AUTHOR. For any group $G$, we define an equivalence relation $\sim$ as below:

\[ \forall g, h \in G \ g \sim h \iff |g| = |h| \]

the set of sizes of equivalence classes with respect to this relation is called the same-order type of $G$ and denote by $\alpha(G)$. In this paper, we give a partial answer to a conjecture raised by Shen. In fact, we show that if $G$ is a nilpotent group, then $|\pi(G)| \leq |\alpha(G)|$, where $\pi(G)$ is the set of prime divisors of order of $G$. Also we investigate the groups all of whose proper subgroups, say $H$ have $|\alpha(H)| \leq 2$.

1. Introduction

Let $G$ be a group, define an equivalence relation $\sim$ as below:

\[ \forall g, h \in G \ g \sim h \iff |g| = |h| \]

the set of sizes of equivalence classes with respect to this relation is called the same-order type of $G$. For instance, the same-order type of the quaternion group $Q_8$ is $\{1, 6\}$. The only groups of type $\{1\}$ are $1, \mathbb{Z}_2$. In [3], Shen showed that a group of same-order type $\{1, n\}(\{1, m, n\})$ is nilpotent (solvable, respectively). Furthermore he gave the structure of these groups. In this paper, we give a partial answer to a conjecture raised by Shen in [3] and we prove that if $G$ is a nilpotent group, then $|\pi(G)| \leq |\alpha(G)|$.

Given a class of groups $\mathcal{X}$, we say that a group $G$ is a minimal non-$\mathcal{X}$-group, or an $\mathcal{X}$-critical group, if $G \not\in \mathcal{X}$, but all proper subgroups of $G$ belong to $\mathcal{X}$. It is clear that detailed knowledge of the structure of minimal non-$\mathcal{X}$-groups can provide insight into what makes a group belong to $\mathcal{X}$. For

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instance, minimal non-nilpotent groups were analysed by Schmidt [2] and proved that such groups are solvable (see also [5]). Suppose that \( t \) be a positive integer and \( \mathcal{Y}_t \) be the class of all groups in which \( \alpha(G) \leq t \). Here, we determine the structure of minimal non-\( \mathcal{Y}_2 \)-group.

Denote by \( \phi \) and \( S_n^G \) the Euler’s function and the number of elements of order \( n \) in a group \( G \) respectively. \( X_n \) is the set of all elements of order \( n \) in a group \( G \). We use symbols \( \pi_e(G) \) for the set of element orders.

2. Shen’s conjecture

In [3], Shen posed a conjecture as follows:

Let \( G \) be a group with same-order type \( \{1, n_2, \ldots, n_r\} \). Then \( |\pi(G)| \leq r \).

Here we give a partial answer to this conjecture. Note that by [4, Lemma 3], we can assume that \( G \) is finite. To prove Shen’s conjecture we need the following interesting lemmas.

**Lemma 2.1.** Suppose that \( G \) is a nilpotent group, \( m, n \in \pi_e(G) \) and \( (m, n) = 1 \). Then

\[
S_{mn}^G = S_m^G S_n^G.
\]

**Proof.** Let \( g \in X_{mn} \). As \( (m, n) = 1 \), so there exist \( y, z \in G \), such that \( o(y) = m \), \( o(z) = n \) and \( g = yz \).

So \( g \in X_m X_n \) and \( X_{mn} \subseteq X_m X_n \). On the other hand, if \( y \in X_m \) and \( z \in X_n \), then, as \( G \) is nilpotent, we can obtain that \( yz = zy \) and so \( o(yz) = o(z) = o(z) o(y) = mn \). It follows that \( X_m X_n \subseteq X_{mn} \) and so \( X_{mn} = X_m X_n \).

**Corollary 2.2.** Let \( G \) be a nilpotent group, \( m \in \pi_e(G) \) and \( m = p_1^{h_1} p_2^{h_2} \cdots p_t^{h_t} \). Then

\[
S_m^G = S_{p_1^{h_1}}^G S_{p_2^{h_2}}^G \cdots S_{p_t^{h_t}}^G.
\]

**Theorem 2.3.** Let \( G \) be a nilpotent group. Then

1. If \( |\pi(G)| \leq 2 \), then \( |\pi(G)| \leq |\alpha(G)| \).
2. If \( |\pi(G)| \geq 3 \), then \( |\pi(G)| \leq |\alpha(G)| - 1 \).

**Proof.** (1). If \( |\pi(G)| = 1 \), then \( G \) is a \( p \)-group and obviously \( |\pi(G)| \leq |\alpha(G)| \). Let \( \pi(G) = \{p, q\} \). Since \( G \) is nilpotent, \( G = P \times Q \), where \( |P| = p^a \) and \( |Q| = q^m \) are \( p \)-Sylow and \( q \)-Sylow subgroups of \( G \), respectively. If \( p = 2 \) and \( n = 1 \), then \( G \cong \mathbb{Z}_2 \times Q \). Clearly \( \alpha(G) = \alpha(Q) \). Now if \( exp(Q) = q \), then \( s_q^G = q^m - 1 \). So \( |\alpha(G)| = |\alpha(Q)| = |\pi(G)| = 2 \). Otherwise if \( exp(Q) \neq q \), then there exists \( x \in Q \) such that \( o(x) = q^2 \) and since \( S_q \neq S_{q^2} \), so \( |\alpha(Q)| \geq 3 \) and \( |\alpha(Q)| \geq 3 > |\pi(G)| \). In other values of \( p \) and \( n \), in view of Lemma 2.1, the conclusion is trivial.

(2). By the hypothesis that \( G \) is nilpotent, we have \( G = P_1 \times \cdots \times P_n \), where \( P_i \)'s are \( p_i \)-Sylow subgroups of \( G \) and \( p_1 < p_2 < \cdots < p_n \). We prove by induction on \( n \). If \( n = 3 \), the \( \alpha(P_1) \cup \alpha(P_2) \cup \alpha(P_3) \supseteq \{r, t\} \), for distinct numbers \( r \) and \( t \), so \( \alpha(G) \supseteq \{1, r, t, rt\} \), as desired.

Now assume the conclusion is true for \( G_{n-1} = P_1 \times \cdots \times P_{n-1} \). Let for any \( 1 \leq i \leq n - 1 \), \( \alpha(P_i) = \{1, n_1^i, \ldots, n_r^i\} \) and \( S_{p_i^{n_i^i}} \) for \( 1 \leq i \leq n - 1 \) be the maximum number of the set \( \alpha(P_i) \). Now
for any \( l \in \pi_e(G_{n-1}) \), assume that \( l = p_1^{\beta_1} \cdots p_r^{\beta_r} \), where \( 1 \leq r \leq n - 1 \). By the maximality of \( S_{p_i} \)'s, we have

\[
S_l = S_{p_1^{\beta_1} \cdots p_r^{\beta_r}} = S_{p_1^{\beta_1}} \cdots S_{p_r^{\beta_r}} \leq S_{p_1^{h_1}} \cdots S_{p_r^{h_r} p_{n-1}^{h_{n-1}}}.
\]

Besides, \( S_{p_n}^G \neq 0 \) and since \( \phi(p_n) = p_n - 1 \mid S_{p_n} \), so \( S_{p_n} \neq 1 \). Hence we have

\[
S_l \leq S_{p_1^{h_1}} \cdots S_{p_{n-1}^{h_{n-1}}} S_{p_n} \leq S_{p_1^{h_1}} \cdots S_{p_{n-1}^{h_{n-1}}} S_{p_n} = S_{p_1^{h_1} \cdots p_{n-1}^{h_{n-1}} p_n}.
\]

It follows that \( S_{p_1^{h_1} \cdots p_{n-1}^{h_{n-1}} p_n} \in \alpha(G_n) \setminus \alpha(G_{n-1}) \). Therefore

\[
|\alpha(G_n)| - |\alpha(G)| \geq |\alpha(G_{n-1})| + 1
\]

and so by induction hypothesis;

\[
|\pi(G)| = n = n - 1 + 1 < |\alpha(G_{n-1})| + 1 \leq |\alpha(G_n)| = |\alpha(G)|.
\]

and the conclusion is proved. \( \square \)

3. On the same-order type of subgroups of a group

In this section, we determine the structure of minimal non-\( \mathcal{Y}_2 \)-group, as follows.

**Theorem 3.1.** Let \( G \) be minimal non-\( \mathcal{Y}_2 \)-group. Then \( G \) is a Frobenius or 2-Frobenius group.

**Proof.** Let \( H \) be a non-trivial proper subgroup of \( G \) and \( p \in \pi(H) \). Suppose, on the contrary, that \( q \in \pi(G) \) and \( q \neq p \). Since \( p \mid 1 + s_p^H \) and \( q \mid 1 + s_q^H \), so \( s_p^H, s_q^H \neq \{0, 1\} \), hence \( s_p^H = s_q^H = n_H \). Now as \( H \) is nilpotent, according to Lemma 2.1, we have \( s_p^H = s_q^H = n_H^2 \), a contradiction. Thus \( H \) is a \( p \)-group. On the other hand, since

\[
p \mid 1 + s_p^H + s_q^H,
\]

so \( s_p^H \neq \{1, n_H\} \), since otherwise \( p \mid 1 \), a contradiction. Hence \( s_p^H = 0 \). It follows that every proper subgroup of \( G \) is \( p \)-group of exponent \( p \). If \( p, q \in \pi(G) \), then \( G \) has no element of order \( pq \). If \( G \) is nilpotent, then \( G \) is a \( p \)-group of exponent \( p \) and it is easy to see that such groups are in \( \mathcal{Y}_2 \), a contradiction. If \( G \) is non-nilpotent, then, as proper subgroup of \( G \) has the same-order type \( \{1, n\} \), Theorem 2.1 of Shen follows that \( G \) is a Schmidt group and so \( |\pi(G)| = 2 \). Now, as \( G \) has no element of order \( pq \), of [1, Theorem A], completes the proof. \( \square \)
REFERENCES


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