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SHEN'S CONJECTURE ON GROUPS WITH GIVEN SAME ORDER TYPE

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Abstract. For any group G, we define an equivalence relation $\sim$ as below:

$$\forall\ g, h \in G \quad g \sim h \Longleftrightarrow |g| = |h|$$

the set of sizes of equivalence classes with respect to this relation is called the same-order type of G and denote by $\alpha(G)$. In this paper, we give a partial answer to a conjecture raised by Shen. In fact, we show that if G is a nilpotent group, then $|\pi(G)| \leq |\alpha(G)|$, where $\pi(G)$ is the set of prime divisors of order of G. Also we investigate the groups all of whose proper subgroups, say H have $|\alpha(H)| \leq 2$.

1. Introduction

Let G be a group, define an equivalence relation $\sim$ as below:

$$\forall\ g, h \in G \quad g \sim h \Longleftrightarrow |g| = |h|$$

the set of sizes of equivalence classes with respect to this relation is called the same-order type of G. For instance, the same-order type of the quaternion group Q_8 is $\{1, 6\}$. The only groups of type $\{1\}$ are 1, $\mathbb{Z}_2$. In [3], Shen showed that a group of same-order type $\{1, n\}(\{1, m, n\})$ is nilpotent (solvable, respectively). Furthermore he gave the structure of these groups. In this paper, we give a partial answer to a conjecture raised by Shen in [3] and we prove that if G is a nilpotent group, then $|\pi(G)| \leq |\alpha(G)|$.

Given a class of groups $\mathcal{X}$, we say that a group G is a minimal non-$\mathcal{X}$-group, or an $\mathcal{X}$-critical group, if $G \notin X$, but all proper subgroups of G belong to $\mathcal{X}$. It is clear that detailed knowledge of the structure of minimal non-$\mathcal{X}$-groups can provide insight into what makes a group belong to $\mathcal{X}$. For

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instance, minimal non-nilpotent groups were analysed by Schmidt [2] and proved that such groups are solvable (see also [5]). Suppose that t be a positive integer and $\mathcal{Y}_t$ be the class of all groups in which $|\alpha(G)| \leq t$. Here, we determine the structure of minimal non-$\mathcal{Y}_2$-group.

Denote by ϕ and S_n^G the Euler's function and the number of elements of order n in a group G respectively. X_n is the set of all elements of order n in a group G. We use symbols $\pi_e(G)$ for the set of element orders.

2. Shen's conjecture

In [3], Shen posed a conjecture as follows:
Let G be a group with same-order type $\{1, n_2, \dots, n_r\}$. Then $|\pi(G)| \leq r$.
Here we give a partial answer to a this conjecture. Note that by [4, Lemma 3], we can assume that G is finite. To prove Shen's conjecture we need the following interesting lemmas.

Lemma 2.1. *Suppose that G is a nilpotent group, $m, n \in \pi_e(G)$ and $(m, n) = 1$. Then*

$$S_{mn}^G = S_m^G S_n^G.$$

Proof. Let $g \in X_{mn}$. As $(m, n) = 1$, so there exist $y, z \in G$, such that $o(y) = m$, $o(z) = n$ and $g = yz$. So $g \in X_m X_n$ and $X_{mn} \subseteq X_m X_n$. On the other hand, if $y \in X_m$ and $z \in X_n$, then, as G is nilpotent, we can obtain that $yz = zy$ and so $o(yz) = o(zy) = o(z)o(y) = mn$. It follows that $X_m X_n \subseteq X_{mn}$ and so $X_{mn} = X_m X_n$. □

Corollary 2.2. *Let G be a nilpotent group, $m \in \pi_e(G)$ and $m = p_1^{h_1} p_2^{h_2} \cdots p_t^{h_t}$. Then*

$$S_m^G = S_{p_1^{h_1}}^G S_{p_2^{h_2}}^G \cdots S_{p_t^{h_t}}^G.$$

Theorem 2.3. *Let G be a nilpotent group. Then*

1. *If $|\pi(G)| \leq 2$, then $|\pi(G)| \leq |\alpha(G)|$.*
2. *If $|\pi(G)| \geq 3$, then $\pi(G)| \leq |\alpha(G)| - 1$.*

Proof. (1). If $|\pi(G)| = 1$, then G is a p-group and obviously $|\pi(G)| \leq |\alpha(G)|$. Let $\pi(G) = \{p, q\}$. Since G is nilpotent, $G = P \times Q$, where $|P| = p^n$ and $|Q| = q^m$ are p-Sylow and q-Sylow subgroups of G, respectively. If $p = 2$ and $n = 1$, then $G \cong \mathbb{Z}_2 \times Q$. Clearly $\alpha(G) = \alpha(Q)$. Now if $exp(Q) = q$, then $s_q^Q = q^m - 1$. So $|\alpha(G)| = |\alpha(Q)| = |\pi(G)| = 2$. Otherwise if $exp(Q) \neq q$, then there exists $x \in Q$ such that $o(x) = q^2$ and since $S_q \neq S_{q^2}$, so $|\alpha(Q)| \geq 3$ and $|\alpha(G)| \geq 3 > |\pi(G)|$. In other values of p and n, in view of Lemma 2.1, the conclusion is trivial.

(2). By the hypothesis that G is nilpotent, we have $G = P_1 \times \cdots \times P_n$, where P_i's are p_i-Sylow subgroups of G and $p_1 < p_2 < \cdots < p_n$. We prove by induction on n. If $n = 3$, the $\alpha(P_1) \cup \alpha(P_2) \cup \alpha(P_3) \supseteq \{r, t\}$, for distinct numbers r and t, so $\alpha(G) \supseteq \{1, r, t, rt\}$, as desired.

Now assume the conclusion is true for $G_{n-1} = P_1 \times \cdots \times P_{n-1}$. Let for any $1 \leq i \leq n-1$, $\alpha(P_i) = \{1, n_1^i, \dots, n_{t_i}^i\}$ and $S_{{p_i}^{h_i}}$ for $1 \leq i \leq n-1$ be the maximum number of the set $\alpha(P_i)$. Now for any $l \in \pi_e(G_{n-1})$, assume that $l = p_1^{\beta_1} \cdots p_r^{\beta_r}$, where $1 \leq r \leq n-1$. By the maximality of $S_{{p_i}^{h_i}}$'s , we have

$$\begin{aligned} S_l = S_{p_1^{\beta_1} \cdots p_r^{\beta_r}} &= S_{{p_1}^{\beta_1}} \cdots S_{{p_r}^{\beta_r}} \\ &\leq S_{{p_1}^{h_1}} \cdots S_{{p_{n-1}}^{h_{n-1}}} \end{aligned}$$

Besides, $S_{p_n}^G \neq 0$ and since $\phi(p_n) = p_n - 1 \mid S_{p_n}$, so $S_{p_n} \neq 1$. Hence we have

$$\begin{aligned} S_l \leq S_{{p_1}^{h_1}} \cdots S_{{p_{n-1}}^{h_{n-1}}} &\lneq S_{{p_1}^{h_1}} \cdots S_{{p_{n-1}}^{h_{n-1}}} S_{p_n} \\ &= S_{{p_1}^{h_1} \cdots p_{n-1}^{h_{n-1}} p_n} \end{aligned}$$

It followes that $S_{{p_1}^{h_1} \cdots p_{n-1}^{h_{n-1}} p_n} \in \alpha(G_n) \setminus \alpha(G_{n-1})$. Therefore

$$|\alpha(G_n)| = |\alpha(G)| \geq |\alpha(G_{n-1})| + 1$$

and so by induction hypothesis;

$$|\pi(G)| = n = n - 1 + 1 < |\alpha(G_{n-1})| + 1 \leq |\alpha(G_n)| = |\alpha(G)|.$$

and the conclusion is proved. □

3. On the same-order type of subgroups of a group

In this section, we determine the structure of minimal non-$\mathcal{Y}_2$-group, as follows.

Theorem 3.1. *Let G be minimal non-$\mathcal{Y}_2$-group. Then G is a Frobenius or 2-Frobenius group.*

Proof. Let H be a non-trivial proper subgroup of G and $p \in \pi(H)$. Suppose, on the contrary, that $q \in \pi(G)$ and $q \neq p$. Since $p \mid 1 + s_p^H$ and $q \mid 1 + s_q^H$, so $s_p^H, s_q^H \neq \{0, 1\}$, hence $s_p^H = s_q^H = n_H$. Now as H is nilpotent, according to Lemma 2.1, we have $s_{pq}^H = s_p^H s_q^H = n_H^2$, a contradiction. Thus H is a p-group. On the other hand, since

$$p \mid 1 + s_p^H + s_{p^2}^H,$$

so $s_{p^2}^H \neq \{1, n_H\}$, since otherwise $p \mid 1$, a contradiction. Hence $s_{p^2}^H = 0$. It follows that every proper subgroup of G is p-group of exponent p. If $p, q \in \pi(G)$, then G has no element of order pq. If G is nilpotent, then G is a p-group of exponent p and it is easy to see that such groups are in $\mathcal{Y}_2$, a contradiction. If G is non-nilpotent, then, as proper subgroup of G has the same-order type $\{1, n\}$, Theorem 2.1 of Shen follows that G is a Schmidt group and so $|\pi(G)| = 2$. Now, as G has no element of order pq, of [1, Theorem A], completes the proof. □

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