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FINITE SIMPLE GROUPS OF LOW RANK: HURWITZ GENERATION AND (2, 3)-GENERATION

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ABSTRACT. Let us consider the set of non-abelian finite simple groups which admit non-trivial irreducible projective representations of degree ≤ 7 over an algebraically closed field \mathbb{F} of characteristic $p \geq 0$. We survey some recent results which lead to the complete list of the groups in this set which are (2, 3, 7)-generated and of those which are (2, 3)-generated.

1. Introduction

The non-abelian finite simple groups admitting non-trivial irreducible projective representations of degree ≤ 7 , over an algebraically closed field \mathbb{F} , are those listed in Table 1, with the exceptions mentioned at the end.

The alternating groups:	$\text{Alt}(5), \text{Alt}(6), \text{Alt}(7), \text{Alt}(8)$ and $\text{Alt}(9)$.
The classical groups:	$\text{PSL}_n(q), 2 \leq n \leq 7, \text{PSU}_n(q^2), 3 \leq n \leq 7,$ $\text{PSp}_4(q), \text{PSp}_6(q)$ and $\Omega_7(q)$.
The exceptional groups of Lie type:	${}^2\text{B}_2(2^{2a+1}), \text{G}_2(q)$ and ${}^2\text{G}_2(3^{2a+1})$.
The sporadic groups:	$\text{M}_{11}, \text{M}_{12}, \text{M}_{22}, \text{J}_1$ and J_2 .
Exceptions with respect to being simple:	$\text{PSL}_2(2), \text{PSL}_2(3), \text{PSU}_3(4), \text{PSp}_4(2),$ ${}^2\text{B}_2(2), \text{G}_2(2)$ and ${}^2\text{G}_2(3)$.

TABLE 1.

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The most recent and most extensive reference for this list is [8] for the groups of Lie type in defining characteristic and [6] for the others. We recall that several of these results were known much earlier (e.g., see [10] and the references therein). Our aim is to say exactly which groups of Table 1 are Hurwitz groups and which are $(2, 3)$ -generated.

We recall that a $(2, 3, 7)$ -group (or a *Hurwitz group*, when finite) is a group generated by an involution and an element of order 3, whose product has order 7. Such groups have been studied intensively, starting from the pioneering works of Hurwitz [7] and Klein [9]. In more recent years many authors focused on finite simple groups. The precise answers, with respect to being Hurwitz or not, are known for alternating groups, sporadic groups and some exceptional groups of Lie type. A positive answer for some series of classical matrix groups of large ranks is also known. There is also an important contribution to intermediate ranks, given by Vsemirnov in [32]. For all these matrix groups explicit Hurwitz generators are constructed, based on Conder's generators for the alternating groups. For the vast literature on this subject we refer to survey articles, e.g., [2] or [27].

The low-rank cases for matrix groups require different methods. A crucial tool is a well known formula of L. Scott [22], which can be applied to any matrix representation of a finitely generated group. The most basic application is the following. Let \mathbb{F} be an algebraically closed field and let $\langle x, y \rangle$ be an irreducible subgroup of $\mathrm{GL}_n(\mathbb{F})$. Set $M = \mathrm{Mat}_n(\mathbb{F})$ and denote by $C_M(g)$ the centralizer of an element g of $\mathrm{GL}_n(\mathbb{F})$ in M . Hence, Scott's formula gives:

$$(1.1) \quad \dim(C_M(x)) + \dim(C_M(y)) + \dim(C_M(xy)) \leq n^2 + 2.$$

When equality holds in (1.1), the triple (x, y, xy) is called *rigid*. As shown in [23], rigid triples with the same similarity invariants (i.e., with the same Jordan forms) are simultaneously conjugate: in particular they generate conjugate subgroups. By an irreducible $(2, 3, 7)$ -triple we mean a triple $(x, y, xy) \in \mathrm{SL}_n(\mathbb{F})^3$ such that x^2 , y^3 and $(xy)^7$ are scalar matrices and the group $\langle x, y \rangle$ is irreducible. For $n \leq 5$ all the irreducible $(2, 3, 7)$ -triples are rigid. Clearly rigidity makes the classification easier, since it restricts the isomorphism types of the generated groups: at most one type for fixed p and n and for a fixed set of similarity invariants, as shown by the results mentioned in Section 3. This aspect is discussed in [14].

But, already for $n = 6$, there are also non-rigid irreducible $(2, 3, 7)$ -triples and this fact tends, asymptotically, to make Hurwitz generation a very common property among finite simple groups of sufficiently large rank (and order divisible by 3 and 7). We illustrate the cases $n = 6, 7$ in the next Section.

Clearly the $(2, 3, 7)$ -generated groups belong to the much wider class of the $(2, 3)$ -generated groups, i.e., of the non-trivial epimorphic images (of order $\neq 2, 3$) of the unimodular group $\mathrm{PSL}_2(\mathbb{Z})$. Also in this field the literature is vast and a lot is known about the $(2, 3)$ -generation of finite simple groups (see [3, 35]). The methods here are essentially of two types: constructive and probabilistic. In particular, by the famous result of [11], almost all finite classical groups are $(2, 3)$ -generated. Indeed the list of exceptions is short! To our knowledge the only classical groups among these exceptions whose degree exceeds 7 are $\Omega_8^+(2)$ and $\mathrm{P}\Omega_8^+(3)$, detected by Vsemirnov in 2012.

2. Some groups generated by non-rigid triples

A series of three papers [28, 29, 36] gives the parametrization, up to conjugation and multiplication by scalars, of the irreducible $(2, 3, 7)$ -triples in $SL_n(\mathbb{F})^3$, $n \leq 7$: in positive characteristic it also gives the isomorphism type of the groups generated by rigid triples. As to the non-rigid ones, for $n = 6, 7$ their similarity invariants are respectively:

$$(2.1) \quad t^2 + 1, t^2 + 1, t^2 + 1; \quad t^3 - 1, t^3 - 1; \quad t^6 + t^5 + t^4 + t^3 + t^2 + t + 1;$$

and

$$(2.2) \quad t + 1, t^2 - 1, t^2 - 1, t^2 - 1; \quad t - 1, t^3 - 1, t^3 - 1; \quad t^7 - 1.$$

Applications of Scott’s formula to the symmetric square $S(\mathbb{F}^n)$, $n = 6, 7$, and to the exterior square power $\Lambda^2(\mathbb{F}^7)$ give that an irreducible triple with similarity invariants (2.1) generates a symplectic group, an irreducible triple with similarity invariants (2.2) exists only for p odd and generates an orthogonal group (see [28]). For $p = 2$ the group $Sp_6(\mathbb{F})$ is isomorphic to $\Omega_7(\mathbb{F})$. In this isomorphism an irreducible triple (2.1) maps to a triple (2.2) which generates a reducible, indecomposable group. Application of Scott’s formula to the space of alternating trilinear forms in dimension 7 gives that the irreducible triples with similarity invariants (2.2) for p odd and those with similarity invariants (2.1) for $p = 2$ generate a subgroup of $G_2(\mathbb{F})$. In the following, we explicitly describe matrices with entries in a finite field \mathbb{F}_q of q elements and similarity invariants (2.1) that generate $Sp_6(q)$ for q odd $\neq 3$ and $G_2(q)$ for $q = 2^a > 4$.

For each $r \in \mathbb{F}_q$ such that $r^2 - 3r + 3 \neq 0$, we set

$$(2.3) \quad h = r^2 - 3r + 3,$$

$$(2.4) \quad a = \frac{2r^2 - 7r + 7}{h}, \quad b = \frac{-r^3 + 7r^2 - 16r + 13}{h}, \quad c = \frac{-r^3 + 2r^2 - r - 1}{h},$$

and define:

$$(2.5) \quad x_6 = \begin{pmatrix} 0 & 0 & -1 & 0 & r & -1 \\ 0 & 0 & 0 & -1 & 0 & a \\ 1 & 0 & 0 & 0 & -1 & -r \\ 0 & 1 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad y_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & c \\ 0 & 1 & 0 & 0 & b & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

The group $H_6 = \langle x_6, y_6 \rangle$ is absolutely irreducible, except when $p = 2$ and $r = 0, 1$, or:

$$(2.6) \quad d(r) = r^{12} - 18r^{11} + 160r^{10} - 915r^9 + 3704r^8 - 11076r^7 + 24918r^6 - 42309r^5 + 53688r^4 - 49644r^3 + 31817r^2 - 12750r + 2437 = 0.$$

By the results of [18] and [30] the following holds:

Theorem 2.1. *Set $q = p^a$, $q \geq 5$. In the above definition of x_6, y_6 take $r \in \mathbb{F}_q$ such that $r^2 - 3r + 3 \neq 0$, $d(r) \neq 0$ in (2.6) and $\mathbb{F}_q = \mathbb{F}_p[r^2 - 3r + 3]$ (such an r always exists). Then $H_6 = \langle x_6, y_6 \rangle$ coincides with $G_2(q)$ when q is even, with $Sp_6(q)$ when q is odd except possibly when $a = 1, 3$. But also in these cases there exists $r \in \mathbb{F}_q$ (satisfying further conditions) such that $H_6 = Sp_6(q)$.*

For each $r \in \mathbb{F}_q$ define

$$(2.7) \quad x_7 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 0 & 0 & r \\ 0 & 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad y_7 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & r+2 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2r+8 \\ 0 & 0 & 1 & 1 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

The group $H_7 = \langle x_7, y_7 \rangle$ is absolutely irreducible, except when $r^2 + 15r + 100 = 0$.

Theorem 2.2. [18] *Let $q = p^a$, p odd, $q \geq 5$. In the above definition of x_7, y_7 take $r \in \mathbb{F}_q$ such that $r^2 + 15r + 100 \neq 0$ and $\mathbb{F}_q = \mathbb{F}_p[r]$ (such an r always exists). Then $H_7 = \langle x_7, y_7 \rangle$ coincides with $G_2(q)$, except possibly when $a = 1, 3$. But also in these cases there exists $r \in \mathbb{F}_q$ (satisfying further conditions) such that $H_7 = G_2(q)$.*

Slightly different explicit generators for $G_2(p)$ when p is a prime ≥ 5 can be found in [33].

The simple groups ${}^2G_2(3^{2a+1})$ are Hurwitz by a result of Malle [13], for all $a \geq 1$. Hurwitz generators for these groups can also be found in [26]. By the above mentioned classification, they are generated by triples with similarity invariants (2.2). Hence they must belong to one of the parametric families in [29].

3. The Hurwitz groups in Table 1

The classification of $(2, 3, 7)$ -triples, combined with the application of Scott's formula to various representations and the knowledge of maximal subgroups of finite simple groups allow to conclude:

Proposition 3.1. *Let G be a non-abelian finite simple group which admits a non-trivial absolutely irreducible projective representations of degree ≤ 7 . Then G is a Hurwitz group if, and only if, it is one of the following:*

- $PSL_2(p)$ if $p \equiv 0, \pm 1 \pmod{7}$;
- $PSL_2(p^3)$ if $p \equiv \pm 2, \pm 3 \pmod{7}$ (see [12]);
- $PSL_5(p)$ if $p \equiv 1, 6 \pmod{35}$;
- $PSL_5(p^3)$ if $p \equiv 11, 16, 26, 31 \pmod{35}$;
- $PSU_5(p^2)$ if $p \equiv 29, 34 \pmod{35}$;

- $\text{PSU}_5(p^4)$ if $p \equiv 8, 13, 22, 27 \pmod{35}$ or $p = 7$;
- $\text{PSU}_5(p^6)$ if $p \equiv 4, 9, 19, 24 \pmod{35}$;
- $\text{PSU}_5(p^{12})$ if $p \equiv 2, 3, 12, 17, 18, 23, 32, 33 \pmod{35}$ (see [31]);

- $\text{PSL}_6(p^{n_6})$, if $p \neq 3$ and $n_6 = o_p(9)$ is odd;
- $\text{PSU}_6(p^{n_6})$, if $p \neq 3$ and $n_6 = o_p(9)$ is even (see [28]);
- $\text{PSp}_6(q)$, for all $q = p^a$, with q odd, $q \geq 5$ (see [30]);
- $\text{G}_2(2^a)$, for all $a \geq 3$ (see [13] and [18]);
- J_2 (see [4]);

- $\text{PSL}_7(p^{n_7})$, if $p \neq 7$ and $n_7 = o_p(49)$ is odd;
- $\text{PSU}_7(p^{n_7})$, if $p \neq 7$ and $n_7 = o_p(49)$ is even (see [28]);
- $\text{G}_2(p^a)$, for all powers $p^a \geq 5$, with p odd (see [13] and [18]);
- ${}^2\text{G}_2(3^{2a+1})$, for all $a \geq 1$ (see [13]);
- J_1 (see [21]).

Here, given an integer u and a prime p , coprime with u , we denoted by $o_p(u)$ the order of $p \pmod u$, i.e., the order of $p + u\mathbb{Z}$ seen as an element of the group $(\mathbb{Z}/u\mathbb{Z})^*$.

4. The groups in Table 1 which are (2, 3)-generated

The study of which groups are (2, 3)-generated, among those in Table 1, was the subject of many papers. We summarize here the complete classification that can be obtained collecting the results of [1, 5, 12, 13, 15, 16, 17, 19, 20, 24, 25, 30, 37, 38].

Theorem 4.1. *Let G be a non-abelian finite simple group which admits a non-trivial absolutely irreducible projective representation of degree ≤ 7 . Then G is (2, 3)-generated, except when G is one of the following groups:*

- $\text{Alt}(6), \text{Alt}(7), \text{Alt}(8)$ (see [15]);
- $\text{PSL}_2(9)$ (see [12]);
- $\text{PSL}_3(4), \text{PSU}_3(9), \text{PSU}_3(25)$ (see [1, 5, 19, 37]);
- $\text{PSL}_4(2), \text{PSU}_4(4), \text{PSU}_4(9), \text{PSp}_4(2^a), \text{PSp}_4(3^a)$ (see [11, 20]);
- $\text{PSU}_5(4)$ (see [34]);
- $\text{M}_{11}, \text{M}_{22}$ (see [38]);
- ${}^2\text{B}_2(2^{2a+1})$.

We note that the orders of the Suzuki groups ${}^2\text{B}_2(2^{2a+1})$ are not divisible by 3.

As mentioned in the Introduction the simple groups $\Omega_8^+(2)$ and $\text{P}\Omega_8^+(3)$ are not (2, 3)-generated.

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