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GROUPS FOR WHICH THE NONCOMMUTING GRAPH IS A SPLIT GRAPH

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ABSTRACT. The noncommuting graph $\nabla(G)$ of a group G is a simple graph whose vertex set is the set of noncentral elements of G and the edges of which are the ones connecting two noncommuting elements. We determine here, up to isomorphism, the structure of any finite nonabelian group G whose noncommuting graph is a split graph, that is, a graph whose vertex set can be partitioned into two sets such that the induced subgraph on one of them is a complete graph and the induced subgraph on the other is an independent set.

1. Introduction

In this paper, we will consider only simple, undirected and finite graphs, i.e., undirected graphs on a finite number of vertices without multiple edges or loops. Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E . Given a set of vertices $X \subseteq V$, the subgraph induced by X is written $\Gamma[X]$. We often identify a subset of vertices with the subgraph induced by that subset, and vice versa. A subset X of V is *complete* if the induced subgraph $\Gamma[X]$ is complete, and it is *independent* (or *stable*) if $\Gamma[X]$ is a null graph (i.e. a graph without edges). A clique is a *maximal* complete subset. As usual, we denote by C_n the cycle and by K_n the complete graph, each on n vertices.

A graph Γ is a *split* graph if and only if there is a partition $V = C \uplus I$, where C is a complete and I an independent set. Thus Γ can be ‘split’ into a complete and an independent set. Any partition of the vertex set of a split graph into a complete and an independent set will be called a *split partition*. In a particular case, a split partition $V = C \uplus I$ is *special* if every vertex in C is adjacent to at least

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one vertex in I . Note that, every split graph has a special split partition, because if there is a vertex in C not adjacent to any element of I , it can be moved to I .

All groups considered here are finite. Let G be a nonabelian group and $Z = Z(G)$ the center of G . The *noncommuting graph* $\nabla(G)$ of G is defined as follows: the vertex set is $G \setminus Z$ (the noncentral elements of G) and two distinct vertices x and y are joined by an edge iff $xy \neq yx$. As illustrated below, this graph can be considered as a multipartite graph. Obviously, we can suppose that $[G : Z] = m + 1 \geq 4$. Let $T = \{t_0 = 1, t_1, \dots, t_m\}$ be a left transversal to Z in G . Then, it is not hard to show that $\nabla(G)$ is an m -partite graph on finite non-empty independent sets t_1Z, t_2Z, \dots, t_mZ .

We denote by $\Delta(G)$ the complement of $\nabla(G)$ and call it the *commuting graph* of G . In the commuting graph of G the vertex-set is again the set of noncentral elements of G and two distinct vertices of $\Delta(G)$ are adjacent if and only if they commute in G . Clearly, a graph and its complement determine each other uniquely.

In the present paper we study the structure of any finite nonabelian group G whose noncommuting (or commuting) graph is a split graph. It is surprising that such groups in fact are Frobenius groups of order $2n$, n is odd, with abelian Frobenius kernel of order n (see Example 2 and Theorem 2.3).

Only basic concepts about graphs and groups will be needed for this paper. They can be found in any textbook about the graph theory or group theory, for instance see [2, 6]. Furthermore, we denote by $\pi_e(G)$ the set of orders of all elements in G . It is clear that the set $\pi_e(G)$ is *closed* and *partially ordered* by divisibility, hence, it is uniquely determined by $\mu(G)$, the subset of its maximal elements. The chromatic number of a graph Γ is denoted by $\chi(\Gamma)$, and the size of a largest clique in Γ by $\omega(\Gamma)$. The maximum size of an independent set of vertices is denoted by $\alpha(\Gamma)$ and the size of a minimum cover of the vertices with cliques by $\theta(\Gamma)$.

2. Groups with split noncommuting graphs

Some results on split graphs will be useful and we recall them now. In [1], the authors provided the following “forbidden subgraph characterization” of split graphs.

Lemma 2.1. (Földes and Hammer [1]). *A graph is a split graph if and only if it contains no induced subgraph isomorphic to $2K_2, C_4$ or C_5 .*

It follows from the definition (or the forbidden subgraph characterization) that the complement, and every induced subgraph of a split graph is split.

In this section, we focus our attention on finite groups whose noncommuting graphs are split graphs. Indeed, in this situation we have

$$G \setminus Z(G) = C \uplus I,$$

where C is a subset of G whose elements do not mutually commute and I is a subset of G whose elements pairwise commute. We are going to describe the structure of such groups.

Before obtaining the structure of such groups, it seems appropriate to point out the following lemma, which will be collected some technical stuff needed to perform our work.

Lemma 2.2. (Hammer and Simeone [3]). *Let Γ be a simple graph with n vertices and degree sequence $d_1 \geq d_2 \geq \dots \geq d_{n-1} \geq d_n$, where d_i is the degree of vertex v_i . Set $p = \max\{i \mid d_i \geq i - 1\}$. Then Γ is a split graph if and only if*

$$(2.1) \quad \sum_{i=1}^p d_i = p(p-1) + \sum_{i=p+1}^n d_i.$$

Moreover, if Γ is a split graph, then $\{v_1, \dots, v_p\}$ and $\{v_{p+1}, \dots, v_n\}$ form a split partition, $\omega(\Gamma) = \chi(\Gamma) = p$, and $\alpha(\Gamma) = \theta(\Gamma) = n - \min\{p, d_p\}$.

Remark. It is routine to check that a maximum independent set is either $\{v_{p+1}, \dots, v_n\}$ if $d_p \geq p$ or $\{v_p, \dots, v_n\}$ if $d_p = p - 1$.

Let us return now to groups whose noncommuting graph is split.

Example 1. Some examples occur when we consider dihedral groups. As a matter of fact, the noncommuting graph associated with dihedral group

$$D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle,$$

with $n > 1$ odd, is a split graph. Actually, this example deals with a centerless group and we have the following special split partition of $\nabla(D_{2n})$:

$$C = \{b, ab, a^2b, \dots, a^{n-1}b\} \quad \text{and} \quad I = \{a, a^2, \dots, a^{n-1}\}.$$

For notational convenience, we write v_i and w_j for $a^{i-1}b$ and a^{j-1} , $1 \leq i \leq n$ and $2 \leq j \leq n$, in what follows. Then, a routine argument shows $d_i = d(v_i) = 2n - 2$, $1 \leq i \leq n$, and $d_{n+j-1} = d(w_j) = n$, $2 \leq j \leq n$. Therefore, we have

$$d_1 \geq d_2 \geq \dots \geq d_n > d_{n+1} \geq \dots \geq d_{2n-1}.$$

With the notation as Lemma 2.2, we see that $p = n + 1$ and the equality (2.1) holds. Therefore, the noncommuting graph of the Dihedral group D_{2n} , n odd, is a split graph, I is a maximum independent set, $\omega(\nabla(D_{2n})) = \chi(\nabla(D_{2n})) = n + 1$, and $\alpha(\nabla(D_{2n})) = \theta(\nabla(D_{2n})) = n - 1$.

Example 2. As we know, the dihedral groups of order $2n$ with n odd, are examples of Frobenius groups with cyclic Frobenius kernel of order n . In the sequel, we will consider a more general case, that is, Frobenius groups of order $2n$, n is odd, with abelian Frobenius kernel of order n . Thus, we assume that $G = K : H$ is a Frobenius group with abelian Frobenius kernel K of order n and Frobenius complement $H \cong \mathbb{Z}_2$. Clearly, G is a centerless group. Put $K = \{1, a_1, a_2, \dots, a_{n-1}\}$ and let x be an involution of G . It follows easily that $G \setminus \{1\}$ is a disjoint union:

$$G \setminus \{1\} = \{x, a_1x, a_2x, \dots, a_{n-1}x\} \uplus \{a_1, a_2, \dots, a_{n-1}\}.$$

As before, it is routine to check that $a_{i-1}x$, $1 \leq i \leq n$, is an involution of G and

$$C_G(a_{i-1}x) = \langle a_{i-1}x \rangle = \{1, a_{i-1}x\}, \quad (1 \leq i \leq n),$$

while

$$C_G(a_j) = K, \quad (1 \leq j \leq n).$$

Therefore, we obtain

$$d_i = d(a_{i-1}x) = 2n - 2, \quad (1 \leq i \leq n),$$

and

$$d_{n+j-1} = d(a_{j-1}) = n, \quad (2 \leq j \leq n),$$

and hence

$$d_1 \geq d_2 \geq \cdots \geq d_n > d_{n+1} \geq \cdots \geq d_{2n-1}.$$

Along lines similar to those used for the case of a dihedral group, we see that G is a split graph with special split partition $G \setminus \{1\} = C \uplus I$, where

$$C = \{x, a_1x, a_2x, \dots, a_{n-1}x\} \quad \text{and} \quad I = \{a_1, a_2, \dots, a_{n-1}\}.$$

Again, with the notation as Lemma 2.2, $p = n + 1$ and the equality (2.1) holds. Moreover, I is a maximum independent set, $\omega(\nabla(G)) = \chi(\nabla(G)) = n + 1$, and $\alpha(\nabla(G)) = \theta(\nabla(G)) = n - 1$.

We mention here an infinite family of such Frobenius groups. Let

$$G_t := \underbrace{(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3)}_{t\text{-times}} : \mathbb{Z}_2,$$

where t is a natural number. By [5], $\pi_e(G_t) = \{1, 2, 3\}$. In fact, G_t is a Frobenius group of order $|G_t| = 2 \cdot 3^t$ with Frobenius kernel $K = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3$ which is an elementary abelian 3-group of order 3^t and the Frobenius complement $H = \mathbb{Z}_2$.

We can now establish our main result on split noncommuting graphs.

Theorem 2.3. *The noncommuting graph of a group G is a split graph if and only if G is isomorphic to a Frobenius group of order $2n$, n is odd, whose Frobenius kernel is abelian of order n .*

Proof. As mentioned above the noncommuting graph associated with a Frobenius group of order $2n$, where n is an odd number and its Frobenius kernel is abelian, is a split graph (see Example 2). Hence, we need only prove the necessity. Therefore, suppose that G is a nonabelian group for which the noncommuting graph $\nabla(G)$ is a split graph. For convenience we break up the proof into a sequence of lemmas. We start with the following simple observation.

Lemma 2.4. *G is a centerless group, that is $Z(G) = 1$.*

Proof. Assume to the contrary that $Z = Z(G) > 1$. Let $[G : Z] = m + 1$ and let $T = \{1, t_1, t_2, \dots, t_m\}$ be a transversal to Z in G . As before, we consider $\nabla(G)$ as an m -partite graph on finite nonempty independent sets t_1Z, t_2Z, \dots, t_mZ . Since $\nabla(G)$ is always connected [4, Proposition 1], we may assume, without loss of generality, that t_1 is adjacent to t_2 in $\nabla(G)$. Now, for a non identity element $z \in Z$, the induced subgraph $\nabla(G)[S]$, where $S = \{t_1, t_1z, t_2, t_2z\}$, is isomorphic to C_4 . This contradicts with Lemma 2.1. \square

From now on we will assume that G is a centerless group and so the vertex-set of $\nabla(G)$ is $G \setminus \{1\}$. In fact, we are faced with a situation where $G \setminus \{1\}$ is a disjoint union of two sets C and I such that no two distinct elements of C commute while the elements of I pairwise commute.

Lemma 2.5. *If $G \setminus \{1\} = C \uplus I$ is a special split partition and $I^* = I \cup \{1\}$, then the following statements hold:*

- (i) *each element of C is an involution.*
- (ii) *I^* is a maximal abelian normal subgroup of G and G/I^* is an elementary abelian 2-group.*
- (iii) *each vertex in C is certainly adjacent to each vertex in I . In particular, if $x \in C$, then $\deg(x) = |G| - 2$, while if $y \in I$, then $\deg(y) = |C|$.*
- (iv) *I^* has odd order.*
- (v) *A Sylow 2-subgroup S of G is isomorphic to \mathbb{Z}_2 . In particular, $|G| = 2|I^*|$ and $G/I^* \cong \mathbb{Z}_2$.*

Proof. (i) Suppose false and let $x \in C$ be an element of order $l > 2$. Let $H = \langle x \rangle$. Clearly, H contains at least $\varphi(l)$ elements of order l , where $\varphi(l)$ is the Euler's totient function. Since $l > 2$, $\varphi(l) \geq 2$. Now suppose that x^m is another element of H of order l . Since x commutes with x^m , it follows that $x^m \in I$. Therefore $I \subseteq C_G(x^m) = C_G(x)$, because $\langle x \rangle = \langle x^m \rangle$. This means that x commutes with the elements of I , or equivalently, x is nonadjacent to all vertices in I . This contradicts the choice of split partition $C \uplus I$ as a special split partition.

(ii) Let x and y be two arbitrary elements of $I^* = I \cup \{1\}$. If $xy \notin I^*$, then $xy \in C$. Since every two elements of I^* commute, it follows that xy commute with all elements of I^* . Thus xy is not adjacent to any element of I , and we see that our choice of a special split partition is contradicted. Therefore, I^* is an abelian subgroup of G . From the fact that, every abelian subgroup of a group is contained in a maximal abelian subgroup, we may assume that $I^* \subseteq A$, where A is a maximal abelian subgroup of G . If there exists $z \in A \setminus I^*$, then $z \in C$ and z commutes with all elements of I , which means that $z \in C$ is not adjacent to any element of I . This contradicts the choice of special split partition $C \uplus I$ and thus proves that I^* is a maximal abelian subgroup of G . All that remains is to show that I^* is normal in G . To do this, we first define $\psi : G \rightarrow \{-1, 1\}$ by

$$\psi(x) = \begin{cases} 1 & \text{if } x \in I^*, \\ -1 & \text{if } x \in C. \end{cases}$$

Note that the group operation on the set $\{-1, 1\}$ is multiplication. We show now that ψ is a homomorphism from G to $\{-1, 1\}$.

Suppose $x, y \in G = C \uplus I^*$. We distinguish three cases:

- (1) $x, y \in I^*$. In this case $xy \in I^*$, and so $\psi(xy) = 1 = 1 \times 1 = \psi(x)\psi(y)$, as required.
- (2) $x \in I^*$ and $y \in C$. In this case $xy \in C$, since otherwise from $xy \in I^*$ and the fact that I^* is a subgroup of G , it follows that $y \in I^*$, a contradiction. Therefore, $\psi(xy) = -1 = 1 \times (-1) = \psi(x)\psi(y)$, as required.

- (3) $x, y \in C$. In this case $xy \in I$, since otherwise xy would be an element of order 2 by (i), and a direct calculation gives

$$[x, y] = x^{-1}y^{-1}xy = xyxy = (xy)^2 = 1.$$

This means that x commutes with y and so x is nonadjacent to y in $\nabla(G)$, which is a contradiction. Thus $\psi(xy) = 1 = (-1) \times (-1) = \psi(x)\psi(y)$, as required.

Finally, I^* as the kernel of ψ is a normal subgroup of G . The rest of the proof follows easily from part (i).

(iii) Let $x \in C$ and $a \in I$ be two arbitrary elements. Then $xa \in G = C \uplus I$, and since I^* is a subgroup of G , $xa \in C$. It follows therefore that

$$1 \neq [xa, x] = [a, x],$$

which means that x and a are adjacent in $\nabla(G)$, as required. The rest follows easily from the definition of split graph.

(iv) Assume by way of contradiction that I^* contains an involution, say a . Let x be an arbitrary element in C . As before, $xa \in C$, which is also an involution by (i), and we have

$$[x, a] = x^{-1}a^{-1}xa = xaxa = (xa)^2 = 1.$$

Thus x and a are nonadjacent vertices in $\nabla(G)$, this would contradict part (iii).

(v) Let S be a Sylow 2-subgroup of G with $|S| \geq 4$. From the fact that the center of every p -group is nontrivial, we have $Z(S) \neq 1$. Let x be an involution in $Z(S)$. On the one hand, by part (iii), we have $\deg(x) = |G| - 2$, which implies that $|C_G(x)| = 2$. On the other hand, we have $S \leq C_G(x)$, since $x \in Z(S)$. This shows that $|C_G(x)| \geq 4$, a contradiction. Therefore, $|S| = 2$ and by parts (ii) and (iv) we conclude that $|G| = 2|I^*|$, as desired. \square

Since $G = C \uplus I^*$, by parts (i) and (v), we see that $2 \in \mu(G)$. Now, from this and Lemma 2.5, we conclude that if x is an involution of G , then $\langle x \rangle$ acts fixed-point-freely on I^* and G is isomorphic to the desired Frobenius group $I^* : \mathbb{Z}_2$, and the proof is complete. \square

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