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SANDWICH CLASSIFICATION THEOREM

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ABSTRACT. The present note arises from the author's talk at the conference "Ischia Group Theory 2014". For subgroups $F \leqslant N$ of a group G denote by $\mathrm{L}(F, N)$ the set of all subgroups of N, containing F. Let D be a subgroup of G. In this note we study the lattice $\mathcal{L} = \mathrm{L}(D, G)$ and the lattice $\mathcal{L}'$ of subgroups of G, normalized by D. We say that $\mathcal{L}$ satisfies sandwich classification theorem if $\mathcal{L}$ splits into a disjoint union of sandwiches $\mathrm{L}(F, N_G(F))$ over all subgroups F such that the normal closure of D in F coincides with F. Here $N_G(F)$ denotes the normalizer of F in G. A similar notion of sandwich classification is introduced for the lattice $\mathcal{L}'$. If D is perfect, i. e. coincides with its commutator subgroup, then it turns out that sandwich classification theorem for $\mathcal{L}$ and $\mathcal{L}'$ are equivalent. We also show how to find basic subroup F of sandwiches for $\mathcal{L}'$ and review sandwich classification theorems in algebraic groups over rings.

1. Introduction

Let G be a group and D a subgroup of G. Put

$$\mathrm{L}(D, G) = \{H \mid D \leqslant H \leqslant G\}.$$

Let $\mathcal{L}$ be a sublattice of $\mathrm{L}(\{1\}, G)$. We say that $\mathcal{L}$ satisfies sandwich classification theorem if

$$\mathcal{L} = \bigsqcup \mathrm{L}(F_i, N_i) \qquad \text{and} \qquad F_i \triangleleft N_i,$$

where i ranges over some index set. Of course, any sublattice satisfies trivial sandwich classification theorem with $N_i = F_i$. In this note we consider two particular kinds of sublattices: subgroups, containing D, and subgroups, normalized by D. For these cases we introduce a more restrictive definitions of sandwich classification.

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The definition of sandwich classification of $L(D,G)$ was introduced by Z. I. Borevich in [8] in connection with the lattice of subgroups of the general linear group over a field, containing the group of diagonal matrices. Group-theoretical aspects of sandwich classification of $L(D,G)$ and related notions were studied in [9, 4]. In that articles the authors wrote that "*D is polynormal in G*"; the term "*sandwich classification*" belongs to A. Bak.

Later Z. I. Borevich and N. A. Vavilov in [11, 12] found some other examples of sandwich classification of $L(D,G)$ in the general linear group $G = \mathrm{GL}_n(R)$ over a ring R. On the other hand, H. Bass [6], J. Wilson [34], I. Z. Golubchik [13] and L. N. Vaserstein [24] obtained a sandwich classification theorem for subgroups of $\mathrm{GL}_n(R)$, normalized by its elementary subgroup $\mathrm{E}_n(R)$. A result of I. Z. Golubchik [14] on subgroups of $\mathrm{GL}_n(R)$ normalized by a block-diagonal elementary subgroup appeared as a common generalization of the results on the normal structure of $\mathrm{GL}_n(R)$ and on the lattice of overgroups of the block-diagonal elementary subgroup. This result suggests that under some additional assumption sandwich classification for overgroups of D and for normal subgroups of certain subgroups, containing D, implies sandwich classification for subgroups, normalized by D. This pure group-theoretical result was obtained in [21, 23] by the author.

The rest of the article is organized as follows. In Section 2 we recall main definitions and state the group-theoretical result mentioned above. Section 3 is a survey of sandwich classification theorems in linear groups over rings.

2. Subgroups normalized by a given subgroup

Let D be a subgroup of a group G. Let $\mathcal{L} = \mathcal{L}_D = \mathrm{L}(D,G)$. For a subgroup $H \leqslant G$ denote by D^H the smallest subgroup, containing D and normalized by H. The normalizer of H in G is denoted by $N_G(H)$. A subgroup $H \in \mathcal{L}$ is called D-full if $D^H = H$.

Definition 1. *We say that the lattice $\mathrm{L}(D,G)$ satisfies sandwich classification if for each subgroup $H \in \mathrm{L}(D,G)$ there exists a unique D-full subgroup F such that*

$$F \leqslant H \leqslant N_G(F).$$

This is equivalent to saying that for any $H \leqslant G$ the subgroup D^H is D-full, i. e.

$$D^{D^H} = D^H.$$

Clearly, sandwich classification holds if D is normal in G or D is a maximal subgroup. More generally, if D is pronormal in G, then $\mathcal{L}$ satisfies sandwich classification.

Now, let us consider the lattice $\mathcal{L}' = \mathcal{L}'_D$ of subgroups of G, normalized by D. For $H \leqslant G$ denote by $[D,H]$ the mutual commutator subgroup, i. e. the subgroup of G generated by commutators $[d,h] = d^{-1}h^{-1}dh$ for all $d \in D$ and $h \in H$. A subgroup $H \in \mathcal{L}'$ is called D-perfect if $[D,H] = H$. A subgroup D is called perfect, if it is D-perfect, i. e. $[D,D] = D$. For $H \in \mathcal{L}'$ denote by $C_{D,G}(H)$ the largest subgroup C of $N_G(H)$ that fits into inclusion $[C,D] \leqslant H$.

Definition 2. *We say that the lattice $\mathcal{L}'$ satisfies sandwich classification if for each subgroup $H \in \mathcal{L}'$ there exists a unique D-perfect subgroup F such that*

$$F \leqslant H \leqslant C_{D,G}(F).$$

This is equivalent to saying that for any $H \in \mathcal{L}'$ the subgroup $[D, H]$ is D-perfect, i. e.

$$\big[[H, D], D\big] = [H, D].$$

Let F be a D-perfect subgroup. Clearly, a subgroup $H \in L\big(F, C_{D,G}(F)\big)$ is normalized by D, and $F = [D, H]$ is uniquely defined by H. Thus, if D satisfies the above definition, then the lattice $\mathcal{L}'$ breaks into a disjoint union of sandwiches $L\big(F, C_{D,G}(F)\big)$, where F ranges over all D-perfect subgroups of G.

If the subgroup D is perfect, then sandwich classification theorems for the lattices $\mathcal{L}$ and $\mathcal{L}'$ are equivalent. This result was obtained by the author in [21]. The following theorem from [23] shows how to find the set of all D-perfect subgroups.

Theorem 1. *Let D be a perfect subgroup of a group G. Suppose that sandwich classification holds for subgroups, containing D. Then sandwich classification holds for subgroups, normalized by D.*

Denote by $\mathcal{P}_F$ the set of all F-perfect subgroups of F. Then the set of all D-perfect subgroups is a union of $\mathcal{P}_F$ over all D-full subgroups F of G.

3. Sandwich classification in linear groups

In this section we give a brief survey of sandwich classification in a linear group G, containing (normalized by) a given subgroup D. We keep writing $\mathcal{L}'$ for the lattice of subgroups of G normalized by D and $\mathcal{L} = \mathrm{L}(D, G)$.

3.1. Subgroups, containing split maximal torus.

Let F be a field, containing at least 7 elements, $G = \mathrm{GL}_n(F)$, and let D be the group of diagonal matrices. Z. I. Borevich in [7] proved sandwich classification theorem for $\mathcal{L}$, where F_σ are net groups. In this case $\mathcal{L}'$ obviously does not satisfy sandwich classification as $[N_G(D), D] = D$ and $[D, D]$ is the trivial group.

The result was extended for semilocal rings by Z. I. Borevich and N. A. Vavilov in [10, 29]. For other [extended] Chevalley groups G over fields the lattice $\mathcal{L}$ for D being split maximal torus was studied in dozens of papers, see [30] and references therein.

3.2. Subsystem subgroups.

Let R be a ring, $G = \mathrm{GL}_n(R)$, and let D be an elementary block-diagonal group with dimensions of diagonal blocks $\geqslant 3$. In [12, 11] Z. I. Borevich and N. A. Vavilov proved sandwich classification theorem for $\mathcal{L}$ if R is commutative or satisfy some stability condition. Again, D-full subgroups are elementary net subgroups. It is not difficult to establish normal structure theorem for an elementary net subgroup and obtain sandwich classification theorem for $\mathcal{L}'$ using theorem 1. However, this result is not published yet.

A counterpart of the group of block-diagonal matrices in a Chevalley group was called a subsystem subgroup. Sandwich classification for overgroups of a subsystem subgroup were studied by N. A. Vavilov and A. Shchegolev. For classical groups it was obtained by N. A. Vavilov in [31, 32]. The result for symplectic group was improved in a recent preprint of A. Shchegolev [19]. For exceptional groups only some (conjecturally all) D-full subgroups are found in [33].

3.3. **Classical groups in natural representations.** Let R be a commutative ring, $G = \mathrm{GL}_n(R)$, $D = \mathrm{ESp}_n(R)$ or $D = \mathrm{EO}_n(R)$. In this case sandwich classification theorem was proved by N. A. Vavilov and V. A. Petrov in [26, 27, 28]. Similar results were independently and simultaneously obtained by You Hong in [35, 36, 37]. Here D-full subgroups $F_I = D \cdot \mathrm{E}_n(R, I)$ are parametrized by ideals I of R.

It follows from Theorem 1 that $\mathcal{L}'$ satisfies sandwich classification, however the normal structure of F_I's is unknown. Recently S. I. Bakulin and N. A. Vavilov in [5] discovered that D-perfect subgroups parametrized by 3 ideals even if $D = \mathrm{EO}_n(R)$ and 2 is invertible in R. This is a challenging problem to show that there are no other D-perfect subgroups in G in this case.

3.4. **Subring subgroups.** Let $(\Phi, \_)$ be a Chevalley–Demazure group scheme with a reduced irreducible root system Φ and $\mathrm{E}(\Phi, \_)$ its elementary subgroup. Let $G = (\Phi, R)$ and $D = \mathrm{E}(\Phi, K)$, where K is a subring of a commutative ring R. First example of sandwich classification theorem for $\mathcal{L}$ in this situation was obtained by N. S. Romanovski in [18] for $(\Phi, \_) = \mathrm{SL}_n$, $n \geqslant 3$, $K = \mathbb{Z}$, and $R = \mathbb{Q}$. Later R. A Shmidt in [20] extended this result for a field of fractions R of a Dedekind domain K. For all Chevalley groups (with some exceptions of small rank and small characteristics) sandwich classification theorem was established by Ya. N. Nuzhin in [15] in the case, where K is a field and R is its algebraic extension, and by Ya. N. Nuzhin and A. V. Yakushevich in [17] in the case, where R is the field of fractions of a euclidean domain K. The author in [23] proved sandwich classification for $\mathcal{L}$ if $\Phi = B_l, C_l, F_4$ and $K \subseteq R$ is an arbitrary pair of rings such that 2 is invertible in K (in case $\Phi = B_{2k+1}$ one requires -1 to be a square in K). On the other hand, in [22] the author shows that for simply laced root systems $\Phi = A_l, D_l, E_l$ sandwich classification theorem fails if R contains a [quasi-]transcendental element over K.

In all these results D-full subgroups are just elementary groups $\mathrm{E}(\Phi, S)$ over intermediate subrings S between K and R. Since the normal structure of $\mathrm{E}(\Phi, S)$ is known by L. N. Vaserstein [25] and E. Abe [1], sandwich classification theorem for $\mathcal{L}'$ follows from Theorem 1.

The case of algebraic field extensions, $\operatorname{char} K = 2$, $\Phi = B_l, C_l, F_4$, or $\operatorname{char} K = 3$, $\Phi = G_2$, was considered by Ya. N. Nuzhin in [16]. In this case D-perfect subgroups are parametrized by a sort of admissible pairs of additive subgroups of R. Formally, sandwich classification for normal subgroups of these D-perfect subgroups is unknown, however it does not seem to be a difficult problem.

3.5. **Tensor product subgroups.** Let $G = \mathrm{GL}_n(R)$ and $D = \mathrm{E}_m(R) \otimes \mathrm{E}_k(R)$, where $mk = n$, $m - 2 \geqslant k \geqslant 3$. In [3, 2] A. S. Ananievskii, N. A. Vavilov, and S. S. Sinchuk found a series of D-full subgroups and described their normalizers. It should be possible now to prove sandwich classification theorem for $\mathcal{L}$ and even for $\mathcal{L}'$ but it is not at all trivial.

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