gets the similar result for the right multiplication. Therefore, d is a two-sided invariant metric on d . The converse follows from the first point of Example 3.1. □

Remark 3.4. *Theorem 3.3 also shows, together with Example 3.1, that in a 1-Lipschitz group the inversion map is forced to be an isometry. Hence 1-Lipschitz groups are particular group objects in Lip .*

One has the following immediate consequence.

Corollary 3.5. *Up to a choice of a topologically equivalent metric, any metrizable topological abelian group is a Lipschitz abelian group.*

Proof. By the theorem of Birkhoff-Kakutani [2], a metrizable topological abelian group G may be equipped with a translation-invariant metric d that defines the same topology as that of G . Then, by Theorem 3.3, (G, d) is a Lipschitz group. \square

3.4. On an analogy with topological nilpotence in a Banach algebra. Given a Lipschitz group (G, d) , it is a commonplace fact from category theory that for each metric space (E, e) , the set $\text{Lip}((E, e), (G, d))$ acquires a group structure with point-wise operations. This is indeed a consequence of [9, Proposition 1, p. 75] which states that the covariant hom-functor $C(-, G)$ is a group object in the category of presheaves over C , once G is a group object in the category C , and thus for each object c of C , the set $C(c, G)$, of morphisms with domain c and codomain G , is a (usual) group. Whence in particular both sets $\text{EndLip}(G, d)$ and $\text{AutLip}(G, d)$ turn to be groups under point-wise operations (hereafter the product is denoted by a point “.”).

For any topologically nilpotent element x in a Banach algebra (this means that $\|x\| < 1$), the element $1 - x$ is invertible (see [13]). For complete 1-Lipschitz groups a similar result holds (which generalizes [1, Theorem 5.1, p. 109]).

Theorem 3.6. *Let (G, d) be a complete 1-Lipschitz group. Let $f \in \text{EndLip}(G, d)$ such that $K_f < 1$. Then, $\text{id}_G \cdot f^{-1} \in \text{AutLip}(G, d)$.*

Proof. If $K_f = 0$, then the result is rather immediate. So, let us assume that $0 < K_f < 1$. For each $g \in G$, the map $f_g: G \rightarrow G$ defined by $f_g(x) = gf(x)$ is K_f -Lipschitz (because $d(f_g(x), f_g(y)) = d(gf(x), gf(y)) = d(f(x), f(y)) \leq K_f d(x, y)$ since d is translation-invariant by Theorem 3.3). Since $0 < K_f < 1$ and G is a complete space, by Banach fixed-point theorem (see [4, Theorem 6]) there exists a unique fixed-point denoted by x_g . Then, $f_g(x_g) = x_g$ is equivalent to $gf(x_g) = x_g \Leftrightarrow g = x_g f(x_g)^{-1}$. By uniqueness of x_g , this means that $\text{id}_G \cdot f^{-1}: x \in G \mapsto xf(x)^{-1} \in G$ is a bijection. Because $\text{EndLip}(G, d)$ is a group under point-wise operations, $\text{id}_G \cdot f^{-1}$ is a Lipschitz map. It remains to prove that $h = (\text{id}_G \cdot f^{-1})^{-1}$ is also a Lipschitz map. Since $gf(x_g) = x_g$, we have $gf(h(g)) = h(g)$. Then we have $d(h(g), h(k)) = d(gf(h(g)), kf(h(k))) = d(k^{-1}g, f(h(k))f(h(g))^{-1}) \leq d(k^{-1}g, 1_G) + d(1_G, f(h(k))f(h(g))^{-1}) = d(g, k) + d(f(h(g)), f(h(k))) \leq d(g, k) + K_f d(h(g), h(k))$ so that $(1 - K_f)d(h(g), h(k)) \leq d(g, k)$ and then $d(h(g), h(k)) \leq \frac{1}{1-K_f}d(g, k)$ (recall that $K_f < 1$). \square

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Laurent Poinsot

University Paris 13, Paris Sorbonne Cité, LIPN, CNRS (UMR 7030) Villetaneuse, France, Secondary adress: CReA, Base aérienne 701, 13661 Salon de Provence, France

Email: laurent.poinsot@lipn.univ-paris13.fr