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CONJUGACY IN RELATIVELY EXTRA-LARGE ARTIN GROUPS

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Dedicated to the memory of my friend David Chillag

Communicated by Patrizia Longobardi

ABSTRACT. In this work we consider conjugacy of elements and parabolic subgroups in details, in a new class of Artin groups, introduced in an earlier work, which may contain arbitrary parabolic subgroups. In particular, we find algorithmically minimal representatives of elements in a conjugacy class and also an algorithm to pass from one minimal representative to the others.

1. Introduction

Let A be an Artin group with standard generators $X = \{x_1, \dots, x_n\}$, $n \geq 1$ and defining graph Γ_A . (See 2.1 for definition.) A *standard parabolic subgroup* of A is a subgroup generated by a subset of X . For elements u and v of A we say (as usual) that u is conjugate to v by an element h of A if $h^{-1}uh = v$ holds in A . Similarly, if K and L are subsets of A then K is conjugate to L by an element h of A if $h^{-1}Kh = L$.

In this work we consider the conjugacy of elements and standard parabolic subgroups of a certain type of Artin groups. Results in this direction occur in [4, 6, 7, 8, 10, 14, 13, 15], for example: Of particular interest are centralisers of elements, and of standard parabolic subgroups, normalisers of standard parabolic subgroups and commensurators of parabolic subgroups. In this work we consider similar problems in a new class of Artin groups, introduced in [11] where the word problem is solved, among other things. Also, intersections of parabolic subgroups and their conjugates are considered. To define this class of Artin groups, let N_A be the adjacency matrix of Γ_A , such that if $v_i, v_j \in V(\Gamma_A)$ and the vertices v_i and v_j are connected with an edge labelled by a natural number $\lambda(v_i, v_j) = n_{i,j}$

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then the (i, j) entry of N_A is n_{ij} . If v_i and v_j are not connected by an edge then the (i, j) entry of N_A is 0.

Let H be a standard parabolic subgroup of A , generated by $\{x_1, \dots, x_k\}$. We say that A is *extra-large relative to H* if N_A decomposes into $\begin{pmatrix} N_H & B \\ C & D \end{pmatrix}$, where N_H is the defining submatrix of H , B and C are block matrices of size $l \times k$ and $k \times l$, respectively, in which all the non-zero entries are at least 4 and D is a block matrix of size $l \times l$ in which all the non-zero entries are at least 3, where $k + l = n$. There are no assumptions on N_H .

We denote by $E_{0,1}$ the edges of Γ_A which have one endpoint in $\{v_1, \dots, v_k\}$ and the other endpoint is in $\{v_{k+1}, \dots, v_n\}$. Similarly, we denote by E_1 the set of edges of Γ_A with both endpoints in $\{v_{k+1}, \dots, v_n\}$ and by E_0 the set of all the edges with both endpoints in $\{v_1, \dots, v_k\}$. As usual, v_i is the vertex of Γ_A which corresponds to the generator x_i , $i = 1, \dots, n$.

To formulate our results we have to introduce some classes of words which describe minimal elements and their conjugacy, (see below). Thus, let $1 \neq u$ be an element of A represented by a reduced word U . Since we are interested in conjugacy, it is convenient to consider the cyclic word \widehat{U} , rather than the linear word U and we shall do so. As mentioned, we are interested, as usual, to represent \widehat{U} by a word which is minimal in some sense. In this work we allow three operations to reduce a word W , $W = x_{i_1}^{\alpha_1} \dots x_{i_m}^{\alpha_m}$: ζ , κ and \mathcal{L} , where

- (1) $\zeta(W)$ - the freely reduced form of W
- (2) $\kappa(W)$ - the sequence of cyclic conjugates of W , starting with $x_{i_1}^{\alpha_1}$ then $x_{i_2}^{\alpha_2}$ and so on
- (3) $\mathcal{L}(W)$ - the sequence of words obtained from $W := W_1W_2W_3$ reduced as written, where W_2 is a subword of a relator R having length $\geq \frac{1}{2}|R|$, by replacing W_2 with its complement W'_2 , $W_2W_2'^{-1} = R$.

Notice that ζ , κ , and \mathcal{L} are well defined.

For a set \mathcal{Y} of words let $\zeta(\mathcal{Y}) = \{\zeta(Y) \mid Y \in \mathcal{Y}\}$, $\kappa(\mathcal{Y}) = \{\kappa(Y) \mid Y \in \mathcal{Y}\}$, and define $\mathcal{L}(\mathcal{Y}) = \{\mathcal{L}(Y) \mid Y \in \mathcal{Y}\}$. Apply $(\zeta \circ \mathcal{L} \circ \zeta \circ \kappa)^m$, $m = 0, 1, 2, \dots$ on W . We get a set of words of length at most $|W|$. Therefore, there is a uniquely defined number m_0 such that $(\zeta \circ \mathcal{L} \circ \zeta \circ \kappa)^{m_0}(W) = (\zeta \circ \mathcal{L} \circ \zeta \circ \kappa)^{m_0+1}(W) = \dots$

Denote the set of words obtained by $\sigma(W)$. Thus $\sigma(W) = \bigcup_{i=1}^{m_0} (\zeta \circ \mathcal{L} \circ \zeta \circ \kappa)^i$ and $\sigma(W)$ is invariant under ζ , κ and \mathcal{L} .

To define the minimal representatives, we first construct a directed rooted tree $\Lambda_0(W)$, the vertices of which are labelled by the words in $\sigma(W)$. The root of $\Lambda_0(W)$ is the vertex v_0 , labelled with W . In the first level are the vertices labelled with the reduced cyclic conjugates of W . In the second level are all the results obtained from the vertices in the first level by applying $\zeta \circ \mathcal{L}$. Denote the tree obtained, by $\Theta(W)$. We consider now Θ as a map from F to directed labelled trees. Apply Θ to all the vertices of the second level. Then we get the third level, which contains all the reduced cyclic conjugates of the words in the second level, and then the fourth level, which contains all the reduced results of the

application of \mathcal{L} on the words in the third level. Continue this way (to at most $2m_0$ levels) until no new words are obtained.

We obtain $\Lambda(W)$ from $\Lambda_0(W)$, as follows:

(*) *If a word in level i is shorter than its immediate predecessor, then erase all its predecessors.*

The different words in each tree (connected component) form an equivalence class, which is the orbit under $\langle \mathcal{K}, \mathcal{L} \rangle$ on any one of them. Denote the equivalence class of a word U by $[U]$ and let U_1, \dots, U_m be the different representatives of the classes, (one representative for each class). Then we define $\rho_0(W) = \bigcup_{i=1}^m [U_i]$. Notice that $\rho_0(W)$ is well defined and can be constructed algorithmically. Notice that all the words in the same component have equal lengths, hence \mathcal{L} , when applied, replaces a subrelator by a subrelator of the same length.

In Theorem A we describe a modification $\rho(W)$ of $\rho_0(W)$ and the interrelation between its elements. Thus, let $W = x_{i_1}^{a_1} \cdots x_{i_m}^{a_m}$ be a word in $F := \langle X | - \rangle$, $x_{i_j} \neq x_{i_{j+1}}$, $j = 1, \dots, m - 1$. We define now the possible sets of words for $\rho(U)$.

$$\mathcal{W}_1 = \{ \widehat{W} \mid W = U_1 W_1 U_2 W_2 \cdots U_k W_k, k = 1, 2, 3, \dots \mid U_i \in F(V_0), W_i \in F(V_i) \}$$

$$\mathcal{W}'_2(a, b, c) = \{ \widehat{W} \mid W = W_1 U_1 \cdots W_k U_k, k = 1, 2, 3, \dots, U_i \in F(V_0) \text{ and either } W_i = a^{\alpha_i} b^{\beta_i} a^{\gamma_i} \text{ or } W_i = a^{\alpha_i} c^{\beta_i} a^{\gamma_i}, \text{ and } U_i = U'_i \text{ in } H, \text{ where } U'_i = c^{\delta_i} \text{ in } F \text{ in the first case and } U'_i = b^{\delta_i} \text{ or } c^{\delta_i} b^{\xi_i} \text{ or } b^{\delta_i} a^{\xi_i} \text{ in } F \text{ in the second case, for all } i, \text{ where } \{a, b, c\} \subseteq X, \lambda(a, b) = \lambda(a, c) = 4 \text{ and } \lambda(b, c) = 2 \text{ and at most one of } |\alpha_i|, |\beta_i|, |\gamma_i|, |\delta_i| \text{ is greater than } 1, \text{ where } \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{Z} \setminus \{0\}. \}$$

$$\mathcal{W}'_3(a, b, c) = \{ \widehat{W} \mid W = W_1 U_1 W_2 U_2 \cdots W_k U_k, k = 1, 2, 3, \dots, U_i \in F(V_0), \text{ and } U_i = U'_i \text{ in } H, W_i = a^{\alpha_i} \text{ and } U'_i = b^{\beta_i} c^{\gamma_i} \text{ in } F \text{ for } i \text{ even and } U'_i = c^{\gamma_i} b^{\beta_i} \text{ for } i \text{ odd. where } \lambda(b, c) = 2 \text{ and } \lambda(a, c) = \lambda(a, b) = 4 \text{ and if one of } |\beta_i| \text{ and } |\gamma_i| \text{ is greater than } 1, \text{ where } |\alpha_i| = 1, \text{ where } \alpha_i, \beta_i, \gamma_i \in \mathbb{Z} \setminus \{0\} \}$$

$$\mathcal{W}_4(a, b, c) = \{ \widehat{W} \mid W = W_1 \cdots W_k, k = 1, 2, \dots, W_i = a^{\alpha_i} b^{\beta_i} \text{ for } i \equiv 1 \pmod{3}, W_i = c^{\gamma_i} a^{\alpha_i} \text{ for } i \equiv 2 \pmod{3} \text{ and } W_i = b^{\beta_i} c^{\gamma_i} \text{ for } i \equiv 0 \pmod{3}, \text{ where } \{a, b, c\} \subseteq X \text{ and } \lambda(a, b) = \lambda(b, c) = \lambda(a, c) = 3 \text{ and } \alpha_i, \beta_i, \gamma_i \in \mathbb{Z} \setminus \{0\} \text{ and at most one of } |\alpha_i|, |\beta_i| \text{ and } |\gamma_i| \text{ are greater than } 1. \}$$

$$\mathcal{W}_5 = \{ a^m \mid a \in V_1, m = 1, 2, \dots \}$$

To obtain $\rho(U)$ from $\rho_0(U)$ we replace U_i by U'_i in $\mathcal{W}'_2(a, b, c)$ and $\mathcal{W}'_3(a, b, c)$. We need to make this modification because we have no information on the reduction procedure of words in H , hence we cannot guarantee in these cases that U_i is a shortest representative. On the other hand, it turns out that U'_i is a shortest representative of the image of U_i in H .

Next, we describe the elements which conjugate two words in $\rho(U)$. For elements of \mathcal{W}_i we denote the set of these words by \mathcal{W}_i^\perp , $i = 1, 2, 3, 4, 5$.

$$\mathcal{W}_1^\perp = \{a^n \mid a \in X, n \in \mathbb{Z}\}$$

$$\mathcal{W}_2^\perp = \{W \mid W = W_1 \cdots W_k, k = 1, 2, \dots, W_i = c^{\alpha_i} b^{\beta_i} a^{\gamma_i}, \text{ where } \{a, b, c\} \text{ is as in } \mathcal{W}_2 \text{ and at most two of } |\alpha_i|, |\beta_i|, |\gamma_i| \text{ is greater than } 1\}$$

$$\mathcal{W}_3^\perp = \{W \mid W = W_1 \cdots W_k, k = 1, 2, \dots, W_i = a^{\alpha_i} b^{\beta_i} a^{\gamma_i} c^{\delta_i}, \{a, b, c\} \text{ as in } \mathcal{W}_3 \text{ and no consecutive exponents in } (\alpha_i, \beta_i, \gamma_i, \delta_i, \alpha_i) \text{ may be both with absolute value greater than } 1\}$$

$$\mathcal{W}_4^\perp = \{W \mid W = W_1 \cdots W_k, k = 1, 2, \dots, W_i = a^{\alpha_i} c^{\beta_i} b^{\gamma_i}, \{a, b, c\} \text{ as in } \mathcal{W}_4 \text{ and as in } \mathcal{W}_3^\perp \text{ no two consecutive exponents may have absolute value at least } 2\}$$

To define \mathcal{W}_5^\perp we need to introduce some notations:

Let Γ_A be the defining graph of A . In what follows we identify the vertices of Γ_A with the generators they represent. Let $N(v) = \{w \in V(\Gamma_A), w \neq v \mid (w, v) \in E(\Gamma_A)\}$. Then $N(v) = N^o(v) \cup N^e(v)$ where $N^o(v) = \{w \in N(v) \mid \lambda(v, w) \text{ is odd}\}$ and $N^e(v) = \{w \in N(v) \mid \lambda(v, w) \text{ is even}\}$. For $u \in N^e(v)$ let $W(u) = (uv)^{\lambda(u,v)-1}u$ and let $\widehat{N}^e(v) = \langle W(u) \mid u \in N^e(v) \rangle$. Now let $\mu = v_1\mu_1v_2\mu_2 \cdots v_m$ be a path in Γ_A , $v_i \in V(\Gamma_A)$, $\mu_i \in E(\Gamma_A)$.

Call μ *admissible* if $v_{i+1} \in N^o(v_i)$, $i = 1, \dots, m - 1$. Denote by $Z(v_1, v_m)$ the set of all the admissible paths from v_1 to v_m . For $\mu \in Z(v_1, v_m)$ let $\mathcal{W}(\mu) = \{W_1 \cdots W_m \mid W_i \in \widehat{N}^e(v_i)\}$ and finally let $J(v_1, v_m) = \bigcup_{\mu \in Z(v_1, v_m)} \mathcal{W}(\mu)$. Thus, $J(v_1, v_m)$ is the set of all words which conjugate v_1 to v_m and are products of m subrelators of lengths $n_i - 1$, where $n_i = \lambda(v_i, v_{i+1})$, $i = 1, \dots, m - 1$.

Define

$$\mathcal{W}_5^\perp(a, b) = J(a, b) \quad \{a, b\} \subseteq X$$

From now on ρ , \mathcal{W}_i and \mathcal{W}_i^\perp , $i = 1, 2, 3, 4, 5$ will denote sets of words, as introduced above. Our main results are Theorems A, B and C.

Let A be an Artin group and let H be a standard parabolic subgroup of A . Suppose that A is extra-large relative to H .

Theorem A *Let W be a cyclically reduced word in F . One of the following holds*

- (a) $\rho(W) \subseteq H$
- (b) $\rho(W) \cap F(V_0) = \emptyset$, $\rho(W) = [U]$ for a cyclically reduced word U . In this case all the elements of $\rho(W)$ have the same length and if U_1 and U_2 are in $\rho(W)$ then U_2 can be obtained from U_1 via the reduction procedure described above.
- (c) $\rho(W) \cap F(V_0) = \emptyset$ and there are U_1 and U_2 in $\rho(W)$ with $[U_1] \neq [U_2]$. In this case there is a uniquely defined number $m \in \{1, 2, 3, 4, 5\}$ and for $m = 2, 3, 4$ a uniquely defined $\{a, b, c\} \subseteq X$,

forming a $(2, 4, 4)$ or $(3, 3, 3)$ triangle in Γ_A such that for certain words $U_1^* \in [U_1]$ and $U_2^* \in [U_2]$ the following hold

- (i) $U_1^*, U_2^* \in \mathcal{W}_m$
- (ii) U_1^* is conjugated to U_2^* by an element of \mathcal{W}_m^\perp

Theorem B deals with conjugacy of elements in H , while Theorem C deals with conjugacy of elements in $A \setminus H$.

Theorem B Assume $u, v \in H$ and $g \in A$.

- (a) If $g^{-1}ug = v$ and $g \notin H$ then u is conjugate in H to x^{m_1} for some $x \in V_0$ and $m_1 \in \mathbb{Z} \setminus \{0\}$ and v is conjugate in H to y^{m_2} , for some $y \in V_0$ and $m_2 \in \mathbb{Z} \setminus \{0\}$.
- (b) If $P \subseteq H$ is a non-cyclic parabolic subgroup of A then $C_A(P) = C_H(P)$, $N_A(P) = N_H(P)$ and the commensurator of P in A is its commensurator in H .
- (c) Let K be a subgroup of H such that $\langle z \rangle \cap K = \{1\}$, for every $z \in V_0$. If K is malnormal in H then K is malnormal in A .

Theorem C

- (a) Let P be a non-cyclic parabolic subgroup of A , $P \not\subseteq H$.
Then $C_A(P) = C_H(P)Z(P)$, where $Z(P)$ is the centre of P .
- (b) Let P and Q be non-cyclic parabolic subgroups of A , let $t \in A \setminus H$ and let $Z = t^{-1}Pt \cap Q$.
Assume $P \not\subseteq H$ and $Q \not\subseteq H$.
Then one of the following holds
 - (i) $Z = \{1\}$
 - (ii) Z is cyclic
 - (iii) $t \in \langle P, Q \rangle$
- (c) Let P be a non-cyclic parabolic subgroup of A , $P \not\subseteq H$. Let $g \in A \setminus H$ and let u, v be elements of P which are not conjugate to the power of $x^\varepsilon \in X$, $\varepsilon \in \{-1, 1\}$.
If $g^{-1}ug = v$ then $g \in P$. In particular, $N_A(P) = PC_H(P)$ and P is its own commensurator.

The work is organised as follows.

Section 2 contains preliminary results on Artin groups and their diagrams, which are the main tools of the work. In particular, we recall results on a small cancellation condition $(V(6))$ which is satisfied by certain modified diagrams (\tilde{M}') . We also discuss the ways conjugacy of elements is expressed in these modified diagrams.

Section 3 recalls known results from [1] on diagrams of Artin groups generated by two elements and also consider a special kind of diagram, needed for the proof of the main results.

Section 4 contains the basic structure theorems of simply connected and of annular diagrams recalled from [10] and [11] and some consequences needed in the sequel.

In Section 5 using a result of S. Orevkov we show that in any Artin group on at least three standard generators the product of powers of three generators is a shortest representative of the element it represents.

In Section 6 we prove the Main Theorems.

2. Preliminary Results

2.1. Artin groups, notation and definitions. (a) Let $X = \{x_1, \dots, x_n\}$, $n \geq 2$, be a set and let $F = F(X)$ be the free group, freely generated by X . For $1 \leq i < j \leq n$ let n_{ij} be natural numbers, let R_{ij} be the empty word if $n_{ij} = 0$ and let R_{ij} be the word (relator) in F defined by $R_{ij} = U_{ij}V_{ij}$, where U_{ij} is the head of length n_{ij} of $(x_i x_j)^{n_{ij}}$ and V_{ij} is a head of length n_{ij} of $(x_j^{-1} x_i^{-1})^{n_{ij}}$, if $n_{ij} > 0$. Thus R_{ij} is $(x_i x_j)^{m_{ij}} x_i (x_j^{-1} x_i^{-1})^{m_{ij}} x_j^{-1}$ if n_{ij} is odd, $n_{ij} = 2m_{ij} + 1$ and R_{ij} is $(x_i x_j)^{m_{ij}} (x_i^{-1} x_j^{-1})^{m_{ij}}$ if n_{ij} is even, $n_{ij} = 2m_{ij}$. Let N be the $n \times n$ matrix defined by $(N)_{(i,j)} = n_{ij}$ if $i < j$, $n_{ji} = n_{ij}$, $n_{ii} = 0$. The group A defined by the presentation

$$(2.1) \quad \langle x_1, \dots, x_n \mid R_{ij}, n_{ij} \neq 0 \rangle$$

is called the *Artin group defined by N* . The x_i are called the *standard generators of A* and the presentation (1) is called *the standard presentation of A defined by N* . The subgroups of A generated by subsets of X are called *standard parabolic subgroups*. An alternative way to define A is by a graph Γ with vertex set $V(\Gamma) = \{v_1, \dots, v_n\}$ in one to one correspondence with x_1, \dots, x_n and the edge set $E(\Gamma) = \{(v_i, v_j) \mid n_{ij} \neq 0\}$ such that the edge (v_i, v_j) is labelled with n_{ij} . We shall identify v_i with x_i and denote $n_{ij} = \lambda(v_i, v_j)$, except in the definition of $J(a, b)$ in the introduction.

Artin groups for which $n_{ij} \geq 4$ for all non-zero n_{ij} are called *extra-large* and those for which $n_{ij} \geq 3$ for all non-zero n_{ij} are called *large*. An Artin group in which all the nonzero n_{ij} are 2 is called a *right-angled Artin group*.

To each Artin group given by (1) there corresponds a Coxeter group obtained from (1) by adding the relations $x_1^2, x_2^2, \dots, x_n^2$. An Artin group is called *of finite type* if the corresponding Coxeter group is finite.

We shall often use the following general results.

Theorem 2.1 ([13] (Parabolics embed)). *Let A be an Artin group given by (1) and let S be a subset of X . Let Γ_S be the subgraph of the defining graph Γ of A . Let $P = \langle S \rangle$. Then P is an Artin group with defining graph Γ_S .*

Theorem 2.2 ([3] (Parabolics are convex)). *Let notation be as in Theorem 2.1 and let $W \in F(S)$ be a word. If U is a shortest representative for W in A then $U \in \langle S \rangle$.*

Finally, when it causes no ambiguity we shall not distinguish between elements of A and their representatives in F and similarly between elements of H and words of $F(x_1, \dots, x_k)$ which represent them.

2.2. Diagrams and small cancellation conditions. For diagrams in general we follow [12, Ch. V]. We denote by Φ the labelling function of a diagram M over a free group F . (Thus, $\Phi : M \rightarrow F$). We denote by $Reg(M)$ the set of all the regions of M . We shall also use the following notation.

Let W be a reduced word in $F(x_1, \dots, x_n)$. Denote, as usual, by $\|W\|$ its *syllable length* and denote by $l(W)$ its *word length*. For a word $W = x_{i_1}^{\alpha_1} \cdots x_{i_m}^{\alpha_m}$, $\alpha_i \in \mathbb{Z} \setminus \{0\}$, we denote $Supp(W) =$

$\{x_{i_1}, \dots, x_{i_m}\}$. Thus, if $W = x_1^2 x_2^3 x_1^{-1} x_4^4$ then $\|W\| = 4$, $l(W) = 2 + 3 + |-1| + 4 = 10$ and $Supp(W) = \{x_1, x_2, x_4\}$. If μ is a path in M we write $l(\mu)$ for $l(\Phi(\mu))$ and for a path μ write $Supp(\mu)$ for $Supp(\Phi(\mu))$. Also, for a region D in M write $Supp(D)$ for $Supp(\Phi(\partial D))$. For a path μ denote by $h(\mu)$ and $t(\mu)$ the endpoints of μ , h for head and t for tail.

It was observed already in [1] that Artin groups on more than two standard generators do not satisfy small cancellation conditions. To overcome this difficulty, a modification has been carried out on the diagrams which declares certain subdiagrams as new regions - which we will call in this work *derived regions* - such that the obtained diagrams - which we will call in this work *derived diagrams* - satisfy the small cancellation condition $C(6)$, for large Artin groups (see [1], for the definition of the $C(6)$ condition see [12, Ch. V]). It turns out that for our work we have to carry out a further modification on the derived diagrams, as described in [11], in order to get a diagram which satisfies a more involved small cancellation condition. We describe both modifications below.

2.2 (a) The underlying small cancellation condition

The main tool we use in this work is the small cancellation condition $V(6)$ and its theory which we recall in Section 4. In this subsection we present the main notions involved.

Definition 2.3 ([11]). *Let M be a map. M satisfies the small cancellation condition $V(6)$ if each of the following holds:*

- (i) *Each inner region has at least 4 neighbours;*
- (ii) *If an inner region has less than 6 neighbours then every vertex on its boundary has valency at least 4.*

Notice that the $C(4)$ & $T(4)$ and also the $C(6)$ small cancellation conditions are special cases of the $V(6)$ condition.

Definition 2.4. *Let D be a boundary region of M . Call D a k -corner region of M , $k \in \{1, 2, 3\}$ if the following hold*

- (1) *$\partial D \cap \partial M$ is connected;*
- (2) *D has k neighbours in M*

Definition 2.5 (one layer maps). (a) *Let M be a connected, simply connected regular map (i.e. every edge is on the boundary of a region). M is called a one layer map if its dual is a line segment.*

(b) *Let A be an annular regular map. A is called a one layer map if its dual is a circle.*

Finally, we describe the basic properties of $V(6)$ diagrams, as we shall need this later.

The main result of [11] on annular diagrams A with the small cancellation condition $V(6)$ is that A splits into annular subdiagrams with boundaries homotopic to the outer boundary $\omega(A)$ and the inner boundary $\tau(A)$ of A , respectively and the regions inside each layer are controlled and behave "nicely".

More precisely, we have the following:

Denote by $d_M(D)$, as usual, the number of neighbours (neighbouring regions) of a region D in M . For a vertex v in M denote, as usual, by $d_M(v)$ the valency of v in M .

We denote for a boundary path μ the number of segments it contains by $|\mu|$, where a segment is a path with endpoints having valency at least 3 and every inner vertex has valency 2.

We have the following:

Theorem 2.6 ([11], Theorem 1). *Let A be a regular annular map with connected interior which satisfies the condition $V(6)$. Let L_0 be an annular submap with boundaries homotopic to the boundary of A , with minimal possible number of regions. Define $S_0 = L_0$ and for $i > 0$ define*

$$L_i = \{D \notin S_{i-1} \mid \partial D \cap \partial L_{i-1} \neq \emptyset\}$$

and define

$$S_i = S_{i-1} \cup L_i$$

Similarly, for $i < 0$ define

$$L_i = \{D \notin S_{i+1} \mid \partial D \cap \partial L_{i+1} \neq \emptyset\}$$

and $S_i = S_{i+1} \cup L_i$.

- (a) *Then, S_i is annular and A is the union of all the S_i . Either L_i is an annular one-layer map, or L_i is the union of simply connected one-layer maps. Also, for $i > 0$ $|\partial S_i| \leq |\partial S_{i+1}|$, and similarly for $i < 0$.*

For $i \geq 0$ we denote by $\omega(L_i)$, $(\omega(S_i))$, the outer boundary of L_i (S_i) and denote by $\tau(L_i)$ ($\tau(S_i)$) the inner boundary of L_i (S_i).

Similar notations used for $i < 0$, accordingly.

- (b) *Let D be a region in L_i , $i \geq 0$. Then D has at most 2 neighbours in L_i . If $d_M(D) < 6$ then D has at most 2 neighbours in L_{i-1} and if it has exactly two neighbours then there is a region E in L_i with $\partial E \cap \omega(L_{i-1}) = \{v\}$, v a point. We call such regions 3-exceptional. If $d(D) \geq 6$ then D has at most 3 neighbours in L_{i-1} and if it has exactly three neighbours in L_{i-1} then there is a region E in L_i which has at most one neighbour in L_{i-1} . We call such regions 2-exceptional. Similar statement holds for $D \in L_i$, $i < 0$.*
- (c) *Let v be a vertex in $\omega(L_i)$. Then $d_{L_i}(v) \leq 4$ and $d_{L_i}(v) = 4$ if and only if v is on the boundary of a region D in L_i with $d_A(D) = 4$ and has two neighbours in L_{i-1} . Similar statements hold for $v \in \tau(L_i)$, $i < 0$.*

For further application for a region D in L_i , $i \geq 1$, denote

$\mathcal{A}(D)$ the set of neighbours of D in L_{i-1} , and $a(D) = |\mathcal{A}(D)|$,

$\mathcal{B}(D)$ the set of neighbours of D in L_i , and $b(D) = |\mathcal{B}(D)|$,

$\mathcal{C}(D)$ the set of neighbours of D in L_{i+1} (L_{i-1} if $i < 0$), and $c(D) = |\mathcal{C}(D)|$.

Now, let M be a simply connected diagram with connected interior, satisfying the small cancellation condition $V(6)$. Then we have the following Theorem.

Theorem 2.7 ([11], Theorem 2). *Suppose M is simply connected with connected interior containing at least two regions. If M satisfies the small cancellation condition $V(6)$, then for every vertex v M has a layer structure with $L_0 = \{v\}$, as described above, with the following properties:*

- (1) *Every vertex v on $\omega(L_i)$ has valency ≤ 3 in L_i .*
- (2) *The last layer contains at least two corner regions of M .*
- (2') *The last layer contains a corner region with $l(\partial M \cap \partial \Delta) \geq \frac{1}{2}l(\partial \Delta)$*

2.2 (b) Derived diagrams

Let M be an \mathcal{R} -diagram, \mathcal{R} a set of Artin relators. Say that two regions are *neighbours* or are *adjacent* if they have a common edge. Say that two adjacent regions D_1 and D_2 are *friends*, if $Supp(D_1) = Supp(D_2)$. Let " \sim " be the transitive closure of friendness. Then " \sim " is an equivalence relation on $Reg(M)$. For a region $D \in Reg(M)$ denote by $[D]$ the equivalence class of D and let

$$\Delta(D) = Int \left(\bigcup_{E \in [D]} \bar{E} \right), \text{ where } \bar{E} \text{ denotes the closure of } E \text{ in } \mathbb{E}^2.$$

It is standard to see that if M is simply connected then $\Delta(D)$ is homeomorphic to an open disc, hence we may consider $\Delta(D)$ as a region. We denote by M' the diagram obtained by considering $\Delta(D)$, for $D \in Reg(M)$, as regions and call M' the *derived diagram* of M . We call $\{\Delta(D) | D \in Reg(M)\}$ the *derived regions* of M' . Observe that $l(\partial M') = l(\partial M)$ and $\Phi(\partial M') = \Phi(\partial M)$, since by definition $\partial M = \partial M'$. The main property of these diagrams is given by the lemma below.

Lemma 2.8 ([1], Lemma 3). *Let notation be as above and let Δ_1 and Δ_2 be derived regions of M' . Then $||\partial \Delta_1 \cap \partial \Delta_2|| \leq 1$.*

As we shall see, combining Lemma 2.8 with Lemma 3.1 (a) implies that derived diagrams of Artin groups of large type satisfy the small cancellation condition C(6).

2.2 (c) Derived Howie Diagrams

The presentation (1) is usually called a *free presentation*, because A is presented as a homomorphic image of a free group. In the sequel we shall also need a *free-product presentation* of A , sometimes also called a relative presentation. See [2]. Thus, let V_0 be a subset of V and let $V = V_0 \dot{\cup} V_1$. Let \mathcal{R}_0 be the symmetric closure (s.c.) of the set of all the relations involving letters of V_0 only, let \mathcal{R}_1 be the s.c. of the set of all the relations involving letters of V_1 only and let $\mathcal{R}_{0,1}$ be the s.c. of the set of all the relations R_{ij} in which $i \in V_0$ and $j \in V_1$.

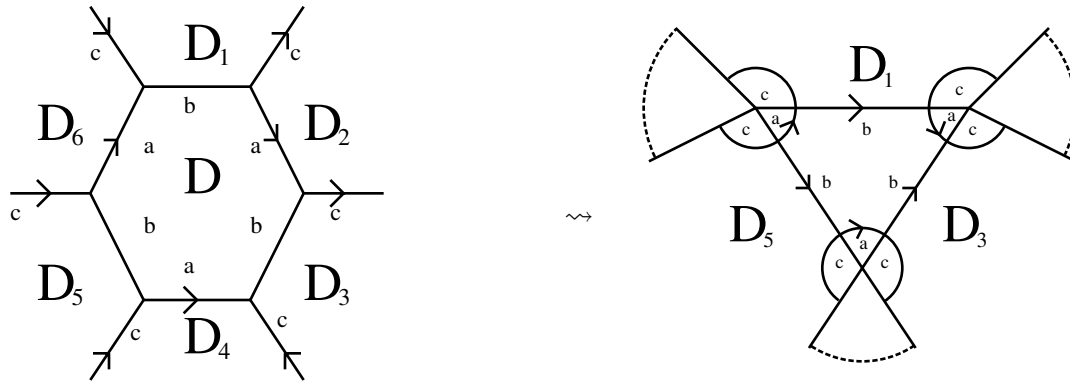
Clearly, $\mathcal{R} = \mathcal{R}_0 \dot{\cup} \mathcal{R}_{0,1} \dot{\cup} \mathcal{R}_1$, where \mathcal{R} is the s.c. of $\{R_{ij}\}$ in (1). Then $F(V_0)/\mathcal{R}_0$ is isomorphic to the subgroup H of A generated by the images of V_0 by the natural projection $\pi : F \rightarrow A$ corresponding to (1) and A is isomorphic to

$$(2.2) \quad \langle H * F(V_1) \mid \mathcal{R}_{0,1} \cup \mathcal{R}_1 \rangle$$

This presentation has been introduced in order to describe extra-large Artin groups relative to H .

Just as van Kampen diagrams correspond to free presentations, relative diagrams, which are also called *Howie diagrams* (see [2]), correspond to relative presentations. Let \tilde{M}' be such a diagram

obtained from M' by shrinking the \mathcal{R}_0 -subdiagrams to a point. (See [11]). Then the regions of \tilde{M}' are $Reg_{E_1}(\tilde{M}') \cup Reg_{E_0,E_1}(\tilde{M}')$, where $Reg_{E_1}(\tilde{M}')$ is the set of regions of \tilde{M}' with boundary label from \mathcal{R}_1 and Reg_{E_0,E_1} is the set of regions of \tilde{M}' with boundary label from $\mathcal{R}_{0,1}$, after shrinking the edges with labels in V_0 to vertices and introducing them as *corner labels*. (This notion has nothing to do with corner regions.) See Figure 1. We call such vertices, with corner labels from V_0 , *coloured vertices*. The main properties of \tilde{M}' for relatively extra-large M are summarised below.



$$c, a \in V_0, b \in V_{0,1}$$

Figure 1

Lemma 2.9 ([11], Lemma 2.1). *Let Δ' be an inner region of M' , Δ' simply connected. Denote by $\tilde{\Delta}'$ the corresponding region in \tilde{M}' , for $\Delta' \in Reg_{E_1}(M') \cup Reg_{E_0,E_1}(M')$.*

- (a) *If $\tilde{\Delta}' \in Reg_{E_1}(\tilde{M}')$ then the corner label of $\tilde{\Delta}'$ at each boundary vertex is 1*
- (b) *If $\tilde{\Delta}' \in Reg_{E_0,E_1}(\tilde{M}')$ with $Supp(\tilde{\Delta}') = \{a_i, a_j\}$, $a_i \in V_0$, $a_j \in V_1$ then at every boundary vertex of $\tilde{\Delta}'$ the corner label is $a_i^{\alpha_i}$, $\alpha_i \in \mathbb{Z} \setminus \{0\}$.*

In particular, every boundary vertex of $\tilde{\Delta}'$ is a coloured vertex.

- (c) *If v is an inner coloured vertex of \tilde{M}' , then $d_{\tilde{M}'}(v) \geq 4$.*

We also have the following

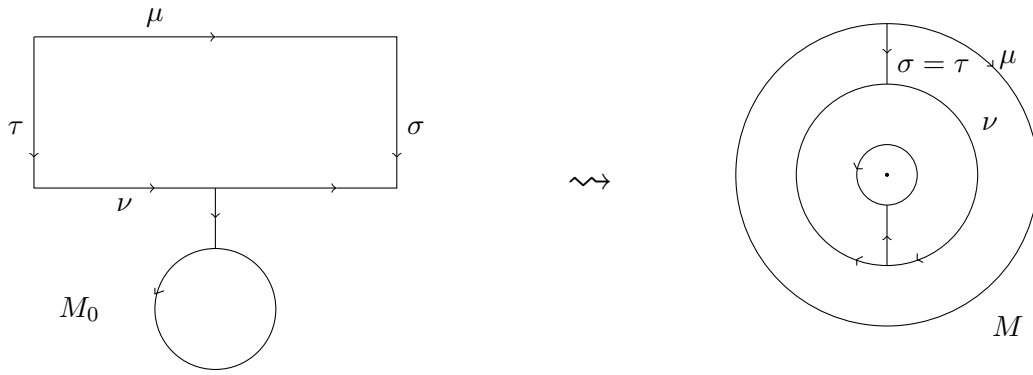
Lemma 2.10 ([11], Lemma 2.2). *Let $\tilde{\Delta} \in Reg(\tilde{M}')$, Δ simply connected.*

- (a) *If $\tilde{\Delta} \in Reg_{E_1}(\tilde{M}')$ and is an inner region of \tilde{M}' then $d_{\tilde{M}'}(\Delta) \geq 6$.*
- (b) *Let $\tilde{\Delta} \in Reg_{E_0,E_1}(\tilde{M}')$. Then*
 - (i) *Every inner vertex of $\partial\tilde{\Delta}'$ has valency at least 4.*
 - (ii) *If $\tilde{\Delta}$ is an inner region of \tilde{M}' then $d_{\tilde{M}'}(\Delta) \geq 4$.*
- (c) *Every region in \tilde{M}' either belongs to $Reg_{E_1}(\tilde{M}')$ or to $Reg_{E_0,E_1}(\tilde{M}')$ and \tilde{M}' satisfies the condition $V(6)$.*

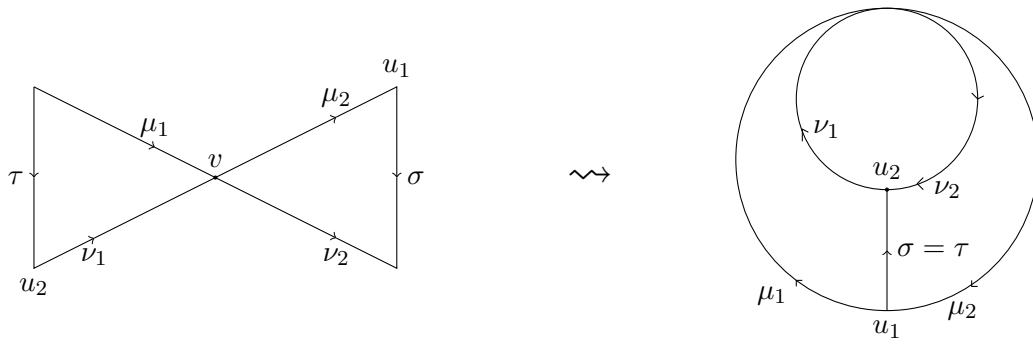
2.3. Conjugacy diagrams. We follow [12]. Let A be a group.

Let $u, v \in A$ and assume that $t^{-1}ut = w$ for some $t \in A$. Let U, W and T be representatives of u, w and t in F , respectively. Then there exists a van Kampen diagram M_0 with boundary cycle $\tau^{-1}\mu\sigma\nu^{-1}$

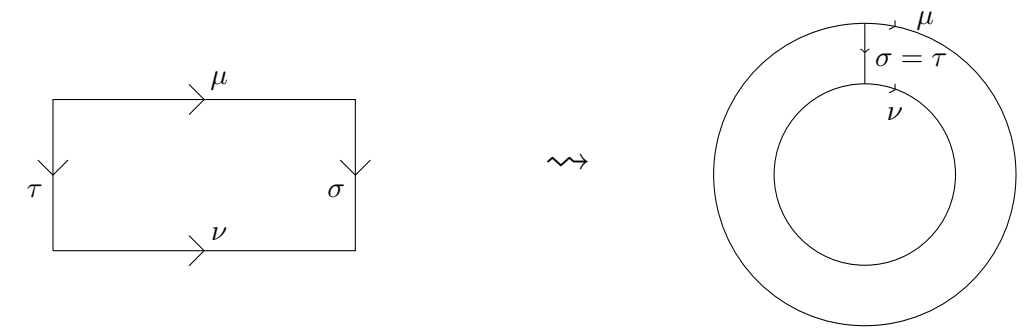
such that $\Phi(\tau) = \Phi(\sigma) = T$, $\Phi(\mu) = U$ and $\Phi(\nu) = W$. Since $\Phi(\sigma) = \Phi(\tau)$, we can glue (identify) σ to τ to get a map M , without distorting $Int(M_0)$ and without leaving the plane. See Figure 2 for possible outcomes.



(a)



(b)



(c)

Figure 2

In Figure 2(a) $Int(M)$ is not connected. As a result ν is not a simple closed curve. This means that W contains a non-empty subword W_0 with $W_0 = 1$ in A . To avoid this situation we shall require that

(\mathcal{H}_1) Neither U nor W contains non-empty subwords which represent 1 in A .

In Figure 2(b) $\mu \cap \nu \neq \emptyset$. As a result we can subdivide μ into $\mu = \mu_1 w \mu_2 u_1$ and subdivide ν into $\nu = u_2 \nu_1 w \nu_2$ such that in M $\Phi(\mu_2 u_1 \mu_1) = \Phi(\nu_2 u_2 \nu_1)$. Hence, if we let $U_i = \Phi(\mu_i)$ and $W_i = \Phi(\nu_i)$, $i = 1, 2$ then $U_2 U_1 =_A W_2 W_1$. Since $U_2 U_1 = U_1^{-1} U U_1$ and $W_2 W_1 = W_1^{-1} W W_1$, we see that a cyclic conjugate of U equals in A to a cyclic conjugate of W . Concerning the cyclic words \hat{U} and \hat{W} this means that $\hat{U} =_A \hat{W}$.

Since we deal with cyclic words \hat{U} and \hat{V} , in what follows we shall assume, without loss of generality, (\mathcal{H}_1) and the following

(\mathcal{H}_2) U and W have no cyclic conjugates U^* and W^* with $U^* =_A W^*$.

(\mathcal{H}_3) U and W are cyclically reduced.

Under these assumptions $Int(M)$ is an annular diagram with boundary labels U and W , respectively. See Figure 2(c).

Now let A be a relatively extra-large Artin group and let M be an annular \mathcal{R} -diagram with connected interior. When forming M' from M then in contrast to the simply connected case it may happen that $\Delta(D)$ is not homeomorphic to an open disc. Due to a standard argument, using the fact that every 2-generated Artin group is torsion free, it follows that in this case $Int(\Delta(D))$ is annular with boundary components homotopic to the boundary components of M . It will turn out (see Section 6) that the structure of derived diagrams with such a derived region is different from the structure of annular derived diagrams with simply connected derived regions.

Next, when forming \tilde{M}' from M' , we shrink some of the edges of the annular diagram M' . As a result, we get

Lemma 2.11. *Let notation be as above. Then \tilde{M}' satisfies one of the following:*

(a) *For every $D \in Reg(M)$, $\Delta(D)$ is simply connected.*

Subcase 1 \tilde{M}' contains an annular subdiagram M_v over $F(V_0)$, maximal with respect to being over $F(V_0)$, which is homotopic to the boundary components of \tilde{M}' . See Figure 3(a).

Then \tilde{M}' is the union of two simply connected diagrams \tilde{M}'_1 and \tilde{M}'_2 with $\tilde{M}'_1 \cap \tilde{M}'_2 = \{v\}$, point v corresponds to M_v in M' . See Figure 3(a).

Subcase 2 M' contains no subdiagram M as in Subcase 1.

Then one of the following holds:

(i) *There is no \mathcal{R}_0 -subdiagram N' in M' with $\partial N' \cap \omega \neq \emptyset$ and $\partial N' \cap \tau \neq \emptyset$, where ω and τ are the boundary components of M . Then \tilde{M}' is an annular diagram with connected interior.*

(ii) *There is an \mathcal{R}_0 -subdiagram N' in M' with $\partial N' \cap \omega \neq \emptyset$ and $\partial N' \cap \tau \neq \emptyset$. Then $Int(\tilde{M}'_1)$ is simply connected, when $\tilde{M}'_1 = \tilde{M}' \setminus \{v\}$. See Figure 3(b).*

(b) $\Delta(D)$ is annular for some $D \in Reg(M)$. See Figure 3(d).

Subcase 1 $\Phi(\partial\Delta(D)) \in \mathcal{R}_0$. Then Subcase 1 of Case a applies.

Subcase 2 $\Phi(\partial\Delta(D)) \notin \mathcal{R}_0$. See Figure 3(c).

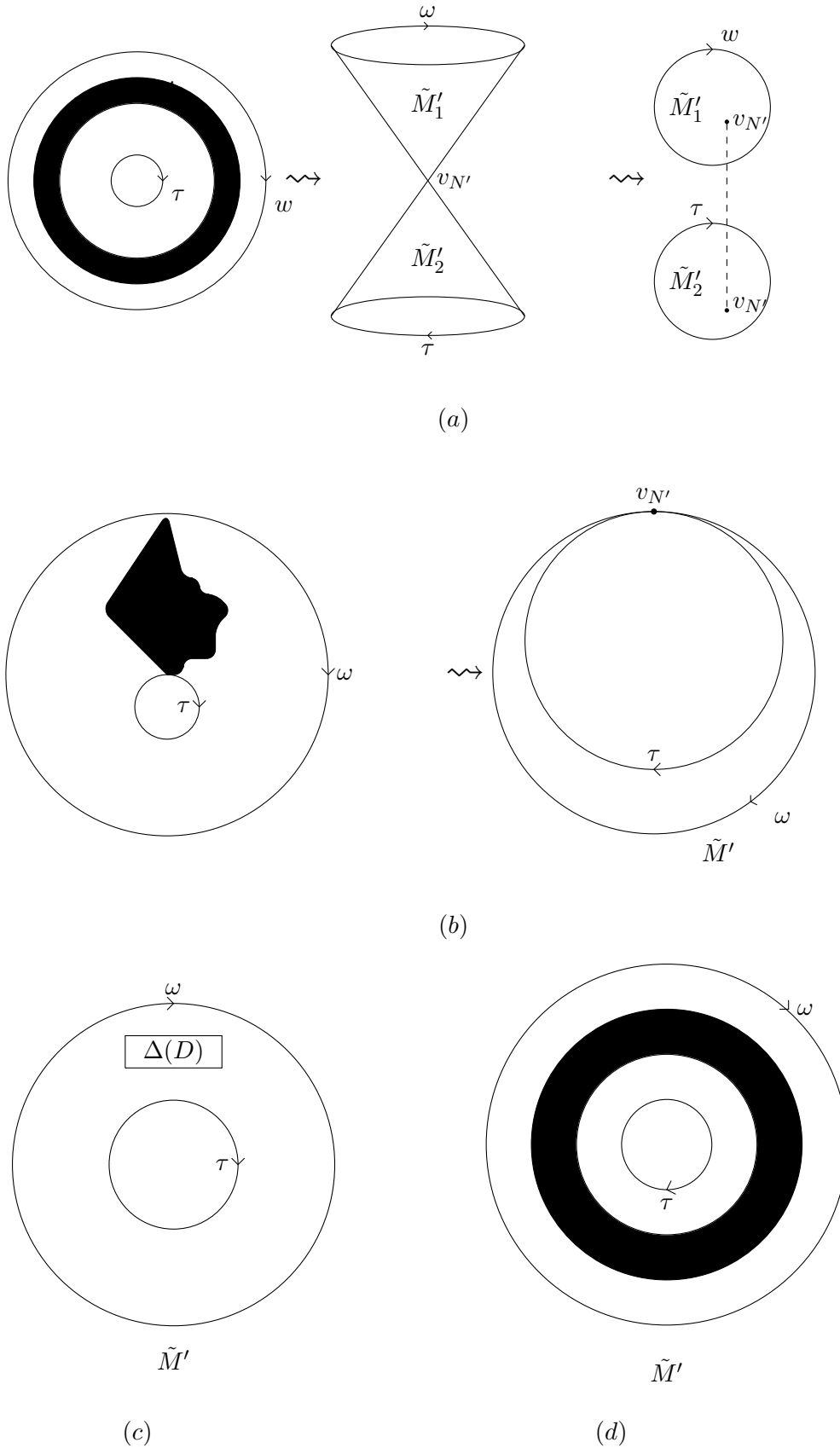


Figure 3

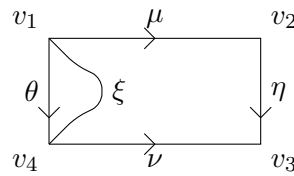
Proof. Clearly all the possibilities are covered. The results follow immediately from the construction of \tilde{M}' . □

Lemma 2.12. *Let $\langle X \mid \mathcal{R} \rangle$ be a presentation of a group and let M_0 be a minimal simply connected \mathcal{R} -diagram with connected interior and boundary cycle $v_1\mu v_2\eta v_3\nu v_4\theta$, such that*

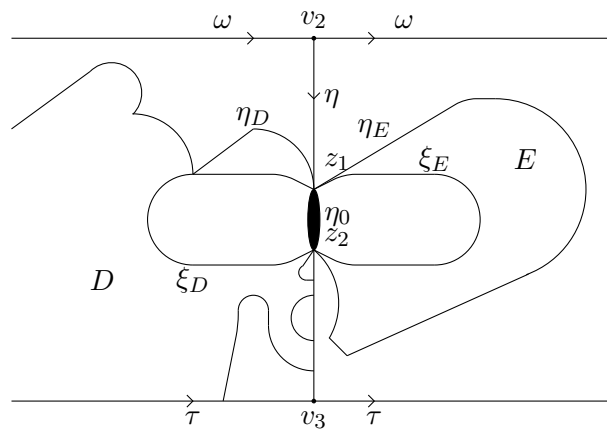
- (1) $\Phi(\mu) = U$, $\Phi(\nu) = W$, $\Phi(\eta) = T$ and $\Phi(\theta) = T^{-1}$
- (2) U, W and T are shortest cyclically reduced representatives.

Since $\Phi(\eta) = \Phi(\theta)^{-1}$, we can glue θ along η to get an annular diagram M_1 . Let M be the annular diagram obtained from M_1 , after cancelling out all cancelling pairs. Then

- (1) $\omega(M) = \omega(M_1)$ and $\tau(M) = \tau(M_1)$
- (2) M contains a path ξ with endpoints v_1 and v_4 such that $\Phi(\xi) =_A T$.



(a)



(b)

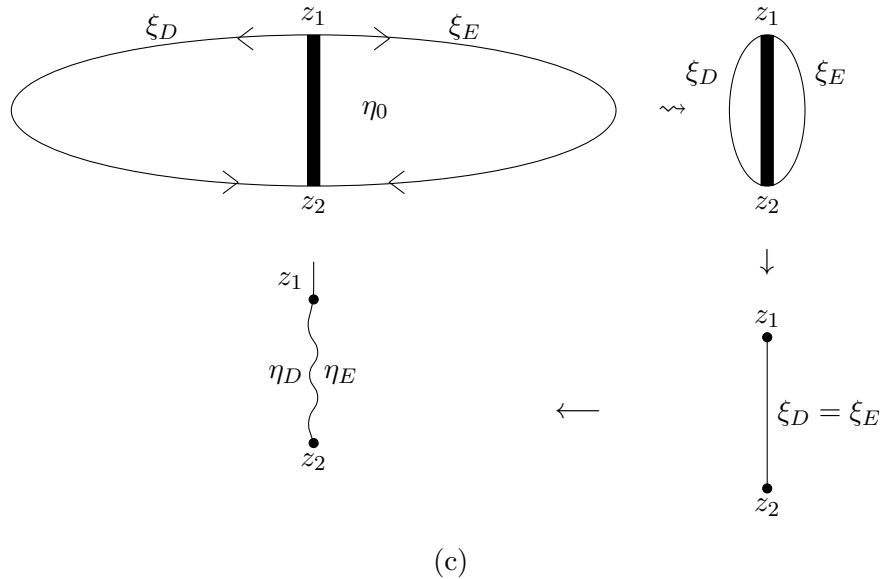


Figure 4

Proof. Let (D, E) be a cancelling pair. Since M_0 is minimal, D and E are boundary regions, which became a cancelling pair due to the identification of η with θ . Hence, after the identification $\partial D \cap \partial E \cap \eta \neq \emptyset$. Let η_0 be a connected component of $\partial D \cap \partial E \cap \eta$. Let η_D be the complement of η_0 in ∂D and let η_E be the complement of η_0 in ∂E . See Figure 4(b). Since (D, E) is a cancelling pair, $\Phi(\eta_D) = \Phi(\eta_E)$. To carry out the cancellation, we add two simple paths inside D and E , ξ_D and ξ_E respectively, with labels $\Phi(\eta_D)$ and $\Phi(\eta_E)$, respectively. Now we replace η_0 with ξ_D and identify ξ_D with ξ_E . See Figure 4(c). (We can do this because $\Phi(\eta_D) = \Phi(\eta_E)$ and $Int(D)$ is homeomorphic to an open disc.) This way we replaced η_0 by $\xi_D (= \xi_E)$. Finally, we identify η_D with ξ_D and η_E with ξ_E . See Figure 4(b). This ends the cancellation of the pair (D, E) towards getting M from M_1 . Since $\Phi(\eta_D) =_A \Phi(\eta_0)$, after carrying out this procedure for all cancelling pairs we did not change $\Phi(\eta) \bmod \mathcal{N}$, where \mathcal{N} is the normal closure of \mathcal{R} in F . Consequently, after carrying out all the cancellations of all the cancelling pairs we get a path η' with $\Phi(\eta') \equiv \Phi(\eta) \bmod \mathcal{N}$. Since the above procedure does not change ω and τ (though may distort the inner structures by identifying certain edges or vertices in course of the procedure), our claim holds. \square

3. Two-generated Artin groups and their diagrams

Two-generated parabolic subgroups of Artin groups are very important ingredients of the study of Artin groups via diagrams. (See [1].) The following two results from [1] are fundamental.

Lemma 3.1 ([1], Lemma 6). *Let A be given by the presentation $\langle x_1, x_2 \mid \mathcal{R}_{12} \rangle$ where \mathcal{R}_{12} is the symmetric closure of $R_{1,2}$ in F , and let $1 \neq W$ be a cyclically reduced word in $F(x_1, x_2)$ which represents 1 in A . Then*

(a) $\|W\| \geq \|R_{1,2}\|$

(b) Every $\mathcal{R}_{1,2}$ -diagram satisfies the small cancellation condition $C(4)$ & $T(4)$. (See [12, Ch. V].)

We shall need a more detailed version of part (a) of the Lemma for the special case when $m := ||R_{1,2}||/2 = 4$. For this we need the Lemma below.

Lemma 3.2 ([1], Lemma 7). *Let $1 \neq W$ be a cyclically reduced word in $F(x_1, x_2)$ which represents 1 in A . Let $W = W_1W_2$, reduced as written.*

- (a) *If $||W_1|| \leq n_{1,2} \left(= \frac{||R_{1,2}||}{2} \right)$ then $|W_1| \leq |W_2|$.*
- (b) *If $||W_1|| < n_{1,2}$ then $|W_1| < |W_2|$.*

Corollary 3.3. *Let A be an Artin group given by (1). Let $x_1^{\alpha_1}x_2^{\alpha_2}W = 1$ in A , W an arbitrary reduced word. Then $|W| \geq |\alpha_1| + |\alpha_2|$.*

Proof. Without loss of generality we may assume that $x_1^{\alpha_1}x_2^{\alpha_2}V$ is cyclically reduced, as written. By Lemma 3.2 $|x_1^{\alpha_1}x_2^{\alpha_2}| \leq |Z|$, for every $Z \in \langle x_1, x_2 \rangle$ with $Z^{-1} =_A x_1^{\alpha_1}x_2^{\alpha_2}$.

Let U be a shortest representative of $x_1^{\alpha_1}x_2^{\alpha_2}$ in A . Then by Theorem 2.2 $U \in \langle x_1, x_2 \rangle$, hence $|x_1^{\alpha_1}x_2^{\alpha_2}| \leq |U|$. But clearly $|U| \leq |x_1^{\alpha_1}x_2^{\alpha_2}|$ by choice of U , hence $x_1^{\alpha_1}x_2^{\alpha_2}$ is a shortest representative for $x_1^{\alpha_1}x_2^{\alpha_2}$ and in particular $|W| \geq |\alpha_1| + |\alpha_2|$. □

Corollary 3.4. *Let $x_1 x_2 x_3 \in X$. Then $x_1^\alpha x_2^\beta x_3^\gamma \neq 1$ in A , if not all of α, β, γ are zero.*

Proof. Suppose $x_1^\alpha x_2^\beta x_3^\gamma = 1$ in A . Then due to Corollary 3.3 we may assume

$$|\{x_1, x_2, x_3\}| = 3$$

Again from Corollary 3.3 we get $|\alpha| = |\beta| + |\gamma|$, $|\beta| = |\alpha| + |\gamma|$ and $|\gamma| = |\alpha| + |\beta|$. Denote $C = |\alpha| + |\beta| + |\gamma|$. Then adding up the equations yields $C = 2C$, i.e. $C = 0$, a contradiction. □

Definition 3.5. (a) *Let D be a region in an $\mathcal{R}_{1,2}$ -diagram M . Then UV^{-1} is a boundary label of D , where U and V are as in subsection 2.1. Let μ be the boundary path with $\Phi(\mu) = U$ and let ν be the boundary path of D with $\Phi(\nu) = V$. Then $\nu\mu\nu$ is a boundary cycle of D . The vertices v and w are called separating vertices and the line segment connecting them is called separating segment. (It does not belong to the diagram.)*

- (b) *Let v be an inner vertex in M with valency $2m$ and let D_1, \dots, D_{2m} be the regions which contain v on their boundary, in this order clockwise.*

Call v alternating if v is a separating vertex for all D_i with i odd (even) and v is not a separating vertex for all D_i with i even (odd).

Lemma 3.6. *Let M be a reduced van Kampen $\mathcal{R}_{1,2}$ diagram with connected interior and with maximal number of vertices with valency 2, among all van Kampen diagrams with the same boundary label. Then every inner vertex with valency 4 in M is alternating.*

Proof. Assume by way of contradiction that the statement of the Lemma is false and let v be an inner vertex with valency 4 which is not alternating. Then v is not a separating vertex for any region which contains it on its boundary, because in a reduced diagram, no adjacent regions may have a common

separating vertex. For each inner vertex v with valency 4 let $\theta(v)$ be the length of a longest common edge of two adjacent regions which contain v on their boundary. Let w be a vertex with valency 4 with maximal $\theta(v)$ among all vertices with valency 4 in all diagrams with boundary label, like M , satisfying the assumption of the Lemma.

Let u_1 and u_2 be the neighbours of w , $u_1 \in \partial D_1$ and $u_2 \in \partial D_2$, where $\theta(w) = |\partial D_1 \cap \partial D_2|$. Carry out a diamond move T around w which identifies u_1 with u_2 . See Figure 5. Then T splits w into v_1 and v_2 , such that $v_1 \in \partial D_3 \cap \partial D_4$ and $v_2 \in \partial D_1 \cap \partial D_2$, both vertices with valency 2. By the maximality assumption, u_1 and u_2 in M should have valency 2. Therefore $u = u_1 = u_2$ (after carrying out T) has valency

$$(d_M(u_1) + 1) + (d_M(u_2) + 1) - 2 = d_M(u_1) + d_M(u_2) = 4$$

and u is not alternating. Therefore u is an inner vertex (after carrying out T) on the common boundary of D_1 and D_2 with valency 4, which is not alternating. But this contradicts the maximality of $\theta(w)$, since $\theta(u) \geq \theta(w) + 1$, (namely, the edge connecting v_2 with u is added to $\partial D_1 \cap \partial D_2$), a contradiction. Hence M contains no non-alternating inner vertices with valency 4. \square

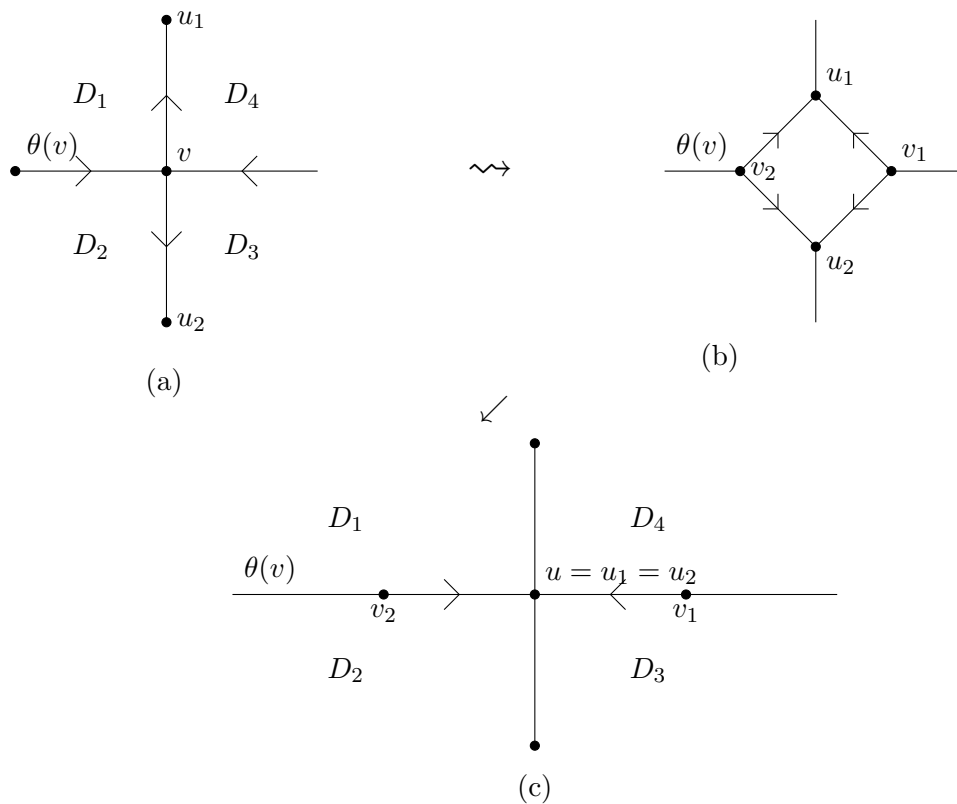


Figure 5

Corollary 3.7. *Let M be a reduced $\mathcal{R}_{1,2}$ -diagram. Then there exists a reduced $\mathcal{R}_{1,2}$ -diagram M_1 with the same boundary labels in which every inner vertex with valency 4 is alternating.*

We shall assume without any further notice that all $\mathcal{R}_{1,2}$ -diagrams we mention have this property.

Definition 3.8. Let M be an \mathcal{R}_0 -van Kampen diagram, \mathcal{R}_0 the symmetric closure of $R(x_1, x_2) = UV$, $n_{x_1, x_2} \geq 3$. Let D be a corner region. Call D complete if $\Phi(\partial D \cap \partial M) \in \{U^{\pm 1}, V^{\pm 1}\}$. Call D alternating if $\partial D \cap \partial M$ contains a separating vertex of D . Finally, call D large alternating if $\partial D \cap \partial M$ contains a separating vertex of D and also $\Phi(\partial D \cap \partial M)$ contains $U^{\pm 1}$ or $V^{\pm 1}$.

Lemma 3.9 (Induction of corner regions). Let $\Lambda = (L_0, \dots, L_p)$ be a layer structure of M , $L_0 = \{v\}$. Let D be a corner region in L_i .

- (a) If D is complete then D induces in L_{i+1} either a complete corner region with the same orientation or a large alternating corner region with the same orientation.
- (b) If D is alternating then D induces in L_{i+1} either an alternating corner region or a large alternating region.
- (c) If D is large alternating then D induces a complete corner region, oriented as $\partial D \cap \omega_i$, and an alternating (possibly large) region.

Proof. (a) Consider $\Delta := \mathcal{C}(D)$.

Case 1 $\text{Int}(\mathcal{C}(D))$ is connected.

If Δ is a single region then since $|\alpha| < n_{a,b}$, β and α are oriented in the same direction, hence Δ is a k -corner region, $k \in \{1, 2\}$ such that γ is either complete or large alternating, with complete part oriented as ∂D . If Δ is not a single region, then $\Delta = \langle E_1, \dots, E_r \rangle$, $r \geq 2$.

If $\partial E_r \cap \partial D$ is a vertex then E_r is a one-corner region, and the result follows as above.

Since M satisfies the condition $T(4)$, $\partial E_r \cap \partial D$ is a vertex

If E_{r-1} has no left neighbour then E_{r-1} is a 1-corner region and the result follows. Assume therefore that E_{r-1} has a left neighbour E . If $\partial E \cap \partial D$ is a vertex then one of E or E_{r-1} is a complete or large alternating, for otherwise E and E_{r-1} cancel each other.

Finally, if $\partial E \cap \partial D$ is not a vertex then the common vertex v of ∂E , ∂E_{r-1} , ∂E_r and ∂D is an inner vertex with valency 4 hence by Lemma 3.6 we may assume it is alternating. But then E_{r-1} is either large alternating with complete part oriented as D or E_{r-1} is complete, oriented as D .

- (b) Let \mathcal{C}_1 be a connected component of $\mathcal{C}(D)$ and let v be the separating vertex on $\partial D \cap w_i$.

Case 1 \mathcal{C}_1 contains v .

If v has valency 4, then \mathcal{C}_1 contains a single region E and due to Lemma 3.6 E is an alternating 2-corner region. If v has valency ≥ 6 then by one of the regions, say E_1 with $\partial E \cap \partial D = \{v\}$ is an alternating 2-corner region.

Case 2 \mathcal{C}_1 does not contain v .

Then either v is a boundary vertex of M in which case L_{i+1} has two consecutive oppositely directed complete 2-corner regions or it contains an alternating 2-corner region.

- (c) Follows from parts (a) and (b).

□

Corollary 3.10. Let notation be as in Lemma 3.9. If M contains an inner vertex v with valency at least 6 then $|\partial M| \geq 2n_{x_1, x_2} + 2$.

Proof. Let $\Lambda = (L_0, \dots, L_p)$ be the layer structure of M with $L_0 = \{v\}$. Then L_1 contains at least 3 complete and 3 alternating corner regions in an alternating manner. Hence

$$||\partial L_1|| \geq 3n_{x_1, x_2} + 3 \cdot 2 - 6 \cdot 1 = 3n_{x_1, x_2} \geq 2n_{x_1, x_2} + 2$$

Now, a successive application of the Lemma implies that the same inequality holds for ∂M . □

The proof of the refinement of Lemma 3.2 below closely follows the way in which Lemma 3.9 is proved, hence we omit it.

Lemma 3.11. *Let M be an \mathcal{R}_0 -van Kampen diagram with connected interior, $|R_0| = 2m$. Suppose that M has a boundary cycle $\mu\nu$ such that $||\mu|| \leq m$. Then M contains a complete corner region D with $\partial D \cap \partial M = \partial D \cap \nu$.*

Lemma 3.12. *Let M be a van Kampen R_0 -diagram which contains an inner vertex v . If $n_{x_1, x_2} \geq 4$ then $||\partial M|| \geq 2n_{x_1, x_2} + 2$.*

Proof. Due to Corollary 3.10 we may assume that every inner vertex has valency 4 and hence by Lemma 3.6 every inner vertex is alternating, Let v be an inner vertex of M and consider the layer structure $\Lambda = \{L_0, \dots, L_p\}$, where $L_0 = \{v\}$. Then every region in L_1 is a 2-corner region and

$$||\partial L_1|| = \bar{a} + \bar{b} + \bar{c} + \bar{d} + 2n_{x_1, x_2} - 4 = 6n_{x_1, x_2} - (a + b + c + d + 4), \text{ where } \bar{x} = n_{x_1, x_2} - x$$

But $a + d = n_{x_1, x_2}$ and $b + c = n_{x_1, x_2}$, hence $||\partial L_1|| = 4n_{x_1, x_2} - 4$. See Figure 6.

Since $n_{x_1, x_2} \geq 4$, by assumption, we get $||\partial L_1|| \geq 2n_{x_1, x_2} + 2$. Now apply the induction argument of Lemma 3.9. □

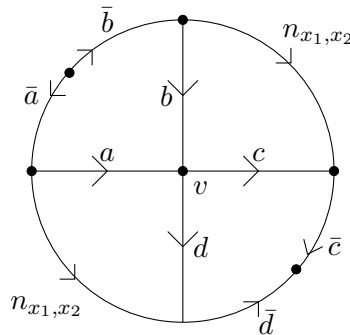


Figure 6

a, b, c and d designate syllable lengths.

Lemma 3.13. *Let M be a van Kampen R_0 -diagram with $||\partial M|| = 8$ and $n_{x_1, x_2} \geq 4$. Then $n_{x_1, x_2} = 4$, M is a one-layer diagram in which each common piece has length 3, and M has a boundary cycle ω with $\Phi(\omega) = x_1^m x_2 x_1 x_2 x_1^{-m} x_2^{-1} x_1^{-1} x_2^{-1}$, $m \geq 1$.*

Proof. By Lemma 3.12 $n_{x_1, x_2} \leq 4$, hence $n_{x_1, x_2} = 4$.

Let M^* be the dual diagram of M . It follows from Lemma 3.12 that M contains no inner vertices. Hence M^* is a tree or a vertex. Each vertex D^* of M^* with valency 1 corresponds to a 1-corner region of M . Hence if M^* has at least 3 vertices with valency 1, then $||\partial M|| \geq 3n_{x_1, x_2} - 3 = 3(n - 1) = 9 > 8$,

hence M^* has at most two boundary vertices of valency 1 and consequently, M is a one-layer diagram. If one of the regions has a common edge of length less than 3 then $|\partial M| \geq 9$. \square

4. Diagrams with the small cancellation condition V(6)

In Section 2.2(a) we introduced the underlying small cancellation condition V(6) and in Theorem 2.6 and Theorem 2.7 described basic properties of V(6) diagrams. We need some further results, which we develop below.

Definition 4.1. *Let M be a diagram and let D be a corner region of M . Call D effective if $\partial D \cap \partial M$ has at least one endpoint with valency 3.*

The reason for introducing this special type of corner regions is the following

Remark 4.2. *Let $\tilde{\Delta}'$ be an effective corner region of \tilde{M}' with $n_{x,y} \geq 4$, where $Supp(\Delta) = \{x, y\}$, let $\mu = \partial \Delta \cap \partial M$ and let ν be the complement of μ on $\partial \Delta$. Then $|\mu| \geq |\nu|$, since then $|\nu| \leq 4$, hence $|\nu| \leq |\mu|$, by Lemma 3.1.*

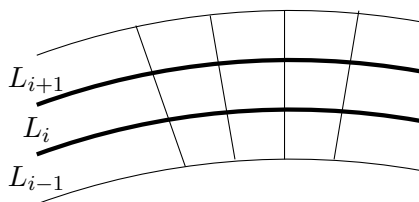
Lemma 4.3. *Let M be a simply connected C(6)-map with connected interior which contains an inner vertex v with valency ≥ 3 . Then M has at least three effective k -corner regions, $k \leq 3$.*

Proof. Let $L_0 = \{v\}$ and let $\Lambda = \{L_0, L_1, \dots, L_p\}$ be the corresponding layer structure. Since $d_M(v) \geq 3$, L_1 contains at least three regions $D_1, D_2, D_k, k \geq 3$ such that $\mathcal{A}(D_i) = \{v\}$ and $b(D_i) = 2$. Hence each D_i is a 2-corner region. Due to Theorem 2.7 each boundary vertex of $\partial L_1 = \partial S_1$ has valency 3 in L_1 , hence each D_i is an effective corner region. Now, by a standard induction argument on i , $1 \leq i \leq p$, it follows along the lines of the proof of Lemma 3.9 that each $S_i, 2 \leq i \leq p$ contains at least 3 effective corner regions. \square

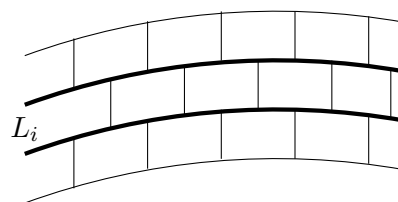
Following the same method we have the C(4) & T(4) and the V(6) analogue of the previous Lemma, the proof of which we omit.

Lemma 4.4. *Let M be a simply connected map with connected interior. Assume that M contains an inner vertex with valency at least 4.*

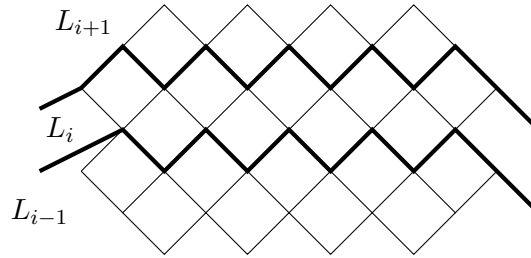
- (a) *If M satisfies the condition C(4) & T(4) then M has at least 4 effective k -corner regions.*
- (b) *If M satisfies the condition V(6) then M has at least two effective corner regions.*



(a)



(b)



(c)

Figure 7

Definition 4.5 (See Figure 7.). (a) Let M be an annular \mathcal{R} -diagram with layer decomposition $\Lambda = (L_{-l}, \dots, L_0, \dots, L_p)$. Call M full if L_i is annular, for every i , $l \leq i < p$.

It follows from the definitions of L_i and S_i that M is full if and only if L_{-l} and L_p are annular.

- (b) Let M_0 be a full annular $V(6)$ diagram. Call L_i 2-homogeneous if every vertex $v \in \omega_i$ has valency 3 in L_i and valency 3 in L_{i+1} , $l \leq i \leq p - 1$ and every region D in L_i has $a(D) = 1$, $b(D) = 2$ and $c(D) = 1$. See Figure 7(a).
- (c) Call L_i 3-homogeneous if every vertex $v \in \omega_i$ has valency 3 in L_i and valency 2 in L_{i+1} , if $l \leq i \leq p - 1$ and $a(D) = b(D) = 2$. See Figure 7(b).
- (d) Call L_i 4-homogeneous if every vertex $v \in \omega_i$ has valency 4 in L_i and valency 2 in L_{i+1} and $a(D) = 0$, $b(D) = 2$ for every region D in L_i . See Figure 7(c).
- (e) Call M r -homogeneous, $r = 2, 3, 4$, if every L_i is.

Lemma 4.6. Let notation be as above. Suppose that L_p is not full. Then L_p contains an effective corner region.

Proof. Let C_1, \dots, C_r , $r \geq 1$, be the connected components of $\text{Int}(L_p)$ occurring in this order, clockwise. Since L_p is not 4-homogenous, there is a connected component C_i such that either $\partial C_{i-1} \cap \partial C_i = \emptyset$ or $\partial C_{i+1} \cap \partial C_i = \emptyset$ ($i - 1$ and $i + 1$ are counted modulo r). Without loss of generality we may assume that $\partial C_{i-1} \cap \partial C_i = \emptyset$. Let v be the left endpoint of $\partial C_i \cap \omega(L_{k-1})$ and let $\{E_1, \dots, E_s\}$ be the set of regions in L_{k-1} which contain v on their boundary. By Theorem 2.6 $1 \leq s \leq 3$. Consider the cases $s = 1, 2, 3$ in turn. Let D_1 be the leftmost region of C_i . Then $b(D_1) \leq 1$, hence if $a(D_1) = 0$ or $C_i = \{D_1\}$ then D_1 is an effective corner region, due to Theorem 2.6. Hence we may assume

$$(*) \quad (i) |C_i| \geq 2 \quad \text{and} \quad (ii) \alpha(D_1) \geq 1$$

Case 1 $s = 1$ (See Figure 8.)

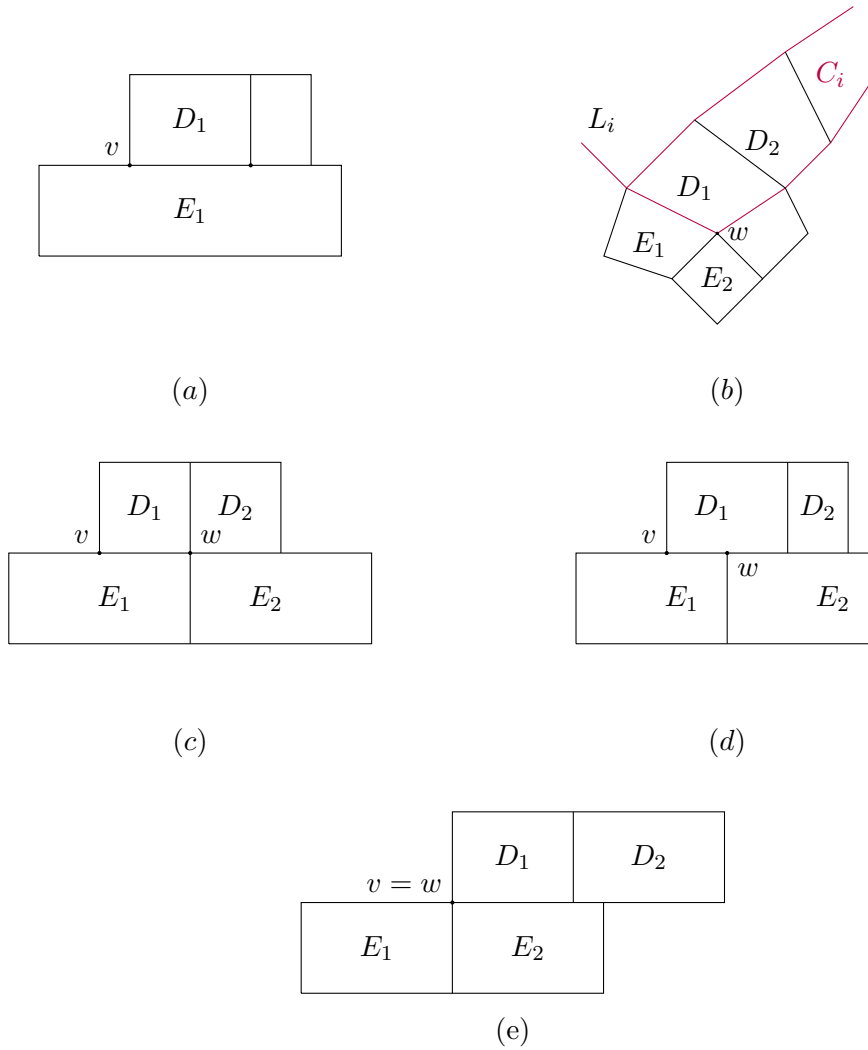


Figure 8

One of the following holds

- (1) E_1 has no right neighbour in L_{i-1} . See Figure 8(a).

Then D_1 is an effective 2-corner region.

- (2) E_1 has a right neighbour E_2 in L_{i-1} and $w := \partial E_1 \cap \partial E_2 \cap \omega(L_{i-1})$ has valency 4 in L_{i-1} .

Then D_2 is an effective 2-corner region, due to Theorem 2.7. See Figure 8(b).

- (3) E_1 has a right neighbour E_2 and w has valency 3.

Then D_1 is an effective 2-corner region or an effective 3-corner region. See Figures 8(c) and 8(d), respectively.

Case 2 $s = 2$

Then D_1 is an effective 2-corner region, (see Figure 8(e)), since $b(D_1) = 1$ and $a(D_1) = 1$, by (*).

Case 3 $s = 3$

Then by Theorem 2.6 E_1 is the left companion of the exceptional 2-corner region E_2 , hence D_1 (and D_2) are effective corner regions. □

Corollary 4.7. *Suppose that M contains no corner regions. Then M is full.*

Proof. Clearly L_p does not contain an effective corner region. Therefore, by the contraposited version of the Lemma, either L_p is annular or L_p is 4-homogeneous. Since L_p contains no corner regions, L_p cannot be 4-homogenous, by definiton. Therefore M is full, by the Lemma. \square

Definition 4.8. *Let M be a diagram over on Artin presentation. Call a boundary path μ of a region Δ of M' with $||\mu|| = 1$ a split edge if Δ has more than one neighbour which intersects μ by an edge.*

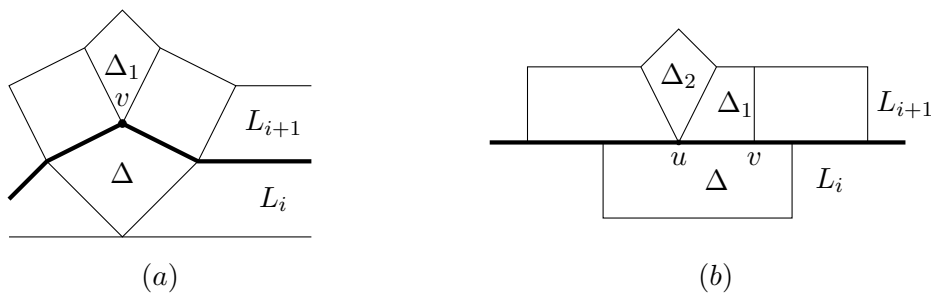
In the next Lemma M is an \mathcal{R} -diagram, as defined in the Introduction. We consider \tilde{M}' , as introduced in Subsection 2.2(c). For simplicity, we denote the regions of \tilde{M}' by Δ , rather than $\tilde{\Delta}'$. This causes no ambiguity.

Lemma 4.9. *Let \tilde{M}' be an annular $V(6)$ diagram and let Δ be a region in L_i , $0 \leq i < p$. Assume that \tilde{M}' is full.*

- (a) *If Δ is a corner region of L_i then $\mathcal{C}(\Delta)$ contains a corner region of L_{i+1} .*
- (b) *In each of the following cases L_{i+1} contains a corner region of L_{i+1} .*
 - (i) *Δ is an exceptional region.*
 - (ii) *$d(\Delta) \geq 7$*
 - (iii) *$d(\Delta) \in \{4, 5\}$ and one of the boundary vertices has valency ≥ 5 .*
 - (iv) *$d(\Delta) = 6$ and one of the vertices on $\partial\Delta$ has valency ≥ 4 .*
 - (v) *Δ has a split edge.*

As a corollary, we have the following

Corollary 4.10. *Suppose that \tilde{M}' is an annular $V(6)$ diagram with more than one layer in which all the regions are simply connected and which contains no corner regions. Then \tilde{M}' is full and none of the assumptions of the Lemma holds. In particular, \tilde{M}' is k -homogeneous for some $k \in \{2, 3, 4\}$.*



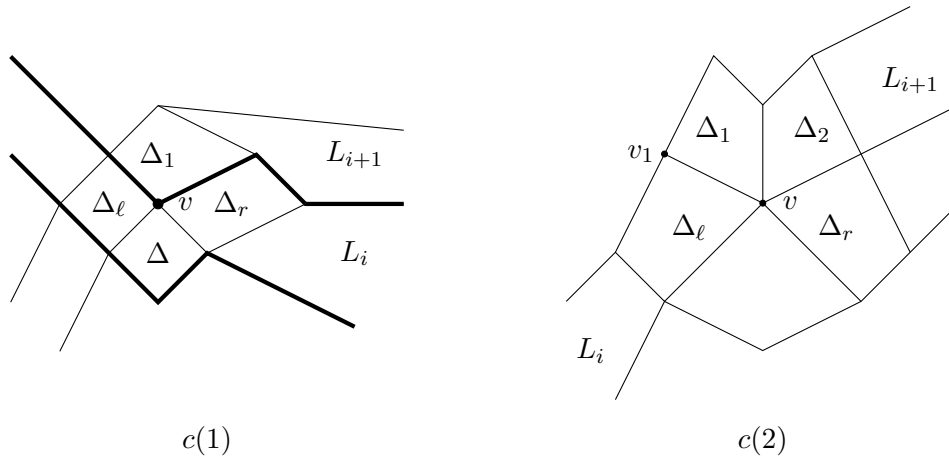


Figure 9

Proof. Due to Corollary 4.7 we may assume that M is full. Hence the result follows from the last Lemma, by inspection. \square

Proof of the Lemma. (a) Suppose first that Δ is a 2-corner region of L_i , $i < p$. If $d_M(\Delta) \leq 5$ then all the boundary vertices of Δ have valency at least 4, hence in particular the inner vertex v in Figure 9(a).

But then $\mathcal{C}(\Delta)$ contains a region Δ_1 with $\partial\Delta_1 \cap \omega_i = \{v\}$, since $d_{L_i}(v) = 2$ and hence $d_{L_{i+1}}(v) \geq 4$. Hence, Δ_1 is a 2-corner region in $\mathcal{C}(\Delta)$. Next, suppose that Δ is a 3-corner region of L_i . Then $d_M(\Delta) \geq 6$, hence $|\partial\mathcal{C}(\Delta) \cap \omega_i| \geq 3$. In particular, there is a region Δ_1 in L_{i+1} which contains the adjacent vertices u and v on its boundary. See Figure 9(b).

If one of u and v has valency at least 4 in M then it has valency at least 4 in L_{i+1} , since $d_{L_i}(w) = 2$, $w \in \{u, v\}$. Hence $\mathcal{C}(\Delta)$ contains a 2-corner region Δ_i which contains it. Finally, if both u and v have valency 3 in L_{i+1} then $d_M(\Delta) \geq 6$, and Δ_1 is a corner region. This also proves the assertion when Δ is a 2-corner region and $d_M(\Delta) \geq 6$.

(b) (i) Let Δ be a 2-exceptional region with companion 2-corner regions Δ_r and Δ_l . See Figure 9(c). Consider $L := \mathcal{C}(\Delta_r \cup \Delta_l \cup \Delta)$.

If there is a unique region Δ_1 in L which contains v on its boundary, then Δ_1 is a 2-corner region by definition, since then $d_{L_i}(v_1) = 2$, hence $d_{L_{i+1}}(v_1) \geq 4$, see Figure 9(c1). Assume therefore, that there are f regions, $f \geq 2$, $\{\Delta_1, \dots, \Delta_f\}$ which contain v on their boundary. If $f \geq 3$ then Δ_2 is a 2-corner region. Therefore, we may assume that $f = 2$, see Figure 9(c2). If $d(\Delta_1) \leq 5$ then v_1 has valency at least 4 in L_{i+1} , again L contains a corner region. Hence $d(\Delta_1) \geq 6$. But then $a(\Delta_1) = 1$, hence again Δ_1 is a 3-corner region. Consequently, $\mathcal{C}(L)$ contains a corner region.

If Δ is 3-exceptional, then a similar argument applies.

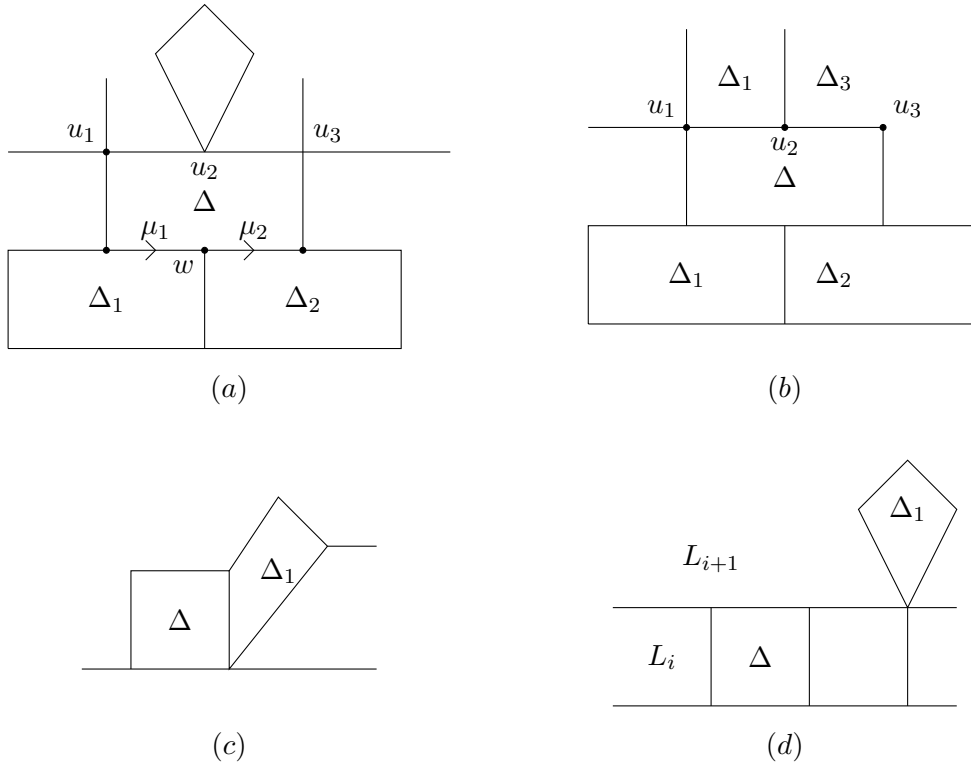


Figure 10

(b) (ii) See Figure 10. We may assume that $a(\Delta) = 2$, otherwise Δ is already a 3-corner region. Let $\mathcal{A}(\Delta) = (\Delta_1, \Delta_2)$, let $\mu_i = \partial\Delta \cap \partial\Delta_i$, $i = 1, 2$. If $h(\mu_1)$ or $t(\mu_2)$ have valency at least 4, then Δ has a neighbour in L_i which is a 2-corner region, hence L_{i+1} contains a corner region, by part (a). Assume therefore, that they both have valency 3 in M . Clearly, the vertex $w := \partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta$ has valency 3 in M . Therefore, one of the vertices u_1, u_2 , and u_3 has valency 4. See Figure 10(a). If this vertex is u_2 then $\mathcal{C}(\Delta)$ contains a region Θ with $\partial\Theta \cap \omega_i = u_2$, hence Θ is a 2-corner region. Assume therefore that u_2 has valency 3. See Figure 10(b). If this is one of u_1 and u_3 then the region Δ_1 of L_{i+1} or Δ_3 of L_{i+1} has $a(\Delta_1) = 1$ or $a(\Delta_3) = 1$, hence $d(\Delta_1) \geq 6$ or $d(\Delta_3) \geq 6$. Consequently, one of them is a 3-corner region of L_{i+1} .

(b) (iii) By Theorem 2.6 $a(\Delta) \leq 2$, hence

$$c(\Delta) = d(\Delta) - (a(\Delta) + b(\Delta)) \geq 7 - (2 + 2) = 3$$

Let $\mathcal{C}(\Delta) = (\Delta_1, \dots, \Delta_f)$, $f \geq 3$. Let $v = \partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta$. If $d_M(v) \geq 4$ then $d_{L_{i+1}}(v) \geq 4$. But then one of Δ_1 and Δ_2 is a 2-corner region of L_{i+1} . If $d_M(v) = 3$ then let $v = \partial\Delta_2 \cap \partial\Delta_3 \cap \partial\Delta$. If $d_M(v) \geq 4$ then one of Δ_2 and Δ_3 is a 2-corner region and if $d_M(v) = 3$ then $d(\Delta_2) \geq 6$ and $a(\Delta_2) = 1$. Hence, Δ_2 is a 3-corner region.

(b) (iv) Assume first that $d(\Delta) = 5$.

By Theorem 2.6 $a(\Delta) \leq 1$, hence

$$c(\Delta) = d(\Delta) - (a(\Delta) + b(\Delta)) \geq 5 - (1 + 2) = 2$$

Let $\mathcal{C}(\Delta) = (\Delta_1, \Delta_2, \dots, \Delta_f)$, $f \geq 2$. Let $v = \partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta$. Then $d_M(v) \geq 4$, by assumption, and since $d_{L_i}(v) = 2$, $d_{L_{i+1}}(v) \geq 4$. Consequently either Δ_1 or Δ_2 is a 2-corner region in L_{i+1} .

Assume now that $d_M(\Delta) = 4$ and one of the boundary vertices of Δ has valency ≥ 5 . By Theorem 2.6 $0 \leq a(\Delta) \leq 2$. If $a(\Delta) = 0$ then Δ is a 2-corner region. If $a(\Delta) = 1$ then either Δ has a neighbour Δ_1 in L_i which is a 2-corner region (see Figure 10(c)) or $\mathcal{C}^+(\Delta)$ contains a region Δ_1 which is a 2-corner region. See Figure 10(d). (Here $\mathcal{C}^+(\Delta)$ is $\mathcal{C}(\Delta)$ together with the regions in L_{i+1} which touch $\partial\Delta$ by a vertex only.) Hence, by part (c) L_{i+1} contains a corner region.

(b) (v) The number of unsplit edges $n(\Delta)$ of Δ is at least 4. Suppose Δ has a split edge. If $n(\Delta) = 4$ then Δ has at least 5 edges and all vertices with valency at least 4. Hence by part (b) (iii) L_{i+1} contains a corner region. If $n(\Delta) = 5$ then Δ has at least 6 edges and each boundary vertex has valency at least 4, hence by part (b) (iv) the result follows. Finally, if $n(\Delta) \geq 7$ again the result follows by part (b) (ii). □

5. Three-generated parabolic subgroups in Artin groups

Theorem 5.1. *Let $x_1, x_2, x_3 \subseteq X$. If $x_1^\alpha x_2^\beta x_3^\gamma W = 1$, where W is a reduced word in A , then $|W| \geq |\alpha| + |\beta| + |\gamma|$.*

Let U be a shortest representative of $x_1^\alpha x_2^\beta x_3^\gamma$ in A . Then $|U| \leq |\alpha| + |\beta| + |\gamma|$. By Theorem 2.2 $U \in \langle x_1, x_2, x_3 \rangle$. Hence if we show that

$$x_1^\alpha x_2^\beta x_3^\gamma \text{ is a shortest representative of itself in } \langle x_1, x_2, x_3 \rangle \tag{*}$$

then the result will follow. We may assume without loss of generality that $\alpha_i \neq 0$, $i = 1, 2, 3$ and $x_1^\alpha x_2^\beta x_3^\gamma U^{-1}$ is cyclically reduced, as written and $A = \langle x_1, x_2, x_3 \rangle$. We prove (*) by checking cases. The first case is when $\langle x_1, x_2, x_3 \rangle$ is of finite type.

The proof of this case is due to Professor S. Orevkov. I am grateful to Professor Orevkov for the proof and for his permission to quote it here.

Let G be an Artin group of finite type on 3 standard generators u, v and w .

Proposition 5.2. *If $u^a v^b w^c = W$ in G where W is a reduced word on u, v and w and a, b and c are non-zero integers, then the length of W is at least as that of $u^a v^b w^c$.*

S. Orevkov. (For undefined terms and general background for the case of braid groups see [5].)

We consider the standard Garside structure on G with the Garside element denoted by Δ . Each element Y of G can be presented in a unique way in the form $Y = N^{-1}P$ (resp. $Y = PN^{-1}$) where N and P are positive elements whose left (resp. right) gcd is 1. If, moreover, N and P are given in the left (resp. right) normal form, then we call this presentation of Y *the left (resp. right) balanced normal form*. The balanced left normal form is used by default in GAP's package Chevie, version 4. □

Lemma 5.3. *Let a, b, c be non-zero integers one of which is negative (we denote it $-m$) and the others are positive (we denote them k, l). Then $\inf(u^a v^b w^c) = -m$.*

Proof. We consider only the case when G is an irreducible Artin group. Thus, G is of the type A_3, B_3 or H_3 , i.e. G is generated by x, y, z subject to the relations $xyx = yxy, xz = zx$ and $\langle yz \rangle^\mu = \langle zy \rangle^\mu$ for $\mu = 3, 4, \text{ or } 5$; here we use Brieskorn-Saito's notation $\langle yz \rangle^\mu = yzyz \dots$ (μ alternating letters), so the Coxeter graph of G is

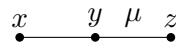


Figure 11

When presenting the left (right) normal forms, we use the notation $A_1 \cdot A_2 \dots$ which means that the A_i 's are simple elements and each pair of consecutive elements is left (right) weighted. If an expression A^n occurs in the left (right) normal form, it stands for $A \cdot A \dots A$ (n times), $\langle A \cdot B \rangle^n$ stands for $A \cdot B \cdot A \cdot B \dots$ (n alternating factors), and t always denotes the last letter of the previous simple element. For example, $(yzy)^2 \cdot \langle yz \cdot zy \rangle^3 t^3$ stands for

$$yzy \cdot yzy \cdot yz \cdot zy \cdot yz \cdot z \cdot z \cdot z$$

To prove the desired result, it is enough to compute the left or right balanced normal form of $u^a v^b w^c$ for all permutations (u, v, w) of (x, y, z) and for all permutations of $(k, l, -m)$. The right balanced normal forms are:

$$\begin{aligned} x^k y^l z^{-m} &= x^{k-1} \cdot xy \cdot y^{l-1} \cdot z^{-m}, \\ z^k y^l x^{-m} &= z^{k-1} \cdot xy \cdot y^{l-1} \cdot x^{-m}, \\ x^k z^l y^{-m} &= x^{k-1} \cdot (xz)^l \cdot y^{-m}, & k \geq l \\ x^k z^l y^{-m} &= z^{l-k} \cdot (xz)^k \cdot y^{-m}, & k \leq l \\ x^k y^{-m} z^l &= x^{k-1} \cdot xzy \cdot \langle yz \cdot zy \rangle^{m-1} \cdot t^{l-m} \cdot \langle yz \cdot zy \rangle^{-m}, & l \geq m, A_3 \\ x^k y^{-m} z^l &= x^{k-1} \cdot xzyz \cdot (zyz)^{m-1} \cdot z^{l-m} \cdot (yzy)^{-m}, & l \geq m, B_3 \\ x^k y^{-m} z^l &= x^{k-1} \cdot xzyzy \cdot \langle yzyz \cdot zyzy \rangle^{m-1} \cdot t^{l-m} \cdot \langle yzyz \cdot zyzy \rangle^{-m}, & l \geq m, H_3 \\ z^k y^{-m} x^l &= z^{k-1} \cdot zxy \cdot \langle yx \cdot xy \rangle^{m-1} \cdot t^{l-m} \cdot \langle yx \cdot xy \rangle^{-m}, & l \geq m \\ x^k y^{-m} z^l &= x^{k-1} \cdot xzy \cdot \langle yz \cdot zy \rangle^{l-1} \cdot (\langle yz \cdot zy \rangle^l \cdot t^{m-l})^{-1}, & l \leq m, A_3 \\ x^k y^{-m} z^l &= x^{k-1} \cdot xzyz \cdot (zyz)^{l-1} \cdot ((yzy)^l \cdot y^{m-l})^{-1}, & l \leq m, B_3 \\ x^k y^{-m} z^l &= x^{k-1} \cdot xzyzy \cdot \langle yzyz \cdot zyzy \rangle^{l-1} \cdot (\langle yzyz \cdot zyzy \rangle^l \cdot t^{m-l})^{-1}, & l \leq m, H_3 \\ z^k y^{-m} x^l &= z^{k-1} \cdot zxy \cdot \langle yx \cdot xy \rangle^{l-1} \cdot (\langle yx \cdot xy \rangle^l \cdot t^{m-l})^{-1}, & l \geq m \end{aligned}$$

We see that the infimum is equal to $-m$ in all listed cases. All other cases are reduced to these ones. Indeed, if $a = -m$ (the first power), then we compute the left balanced normal form. If x^{-m} or z^{-m} occurs in the middle, then we exchange it with the commuting letter.

The fact that the expressions in the right hand sides are right weighted is evident. The fact that the equalities hold can be easily proved by induction. □

Proof of Proposition. If $a, b, c > 0$ or $a, b, c < 0$, then the statement is evident. If two of a, b, c are negative and the other is positive, then we consider $(-a, -b, -c)$ instead of (a, b, c) . So, we may assume

that two of the numbers a, b, c are positive (we denote them by k and l) and the third one is negative (we denote it by $-m$).

Let p and n be the number of positive and negative letters in W respectively. Since the relations in the Artin groups are homogeneous, we have $p - n = k + l - m$. We have $\text{inf}(W) \geq -n$. By Lemma 5.3, this means $m \leq n$. Thus, $p + n = (p - n) + 2n \geq (p - n) + 2m = (k + l - m) + 2m = k + l + m$. \square

We turn now to the remaining cases. Let Γ be the defining graph of A with edges $E = E(\Gamma)$, with respect to the standard generators x_1, x_2 and x_3 .

If $E = \emptyset$ then A is a free group and the result is clear. If E consists of a single edge, say (x_1, x_2) then $A = \langle x_3 \rangle * \langle x_1, x_2 \mid R_{1,2} \rangle$ and the result follows from the Normal Form Theorem for free products (see [12]) and Corollary 3.3 and Corollary 3.4. If E consists of two edges, say (x_1, x_2) and (x_2, x_3) , then

$$A = \langle x_1, x_2 \mid R_{1,2}(x_1, x_2) \rangle * \langle t_2, x_3 \mid R_{2,3}(t_2, x_3) \rangle$$

$$x_2 = t_2$$

and again the result follows from the Normal Form Theorem for amalgamated free products (see [12]) and Corollary 3.3 and Corollary 3.4.

So assume that Γ is a triangle.

Let $x_1^\alpha x_2^\beta x_3^\gamma U^{-1} = 1$ in $A = (\langle x_1, x_2, x_3 \rangle)$, where U is a shortest representative of $x_1^\alpha x_2^\beta x_3^\gamma$ in A . Then there exists a van Kampen diagram M with boundary label $x_1^\alpha x_2^\beta x_3^\gamma U^{-1}$. We propose to show that $\text{Reg}(M) = \emptyset$, i.e. M contains no regions. This implies $U = x_1^\alpha x_2^\beta x_3^\gamma$ in F . Assume by way of contradiction that $\text{Reg}(M) \neq \emptyset$. M has a boundary cycle $u\mu\nu\nu$, u and ν vertices, such that $\Phi(\mu) = x_1^\alpha x_2^\beta x_3^\gamma$ and $\Phi(\nu) = U$. Thus, we shall assume

(\mathcal{H}_1) W is a shortest word for which $\text{Reg}(M) \neq \emptyset$, where $w = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} x_{i_3}^{\alpha_3}$, $|\alpha_i| \neq 0$ $\{i_1, i_2, i_3\} = \{1, 2, 3\}$ and M is the corresponding van Kampen diagram.

(\mathcal{H}_2) U is a shortest representative of W .

(\mathcal{H}_3) M contains minimal number of regions among all diagrams with boundary cycle $u\mu\nu\nu$, where $\Phi(\mu) = W$ and $\Phi(\nu)$ is a shortest representative of W , for which (\mathcal{H}_1) and (\mathcal{H}_2) hold.

We shall derive a contradiction to one of (\mathcal{H}_1), (\mathcal{H}_2) and (\mathcal{H}_3), showing that $\text{Reg}(M) = \emptyset$. (For if $\text{Reg}(M) \neq \emptyset$ then there is a diagram which satisfies all of (\mathcal{H}_1), (\mathcal{H}_2) and (\mathcal{H}_3), this however leads to a contradiction, as we show.)

First observe that if μ contains a subpath $\mu_0 \neq \emptyset$ such that $\Phi(\mu_0) = 1$ in A then by $\|\mu_0\| \geq 4$, a contradiction, since $\|\mu\| = 3$. Hence, μ cannot contain such a subpath. Also, ν may not contain a subpath $\nu_0 \neq \emptyset$ with $\Phi(\nu_0) = 1$ in A , since then (\mathcal{H}_2) is violated. Consequently, if M_1, \dots, M_r , $r \geq 1$, are the connected components of $\text{Int}(M)$ with $\text{Reg}(M_i) \neq \emptyset$, then $\mu = \mu_1 \delta_1 \mu_2 \delta_2 \dots \mu_r$, $\nu = \nu_1 \delta_1 \nu_2 \delta_2 \dots \nu_r$, where $\mu_i = \partial M_i \cap \mu$, $\nu_i = \partial M_i \cap \nu$ and δ_i are (possibly empty) paths. See Figure 12.

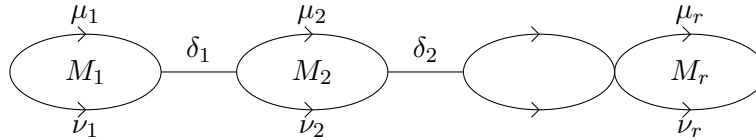


Figure 12

We concentrate on M_r and show that $Reg(M_r) = \emptyset$. This contradiction implies that $Reg(M) = \emptyset$, hence we may write M for M_r without loss of generality.

Case 1 Γ is a (c_1, c_2, c_3) triangle with $c_i \geq 3, i = 1, 2, 3$.

In this case M' is of large type, hence due to Lemma 3.1 it satisfies the condition C(6). Hence, if M' contains an inner vertex with valency greater than 2 then M' contains at least three (effective) corner regions, due to Lemma 4.3. Since M' is planar, u and v may occur on the boundary of at most one of them (each), hence at least one of the corner regions, say Δ , satisfies $\theta := \partial\Delta \cap \partial M = \partial\Delta \cap \xi$, where $\xi \in \{\mu, \nu\}$. We claim that this is not possible. Let η be the complement of θ on $\partial\Delta$. Then due to Lemma 3.2 $|\eta| \leq |\theta|$, hence if $\xi = \nu$ then we can remove Δ from M' and get a smaller diagram with the required properties, thereby violating (\mathcal{H}_3) . On the other hand, if $\xi = \mu$ then $\Phi(\mu)$ contains a subword $x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} x_{i_1}^{\beta_3}$, $x_{i_1} \neq x_{i_2}, \beta_i \neq 0$, which is not possible, since $\Phi(\mu) = x_1^\alpha x_2^\beta x_3^\gamma$. Consequently, M' has no inner vertices, hence its dual is a tree. Therefore, all the corner regions of M' are 1-corner regions. Hence, if Δ is a corner region then $\partial\Delta \cap \partial M$ again cannot be a subpath of μ or of ν . Consequently, M' is a one layer diagram, $M' = \langle \Delta_1, \dots, \Delta_t \rangle, t \geq 1$. We concentrate on Δ_t and show that $Reg(\Delta_t) = \emptyset$. This proves our claim. Assume $t \geq 2$. Let $\rho = \partial\Delta_t \cap \partial\Delta_{t-1}$. Then due to Lemma 2.8 $|\rho| = 1$. Clearly $|\mu_t| \leq 2$, hence $|\mu_t^{-1}\rho| \leq 3$. But $\mu_t^{-1}\rho\nu_t$ is a boundary cycle of Δ_t , hence by Lemma 2.8 and Lemma 3.2 $|\mu_t^{-1}\rho| \leq |\nu_t|$. Consequently, we may remove Δ_t and replace ν_t by $\mu_t^{-1}\rho$, violating (\mathcal{H}_3) . Therefore $t = 1$. But then, the above argument with $\rho = \emptyset$ leads to a violation of (\mathcal{H}_3) . Hence $Reg(M') = \emptyset$, as required.

Case 2 Γ is a $(2, c_1, c_2)$ triangle, $c_1, c_2 \geq 4$.

In this case we consider M as a relative diagram with $H = \langle x_1, x_2 \rangle, R_0 = x_1 x_2 x_1^{-1} x_2^{-1}$. Then M' is extra-large relative to H . Consequently, it follows from Lemma 2.9 that \tilde{M}' is a C(4) & T(4) diagram, hence if \tilde{M}' contains an inner vertex, then by Lemma 3.2 \tilde{M}' has at least 4 effective k -corner regions, $k \in \{1, 2\}$. Let Δ be such a corner region and assume $\theta := \partial\Delta \cap \partial M = \partial\Delta \cap \xi, \xi \in \{\mu, \nu\}$. Let η be the complement of θ on $\partial\Delta$. Then $|\eta| \leq 4$, since Δ is effective. Consequently, $|\eta| \leq |\theta|$ and $|\theta| \geq 4$, since $c_1, c_2 \geq 4$. This however, violates (\mathcal{H}_3) , if $\xi = \nu$ and the fact that $||W|| = 3$, if $\xi = \mu$. Consequently, \tilde{M}' has no inner vertices and as in Case 1, the dual of \tilde{M}' is a tree. From this it follows easily, as in Case 1, that \tilde{M}' is a one layer diagram $\langle \Delta_1, \dots, \Delta_t \rangle$. Again, concentrate on Δ_t . Assume first that $t \geq 2$ and let $\rho = \partial\Delta_t \cap \partial\Delta_{t-1}$. Then $|\rho| = 1$. Let z_1 and z_2 be the endpoints of ρ , such that $z_1 \in \mu$ and $z_2 \in \nu$. Since \tilde{M}' is planar, at least one of z_1 and z_2 has valency 3 in \tilde{M}' . Assume first $d_{\tilde{M}'}(z_1) = 3$ and let $\xi_t = \partial M_{z_1} \cap \partial\Delta_t, \xi_{t-1} = \partial M_{z_1} \cap \partial\Delta_{t-1}$. See Figure 13(a).

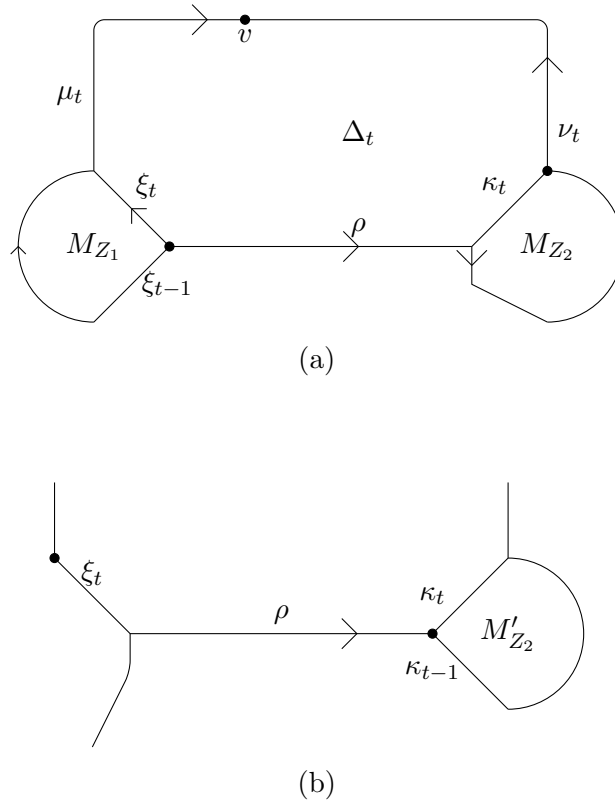


Figure 13

Then $||\xi_{t-1}|| = ||\xi_t|| = 1$, hence $||\partial M_{z_1} \cap \mu|| \geq ||\xi_{t-1}\xi_t|| = 2$, i.e. $||\partial M_{z_1} \cap \mu|| \geq 2$. Now $Supp(M_{z_1}) = \{x_1, x_2\}$, while μ_t starts with $x_3^{\pm 1}$, hence $||\mu_t|| = 1$. Consequently $||\mu_t^{-1}\xi_t^{-1}\rho\kappa_t|| \leq 4$, where $\kappa_t = \partial M_{z_2} \cap \partial \Delta$. But $\mu_t^{-1}\xi_t^{-1}\rho\kappa_t\nu_t$ is a boundary cycle of Δ_t . Therefore $|\mu_t^{-1}\xi_t^{-1}\rho\kappa_t| \leq |\nu_t|$ and as in Case 1, we may remove Δ_t and violate (\mathcal{H}_2) . If $t = 1$ then the same argument works with $\rho = \emptyset$.

Assume now that $d_{\tilde{M}'}(z_2) = 3$. See Figure 13(b).

Then $||\kappa_t\kappa_{t-1}|| = 2$, hence $|\kappa_t\kappa_{t-1}| \leq |\partial M'_{z_2} \cap \nu|$, by Lemma 3.2. But then the removal of \tilde{M}'_{z_2} violates (\mathcal{H}_3) . Consequently, $Reg(M'_{z_2}) = \emptyset$. But then $|\mu_t^{-1}\xi_t^{-1}\rho| \leq |\nu_t|$, as above, since $||\mu_t|| \leq 2$, $||\xi_t|| = ||\rho|| = 1$. Hence we may replace ν_t with $\mu_t^{-1}\xi_t\rho$ and get a smaller diagram with the required properties, again violating (\mathcal{H}_3) . A similar argument applies if $t = 1$. Consequently $Reg(\Delta_t) = \emptyset$, as required.

Case 3 Γ is a $(2, 3, c)$ triangle, $c \geq 6$.

In this case we take $H = \langle x_2, x_3 \mid x_2x_3x_2 = x_3x_2x_3 \rangle$. The relative diagram \tilde{M}' obtained is not relatively extra-large, however, we may use the ideas used in Case 2 in order to show that $Reg(M') = \emptyset$. Now, $Reg(\tilde{M}')$ consists of two types of regions: n -gons Δ_i with $n \geq 6$ having $Supp(\Delta_i) = \{x_1, x_3\}$ and bigons Λ_j having $Supp(\Lambda_j) = \{x_1, x_2\}$, $(x_1x_2 = x_2x_1)$. Since $||\partial M_v|| \geq 6$, for every vertex v for which $Reg(M_v) \neq \emptyset$, every inner vertex in \tilde{M}' has valency at least 6. (See proof of Lemma 2.3 in [11]). Now, all the neighbours of Δ_i are bigons and all the neighbours of Λ_j are n -gons, $n \geq 6$. It follows that if we shrink every bigon to an edge then \tilde{M}'' , the diagram obtained from \tilde{M}' by shrinking bigons, will satisfy the condition $C(6)$ and $T(\frac{6}{2})$, i.e. $C(6)$ & $T(3)$. See Figure 14.

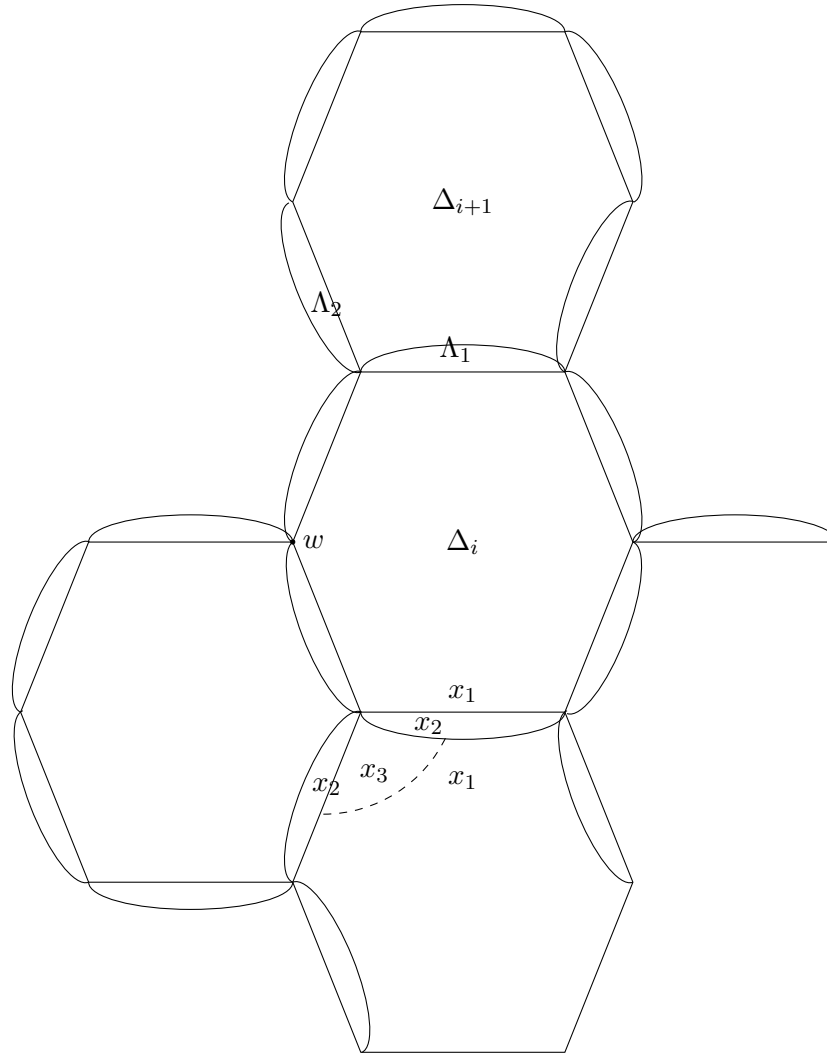


Figure 14

Let w be a boundary vertex of \tilde{M} with valency 3 and consider M_w . Let Δ_i , Δ_{i+1} and Λ_j be the regions of \tilde{M}' containing w on their boundary. Let ξ_i , ξ_{i+1} and ξ_j be the common edges of Δ_i , Δ_{i+1} and Λ_j , respectively, with ∂M_w , and let $\eta = \partial M_w \cap \partial \tilde{M}'$. See Figure 15.

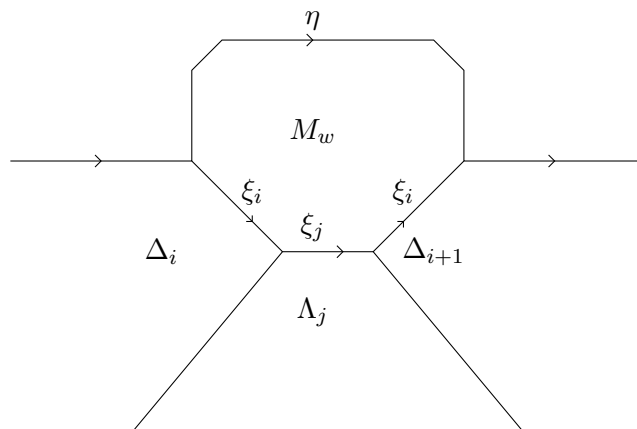


Figure 15

Then $\xi_i \xi_j \xi_{i+1} \eta^{-1}$ is a boundary cycle of M_w . Since $\|\xi_i\| = \|\xi_j\| = \|\xi_{i+1}\| = 1$ and since $Supp(Reg(M_w)) = \{x_2, x_3\}$, we get that

- (i) $\|\eta\| \geq 3$
- (ii) $|\eta| \geq |\xi_i \xi_j \xi_{i+1}|$

Consequently w cannot be a vertex in μ , due to (i) and cannot be a vertex in ν , due to (ii), without violating one of (\mathcal{H}_1) , (\mathcal{H}_2) and (\mathcal{H}_3) , as we have seen several times above. Thus,

$$\text{neither } \mu \text{ nor } \nu \text{ may contain a vertex with valency 3 in } \tilde{M}'' \tag{*}$$

It follows as in the previous cases, that if \tilde{M}' is not a one-layer diagram then either μ or ν contains an effective corner region. But by definition then μ or ν contains a vertex with valency 3 in \tilde{M}' , a contradiction to (*). Therefore, \tilde{M}' is a one layer diagram, $\tilde{M}' = \langle \Delta_1, \dots, \Delta_t \rangle$. If $t = 1$ then since $\|\mu \cap \partial \Delta_1\| \leq 2$, we get that $|\mu \cap \partial \Delta_1| < |\nu \cap \partial \Delta_1|$, violating (\mathcal{H}_2) . Consequently, $t \geq 2$.

Let z_1 and z_2 be the endpoints of $\theta := \partial \Delta_t \cap \partial \Delta_{t-1}$, $z_1 \in \mu$ and $z_2 \in \nu$. Then one of them should be with valency 3 in \tilde{M}'' , since \tilde{M}'' is planar. This however violates (*), proving that $Reg(\Delta_t) = \emptyset$, completing the proof of the Theorem. □

Remark 5.4. (1) *In Case 3 we used \tilde{M}'' only in order to use the C(6) condition which makes the machinery applicable.*

(2) *In Case 2 we could take $H = \langle x_1 x_3 \mid x_1 x_3 x_1 x_3 = x_3 x_1 x_3 x_1 \rangle$ and then the regions of \tilde{M}' would be n -gons, $n \geq 4$ and bigons. The construction of \tilde{M}'' would lead again to a C(4) & T(4) diagram.*

Corollary 5.5. *If $x_1^\alpha x_2^\beta x_3^\gamma x_4^\delta = 1$ in A , then $x_3 = x_1$, $x_2 = x_4$ and $\alpha + \gamma = 0$, $\beta + \delta = 0$.*

Proof. Assume by way of contradiction that $x_3 \neq x_1$, or $x_2 \neq x_4$.

Then one of the following holds:

Case 1 $x_3 \neq x_1$ and $x_2 = x_4$.

By the Theorem $|\alpha| + |\beta| + |\gamma| \leq |\delta|$ and by Theorem 2.1 $|\delta| \leq |\alpha| + |\beta| + |\gamma|$. Consequently,

$$(5.1) \quad |\alpha| + |\beta| + |\gamma| = |\delta|$$

By two applications of Theorem 2.1 we get

$$(5.2) \quad |\alpha| + |\beta| = |\gamma| + |\delta|$$

Substitution of (2) in (1) implies $\gamma = 0$, a contradiction.

Case 2 $x_3 = x_1$ and $x_2 \neq x_4$.

This is Case 1, with the x_i renamed.

Case 3 $x_3 \neq x_1$ and $x_2 \neq x_4$.

Clearly every two adjacent x_i ((4, 1) too) are different, due to Corollary 3.4. Hence, by assumption all the x_i are different. Then by the Theorem

$$|\alpha_i| = |\alpha_{i+1}| + |\alpha_{i+2}| + |\alpha_{i+3}|$$

$i = 1, 2, 3, 4$.

Hence, if we write $\alpha = \sum |\alpha_i|$ we get $\alpha = 3\alpha$, i.e. $\alpha = 0$, a contradiction. □

Hence, $x_1^\alpha x_2^\beta x_1^\gamma x_2^\delta = 1$. It follows from Lemma 3.1 that $x_1 x_2 = x_2 x_1$. Consequently, $\alpha + \gamma = 0$ and $\beta + \delta = 0$, as required. □

6. Proofs of the Main Results

6.1. Preliminary Results. In this section we assume that A is an Artin group, extra-large relative to H , as defined in the introduction. In particular, \tilde{M}' satisfies the condition V(6) for every annular \mathcal{R} -diagram M , by Lemma 2.10.

In Lemma 6.1 and Lemma 6.3 V denotes a word in F and not the vertex set of Γ_A .

Lemma 6.1. *Let U and V be elements of F and suppose that for some W in F , representing an element g of A , $g^{-1}ug = v$ in A , where u and v are represented by U and V , respectively. Let \tilde{M}' be the corresponding relative conjugacy diagram. If $g \notin H$ then $Reg(\tilde{M}') \neq \emptyset$.*

Proof. There is a connected simply connected van Kampen diagram M_0 with boundary label $\eta\mu\theta^{-1}\nu^{-1}$ $\Phi(\eta) = \Phi(\theta) = W$, $\Phi(\mu) = U$, $\Phi(\nu) = V$.

Since $g \notin H$, by Lemma 2.12 the reduced annular diagram M obtained by identifying η with θ and carrying out possible cancellations of regions, the path obtained from η has label which represents g . Hence, we get a reduced annular diagram M which contains a region in $Reg_{0,1} \cup Reg_1$. □

Lemma 6.2. *Let notation be as in Lemma 6.1. Suppose that $Reg(\tilde{M}') \neq \emptyset$ and \tilde{M}' contains a simply connected region $\tilde{\Delta}'$.*

Then either U or V contains a letter not in H .

Proof. Let $\tilde{\Delta}'$ be a simply connected region in \tilde{M}' . Suppose $\tilde{\Delta}'$ occurs in L_i for some $i > 0$. Then one of the following holds

- (i) $\Delta' \in Reg_1$ and $d_{\tilde{M}'}(\tilde{\Delta}') \geq 6$, $\alpha(\tilde{\Delta}') \leq 2$ and $\beta(\tilde{\Delta}') \leq 2$.
- (ii) $\Delta' \in Reg_{0,1}$ and $d_{\tilde{M}'}(\tilde{\Delta}') \geq 4$, $\alpha(\tilde{\Delta}') \leq 1$ and $\beta(\tilde{\Delta}') \leq 2$.

We claim that L_{i+1} contains a simply connected region from $Reg_1 \cup Reg_{0,1}$.

In case (i) $\gamma_{\tilde{M}'}(\tilde{\Delta}') = d_{\tilde{M}'}(\tilde{\Delta}') - \alpha_{\tilde{M}'}(\tilde{\Delta}') - \beta_{\tilde{M}'}(\tilde{\Delta}') \geq 2$. Consequently, $\|\Phi(\omega_i)\| \geq 2$ due to Lemma 2.8. In particular, if L_{i+1} exists, then it cannot consist of a single annular region, by the definition of derived regions. Hence L_{i+1} contains a simply connected region.

In case (ii) $\gamma_{\tilde{M}'}(\tilde{\Delta}') \geq d_{\tilde{M}'}(\tilde{\Delta}') - \alpha_{\tilde{M}'}(\tilde{\Delta}') - \beta_{\tilde{M}'}(\tilde{\Delta}') \geq 8 - 1 - 2 = 5$.

Taking into account the labelled vertices we get that

$$\gamma_{M'}(\Delta') \geq \gamma_{\tilde{M}'}(\tilde{\Delta}') - 2 \geq 5 - 2 = 3.$$

Hence, as in case (i), L_{i+1} contains a simply connected region, proving our claim. Now, by induction on i we get that $\|\Phi(\omega(\tilde{M}'))\| \geq 2$. But since $Reg\tilde{M}' = Reg_1 \cup Reg_{0,1}$, it follows that $\omega(M)(= \omega(\tilde{M}'))$ contains a letter outside H . If $i \leq 0$, then the same argument applies. □

Lemma 6.3. *Let notation be as in Lemma 6.1. If $\text{Reg}(\tilde{M}') \neq \emptyset$ and every region of \tilde{M}' is annular then $U = x^{m_1}$ and $V = y^{m_2}$ for some $x, y \in X$, $m_1, m_2 \in \mathbb{Z} \setminus \{0\}$.*

Proof. First notice that if $\tilde{\Delta}'$ is an annular region occurring in L_i , then L_i consists of $\tilde{\Delta}'$. Consequently, $\omega_i = \omega(\tilde{\Delta}')$ and hence if L_{i+1} exists then due to the definition of derived regions, $\|\Phi(\omega_i)\| = 1$.

If L_{i+1} is the last layer then $\omega(M) = \omega(L_{i+1})$ and one of the following holds

- (i) $\|\Phi(\omega(L_{i+1}))\| = 1$
- (ii) $\text{Supp}(\omega(L_{i+1})) = \text{Supp}(\Delta)$, where $L_{i+1} = \{\Delta\}$ and $\Phi(\omega(L_{i+1}))$ is conjugate to x^i in $\langle \text{Supp}(\Delta) \rangle$, where $x \in \text{Supp}(\Delta)$ and $i \in \mathbb{Z} \setminus \{0\}$.

The same argument applies if $i \leq 0$ and also when \tilde{M}' is one-layer. □

Definition 6.4. *Let $g \in A$. Say that g is non-elementary if $g \neq 1$ and g is not conjugate in A to a power of a standard generator. Call a subgroup P of A non-elementary if it contains a non-elementary element.*

Lemma 6.5. *Suppose that P is a non-cyclic parabolic subgroup of A , $P \not\subseteq H$.*

Then P is non-elementary.

Proof. Let V_0 and V_1 be sets of vertices of Γ_A , as defined in the Introduction. Since P is not cyclic, there are $x, y \in P \cap V$, $x \notin \langle y \rangle$, $y \in V_1$. Let $W = xy$. Then $\rho(W) = \{\hat{W}\}$, since xy is cyclically reduced and since W contains no half relators, as $y \in V_1$. On the other hand, $\rho(z^n) = \{z^n\}$ for every $z \in V$. Since clearly $\rho(z^n) \cap \rho(W) = \emptyset$, W is not conjugate to a power of a generator. □

Lemma 6.6. *Suppose that \tilde{M}' has corner regions but only non-effective corner regions. Then \tilde{M}' is full and L_i is an alternating sequence of 2-exceptional and their companion 2-corner regions (i.e. 4-homogeneous).*

Proof. If \tilde{M}' is not full then by Lemma 4.6 \tilde{M}' has an effective corner region, violating our assumption. Hence, M is full.

If $d_M(v) = 3$ for every boundary vertex of \tilde{M}' then all the corner regions of \tilde{M}' are effective, by definition. Hence $d_M(v) = 4$, for at least one boundary vertex v of \tilde{M}' . Therefore L_p contains a 2-exceptional region D with its companion 2-corner regions E_l and E_r , by Theorem 2.6.

We claim that the left neighbour of E_l, K_l , is a 2-exceptional region and the right neighbour, K_r of E_r , is a 2-exceptional region. Suppose that K_l is not 2 exceptional and let $w = w \cap \partial K_l \cap \partial E_l$. Then $d_{\tilde{M}'}(v) = 3$, hence E_l is an effective corner region, violating our assumption. Hence K_l , and similarly K_r are 2-exceptional.

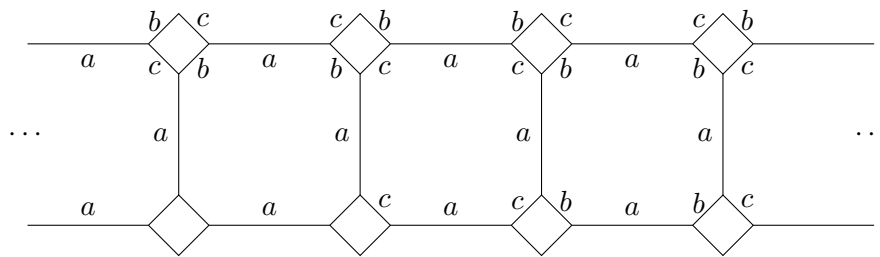
Next, let u be the inner vertex of $w_p \cap \partial D$, then $d_M(u) \geq 4$, while $d_{L_p}(u) = 2$. Hence $d_{L_{p-1}}(u) \geq 4$, hence $(d_M(u) = d_{L_p}(u) + d_{L_{p-1}}(u) - 2) d_p(u) = 4$ and L_{p-1} contains a

Since L_p (and here $L_i, i \leq p$) is full it follows from the claim that M is 4-homogenous. □

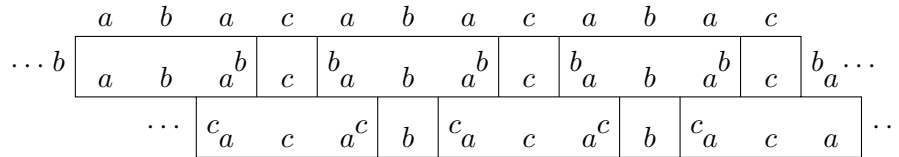
Corollary 6.7. *Suppose \tilde{M}' has no effective corner region. Then \tilde{M}' is full and is k -homogeneous for some $k \in \{2, 3, 4\}$.*

- Lemma 6.8.** (a) If \tilde{M}' is full and every region is simply connected and contains more than one layer and is k -homogeneous for some k , $k \in \{2, 3, 4\}$ then L_i has boundary cycles $v_i w_i v_i$ and $u_i \tau_i u_i$, u_i, v_i vertices, such that $\Phi(\omega_i), \Phi(\tau_i) \in \mathcal{W}_k$ and such that the shortest path θ_i connecting v_i with u_i satisfies $\Phi(\theta_i) \in \mathcal{W}_k^+$.
- (b) If \tilde{M}' is full and for some D in M $\Delta(D)$ is annular, then $\Delta(E)$ is annular for every region E of M and all the layers have boundary labels x_i^m , $x_i \in X$ $m \in \mathbb{Z} \setminus \{0\}$ and a shortest path connecting w_i to τ_i has label in \mathcal{W}_5 .

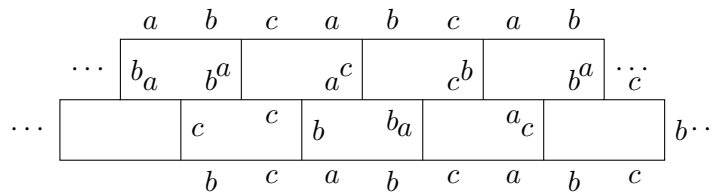
\mathcal{W}_3



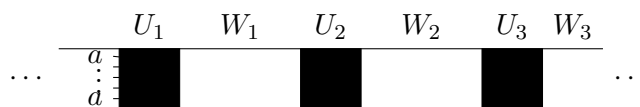
\mathcal{W}_2



\mathcal{W}_4



\mathcal{W}_1



\mathcal{W}_5

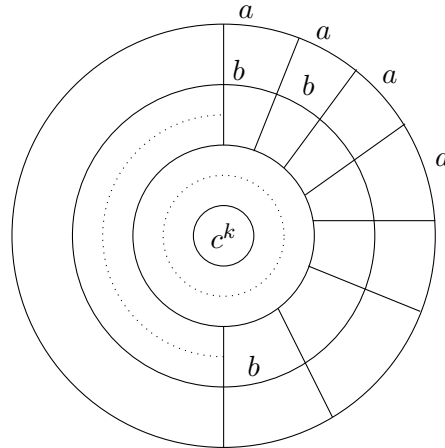


Figure 16

Proof. (a) By inspection using Lemma 3.13 and Corollary 4.10. See Figure 7 and Figure 16.

- (b) Suppose that L_i consists of a single annular region Δ and L_{i+1} consists of simply connected regions $\Delta_1, \dots, \Delta_k, k \geq 1$. Then $a(\Delta_i) \leq 1$. If $\Delta_i \in Reg_{0,1}(\tilde{M}')$ then all the inner (in \tilde{M}') boundary vertices of Δ_i have valency at least 4, by Lemma 2.9 (c), hence either $b(\Delta_i) \geq 1$, in which case Δ_i has a neighbour Δ_j with $a(\Delta_j) = 0$, hence Δ_j is a 2-corner region, or $b(\Delta_i) = 0$, in which case Δ_i is a k -corner region with $k \leq 1$. It follows by Lemma 4.9 that \tilde{M}' is not full. If $\Delta_i \in Reg_1(\tilde{M}')$, then Δ_i is a k -corner region with $k \leq 3$. Thus, again by Lemma 4.9 \tilde{M}' is not full, a contradiction.

Hence, every region of \tilde{M}' is annular.

The label of θ is found by inspection, using Lemma 6.3.

The Lemma is proved. □

Lemma 6.9. *Let $U \in \rho(W)$. Then*

- (a) $Supp(U) \subseteq Supp(W)$.
- (b) $\pi(U)$ is conjugate in A to $\pi(W)$.

Proof. Since U is obtained from W by a sequence of the elementary operators ξ, K and \mathcal{L} , it is enough to show that

- (i) $Supp(\xi(W)) \subseteq Supp(W)$
- (ii) $Supp(K(W)) \subseteq Supp(W)$
- (iii) $Supp(\mathcal{L}(W)) \subseteq Supp(W)$

Now, (i) is immediate by the definition. Also, (ii) follows from the fact that $Supp(U) = Supp(U^*)$, for every cycle conjugate U^* of U . Finally, if $R = Z\tilde{Z}$ with $|Z| \geq \frac{1}{2}|R|$ then $|Z| \geq 2$, hence Z contains both letters of R and consequently

$$Supp(\tilde{Z}) \subseteq Supp(Z)$$

- (b) (i) and (iii) yield equal elements in A , while (ii) is conjugation.

□

Theorem 6.10. *Let $U_1, U_2 \in \rho(U)$, for a cyclically reduced word U and assume that $[U_1] \neq [U_2]$, $U \notin H$. Then there exist $U_1^* \in [U_1]$ and $U_2^* \in [U_2]$ such that U_1^* and U_2^* belong to \mathcal{W}_i , for some i , $i = 1, 2, 3, 4, 5$. If $U_1^{**} \in [U_1]$ and $U_2^{**} \in [U_2]$ such that U_1^{**} and $U_2^{**} \in \mathcal{W}_j$ then $i = j$. If U_1^* and U_2^* belong to \mathcal{W}_i then they are conjugate by an element of \mathcal{W}_i^\perp .*

Proof. Clearly, U_1 and U_2 are conjugate in A , hence there exists an annular diagram M_0 with connected interior such that $\Phi(\omega(M)) = U_1$ and $\Phi(\tau(M)) = U_2$, where $\omega(M)$ and $\tau(M)$ are the outer and inner boundaries of M , respectively. Denote it $M(U_1, U_2)$. Among all $U'_1 \in [U_1]$ and $U'_2 \in [U_2]$ choose a pair (U'_1, U'_2) such that $M := M(U'_1, U'_2)$ contains minimal number of regions. Then M' may have no corner regions, because if M' has a corner region Δ then by Lemma 3.11 Δ contains a corner region D with $\partial D \cap \partial M' = \mu$ and μ has a complement ν on ∂D , (such that $\mu\nu^{-1} = \partial D$) then replacing μ by ν is an application of \mathcal{L} . Hence the word obtained by this replacement belongs to $[U_1]$, say, yields $M_1^* := M' \setminus \{\Delta\}$ as the corresponding annular diagram which still has connected interior, violating our assumption.

Hence by passing to \tilde{M}' , we see that due to Remark 4.2

$$\tilde{M}' \text{ has no effective corner region} \tag{*}$$

We consider now the possible structures of \tilde{M}' , according to Lemma 2.11 when \tilde{M}' satisfies (*).

Case (1, 1) Then \tilde{M}' is the union of two simply connected diagrams \tilde{M}'_1 and \tilde{M}'_2 with $\tilde{M}'_1 \cap \tilde{M}'_2 = \{w\}$, w a vertex. Since \tilde{M}' has no effective corner regions by (*), \tilde{M}'_1 and \tilde{M}'_2 consist of w , due to Lemma 4.4 (b), hence $M' = M_w$. Consequently, $U'_1, U'_2 \in H$, hence $U \in H$, contrary to assumption.

Case (1, 2)

- (i) Assume first that \tilde{M}' has more than one layer. Since \tilde{M}' contains no effective corner region by (*), Corollary 6.7 and Lemma 6.8 apply and $\Phi(\omega(M)) \in \mathcal{W}_k$ and $\Phi(\tau(M)) \in \mathcal{W}_k$, for some k , $k \in \{1, 2, 3, 4, 5\}$, and elements of \mathcal{W}_k^\perp conjugate $\Phi(\omega)$ to $\Phi(\tau)$, as required.

Assume now that \tilde{M}' has one layer. We claim that either U_1^* or U_2^* belong to \mathcal{W}_2 and are conjugate by an element of \mathcal{W}_2^\perp or U_1^*, U_2^* belong to \mathcal{W}_1 and are conjugate by elements of \mathcal{W}_1^\perp .

Let v be a vertex in \tilde{M}' , say $v \in \omega(\tilde{M}')$. If $d_{\tilde{M}'}(v) \geq 5$ then there is a region $\tilde{\Delta}'$ in \tilde{M}' which contains v on its boundary and one endpoint of $\partial\tilde{\Delta}' \cap \tau(\tilde{M}')$ has valency 3. Since $b(\tilde{\Delta}') = 2$, $\tilde{\Delta}'$ is an effective corner region, violating (*). Hence $d_{\tilde{M}'}(v) \leq 4$, for every vertex in \tilde{M}' . We claim that if $Reg(M_v) \neq \emptyset$ then $d_{\tilde{M}'}(v) = 4$, for every vertex v . If $d_{\tilde{M}'}(v) = 3$ for some vertex v and $\tilde{\Delta}'_1$ and $\tilde{\Delta}'_2$ are the regions of \tilde{M}' which contain v on their boundary, then $|\partial M_v \cap \partial\tilde{\Delta}'_i| \leq 1$, $i = 1, 2$, hence by Corollary 3.3 and Lemma 2.8

$$\left| \partial M_v \cap \left(\partial\tilde{\Delta}'_1 \cup \partial\tilde{\Delta}'_2 \right) \right| \leq |\partial M_v \cap \omega(M)|$$

Therefore due to the minimality assumption $Reg(M_v) = \emptyset$. It follows that either all the vertices have valency 3, in which case $Reg(M_v) = \emptyset$ for every vertex and hence U_1^* and U_2^* are conjugate by x^α , where $x \in X$ and $\alpha \in \mathbb{Z} \setminus \{0\}$, hence U^*, V^* are in \mathcal{W}_1 and are conjugate in

\mathcal{W}_1^\perp , or \tilde{M}' has a vertex v with valency 4. Let $\tilde{\Delta}'_1, \tilde{\Delta}'_2$ and $\tilde{\Delta}'_3$ be the regions which contain v on their boundary and let $\eta_i = \partial\tilde{\Delta}'_i \cap \partial M_v$.

If $Reg(M_v) = \emptyset$ then relying on Remark 4.2 we may remove $\tilde{\Delta}_1$ (and $\tilde{\Delta}_3$) to get a smaller diagram by an \mathcal{L} -operation, a contradiction to the minimality assumption. Consequently, $Reg(M_v) \neq \emptyset$. Hence, $\Phi(\eta_i) = x_{j_i}^{\alpha_i}$, $x_{j_i} \in X$ and $\alpha \in \mathbb{Z} \setminus \{0\}$, $i = 1, 2, 3$. If $|\{x_{j_1}, x_{j_2}, x_{j_3}\}| = 3$ then by Theorem 5.1 $|\Phi(\eta_1\eta_2\eta_3)| \leq |\Phi(\theta)|$, where $\theta = \partial M_v \cap \omega(M)$, violating the minimality assumption. Consequently $x_{j_1} = x_{j_3}$, and again due to the minimality assumption $\lambda(x_{j_1}, x_{j_2}) = 2$. (We used Theorem 2.1 and Lemma 3.1.) It follows by repetition of this argument that in this case all the vertices have valency 4, $Reg(M_v) \neq \emptyset$ and $\lambda(x_{j_1}, x_{j_2}) = 2$. Hence U_1^* and U_2^* are in \mathcal{W}_2 and are conjugate by an element of \mathcal{W}_2^\perp .

- (ii) Let $\tilde{M}'_1, \dots, \tilde{M}'_r$ be the connected components of $Int(\tilde{M}')$. Since M' has connected interior and \tilde{M}' is obtained from M' by shrinking H -labelled edges to points, it follows that $\partial\tilde{M}'_i \cap \partial\tilde{M}'_{i+1}$ is a coloured vertex v_i , (i taken modulo r) $i = 1, \dots, r$. See Figure 17.

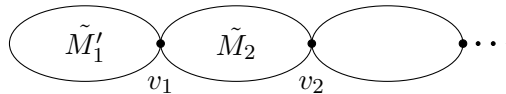


Figure 17

Now, \tilde{M}'_i is a simply connected diagram with the condition C(6). Now, \tilde{M}'_i is a simply connected diagram with connected interior (otherwise $U_1^* = U_2^*$ in A) with $Reg(M'_i) \subseteq Reg_{0,1} \cup Reg_1$. Let $\theta_i = \partial\tilde{M}'_i \cap \partial M'_{v_i}$. Then $\Phi(\theta_i) \in H$ and M'_i has a boundary cycle $\theta_i\eta_i\theta_{i+1}\xi_i$, where $\eta_i = \partial M'_i \cap \omega(M')$ and $\xi_i = \partial M'_i \cap \tau(M')$. We propose to show that $|\theta_i| = 1$, $i = 1, \dots, r - 1$. This is clear if $|M'| = 1$. Suppose first that M'_i is a one-layer diagram, $M'_i = \langle \Delta_1, \dots, \Delta_t \rangle$, $t \geq 2$. If for some j , $j \neq 1$ and $j \neq t$ we have $\partial\Delta_j \cap \theta_i \neq \emptyset$ and also $\partial\Delta_j \cap \theta_{i+1} \neq \emptyset$ then

$$(**) \quad \|\partial\Delta_j \cap \theta_i\| = \|\partial\Delta_j \cap \theta_{i+1}\| = 1$$

Hence, if $\partial\Delta_j \cap \eta_i = \emptyset$ and $\partial\Delta_j \cap \xi_i = \emptyset$ then due to Lemma 2 $\|\partial\Delta_j\| = 4$, a contradiction since $\|\Delta_j\| \geq 8$, as $\Delta_j \in Reg_{0,1}$. Hence, if for some j , $\partial\Delta_j \cap \theta_i \neq \emptyset$ and $\partial\Delta_j \cap \theta_{i+1} \neq \emptyset$ then either $\theta \subseteq \partial\Delta_j$ in which case $|\theta_i| = 1$, as required or $j = 1$ or $j = t$. Hence $t = 2$. But then due to (**), if κ is the complement of $\partial\Delta_1 \cap \eta_i$ on $\partial\Delta$, then $\|\kappa\| \leq 3$, hence due to Lemma 3.1 we can reduce $|M|$, as above.

Similar argument applies for $\partial\Delta_i \cap \xi_i$, $\partial\Delta_t \cap \xi_i$ and $\partial\Delta_t \cap \eta_i$. Thus, if M'_i is a one-layer diagram then either $\partial\Delta_j \cap \theta_i = \emptyset$ or $\partial\Delta_j \cap \theta_{i+1} = \emptyset$, for every j , $j = 1, \dots, t$. It follows that $\theta_i \subseteq \partial\Delta_1$ and $\theta_{i+1} \subseteq \partial\Delta_t$, hence $|\theta_i| = |\theta_{i+1}| = 1$, as required. So we assume now that M' is not a one-layer diagram. Observe first that if Δ is a corner region of M'_i then $\partial\Delta \cap \partial M'_i \not\subseteq \eta_i$, for then by a standard application of Lemma 3.1, $|M|$ can be reduced, as above, violating the minimality assumption on $|M|$. By the same argument $\partial\Delta \cap \partial M'_i \not\subseteq \xi_i$. Thus

$$(***) \quad \partial\Delta \cap \partial M'_i \not\subseteq \eta_i \text{ and } \partial\Delta \cap \partial M'_i \not\subseteq \xi_i$$

Since by assumption M'_i is not a one-layer diagram, it has at least 3 corner regions, each of which satisfies $(***)$. Consequently, for every corner region Δ of M'_i either $\partial\Delta \cap \theta_i$ contains an edge or $\partial\Delta \cap \theta_{i+1}$ contains an edge. Observe that since M' is planar, there are at most 2 corner regions for which both conditions hold. Consequently, M' contains a corner region Δ such that either $\partial\Delta \cap \theta_i = \emptyset$ or $\partial\Delta \cap \theta_{i+1} = \emptyset$.

Suppose $\partial\Delta \cap \theta_{i+1} = \emptyset$. Then $\partial\Delta \cap \theta_i$ contains an edge, due to $(***)$. If $\theta_i \subseteq \partial\Delta$ then we are done. Assume therefore that $\theta_i \not\subseteq \partial\Delta$. Then we may assume without loss of generality that $\partial\Delta \cap \partial M'_i = (\partial\Delta \cap \theta_i) \cup (\partial\Delta \cap \eta_i)$. Let κ be the complement of $\partial\Delta \cap \partial M'_i$ on $\partial\Delta$. Then $|\kappa| \leq 3$, since Δ is a corner region in a C(6) diagram, hence $|\partial\Delta \cap \theta_i| + |\kappa| \leq 4$.

Consequently, Δ has a complete corner region D with $\partial D \cap \partial M'_i = \partial D \cap \eta_i$, due to Lemma. But this violates the minimality of $|M|$, as we have seen above several times. Hence this case cannot occur and hence due to $(***)$ the only possibility is that $\theta_i \subseteq \partial\Delta$. Hence $|\theta_i| = 1$, as required.

Consequently $\partial M_{v_i} \cap \partial M'_i$ is labelled by a^n , for some $a \in V_0$ and $n \in \mathbb{Z} \setminus \{0\}$. Hence $\Phi(\omega(M)) \in \mathcal{W}_1$ and $\Phi(\tau(M)) \in \mathcal{W}_1$.

Case (2,1) Let $\Delta \in \text{Reg}(M)$ with $\Delta := \Delta(D)$ annular. If $D \in \text{Reg}_0$ then the arguments of Case (1,1) apply.

Case (2,2) Follows by Lemma 6.8 (b). □

6.2. Proof of Theorem A. Suppose case (a) does not hold.

- (b) If $\rho(W) = [U]$ then the statement follows from the definition of $\rho(W)$. Assume $\rho(W) \neq [U]$.
- (c) For each $U'_1 \in [U_1]$ and $U'_2 \in [U_2]$ denote by $M(U'_1, U'_2)$ the annular diagram with boundary labels U'_1 and U'_2 . (Notice that U'_1 is conjugate to U'_2 in A).

Let $U_1^* \in [U_1]$ and $U_2^* \in [U_2]$ be such that $M(U_1^*, U_2^*)$ contains minimal number of regions among $\{M(U'_1, U'_2) \mid U'_1 \in [U_1], U'_2 \in [U_2]\}$.

Then by Theorem 6 U_1^* and U_2^* belong to exactly one of \mathcal{W}_i , $i = 1, 2, 3, 4, 5$, and they are conjugate by an element of \mathcal{W}_i^\perp .

The Theorem is proved.

6.3. Proof of Theorem B.

- (a) Since $g \notin H$, due to Lemma 6.1 and Lemma 2.12 any conjugating diagram $M := M(U, W)$, where U is a shortest representative of u and W represents w , $U, W \in H$ contains a region outside \mathcal{R}_{E_0} . Consequently \tilde{M}' contains at least one region. If \tilde{M}' contains a region $\tilde{\Delta}'$ which is simply connected then by Lemma 6.2 either U or W contains a letter from V_1 , violating our assumptions. Consequently, every layer of \tilde{M}' consist of a single annular region, hence by Lemma 6.3 $u = u_0^{m_1}$ and $w = w_0^{m_2}$, $u_0, w_0 \in V$, $m_1, m_2 \in \mathbb{Z} \setminus \{0\}$, as required.
- (b) By part (a), if Z is an element in P with $||Z|| \geq 2$ and $L^{-1}ZL = Z$, for some $L \in F$, then $L \in H$. Consequently, since P is not cyclic,

$$C_A(P) = \bigcap_{t \in P} C_A(t) \subseteq C_A(z) \subseteq H$$

where Z in A is represented by Z , hence $C_A(P) \subseteq C_H(P)$. Since clearly $C_H(P) \subseteq C_A(P)$, we get $C_A(P) = C_H(P)$, as required.

The other two claims follow by similar arguments.

- (c) Suppose that $g^{-1}Kg \cap K \neq 1$ for some $g \in A$. Then $g^{-1}ug = v$ for $u, v \in K$, hence $g \in H$, by part (a), since $\langle x \rangle \cap K = (1)$ for every $x \in V_0$. But then $g \in K$, since K is malnormal in H . The Theorem is proved.

6.4. Proof of Theorem C.

- (a) By Lemma 6.5 P contains a non-elementary element z . Let $t \in C_A(z)$. If $z \in H$, then it follows from Theorem B and Lemma 2.12 that $t \in H$.

Hence $C_A(z) = C_H(z)$, in this case.

If $z \notin H$ then it follows from Theorem A and Lemma 2.12 that $t \in P$.

Hence $C_A(z) = C_P(z)$, in this case.

Since $C_A(z) \supseteq C_A(P)$, it follows that $C_A(P) \subseteq C_H(P)Z(P)$, where $Z(P)$ is the centre of P .

Clearly, $C_H(P)Z(P) \subseteq C_A(P)$, hence $C_A(P) = C_H(P)Z(P)$.

- (b) Suppose (i) and (ii) do not hold. Then by Lemma 6.6 Z contains a non-elementary element $q \in Q$, such that $q = t^{-1}pt$, $p, q \notin H$, $p \in P$.

Let U, W and T be shortest representatives of p, q and t , respectively.

Then $T^{-1}UTW^{-1} \equiv 1 \pmod{\mathcal{N}}$ hence there is a van Kampen diagram with boundary cycle labelled with $T^{-1}UTW^{-1}$. Let T' be the word obtained from T after forming the reduced annular diagram \tilde{M}' by the identification of the two occurrences of T . Then by Lemma 2.12 $T' \equiv T \pmod{\mathcal{N}}$. Now, T' is the label of a path in \tilde{M}' , hence it follows by Lemma 6.2 and Lemma 6.3 that $\text{Supp}(T') \subseteq \text{Supp}(P) \cup \text{Supp}(Q)$.

Hence $t \in \langle P, Q \rangle$.

- (c) Follows from parts (a) and (b) with $Q = P$.

The Theorem is proved.

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