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INTERSECTIONS OF PREFRATTINI SUBGROUPS IN FINITE SOLUBLE GROUPS

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Abstract. Let H be a prefrattini subgroup of a soluble finite group G. In the paper it is proved that there exist elements $x, y \in G$ such that the equality $H \cap H^x \cap H^y = \Phi(G)$ holds.

1. Introduction

The notation in the paper is standard and agree with that of [1, 2]. In this paper we consider finite groups only, so the term "group" always means "finite group".

D. S. Passman in [3] proved that a p-soluble group G always possesses three Sylow p-subgroups such that their intersection is equal to $O_p(G)$. Later V. I. Zenkov proved the same statement for an arbitrary group (see [4], Corollary C). In [5] S. Dolfi proved that if $2 \notin \pi$, then every π-soluble group G possesses three Hall π-subgroups such that their intersection is equal to $O_\pi(G)$. In [6] S. Dolfi proved that for any Hall π-subgroup H in every π-soluble group G there exist elements $x, y \in G$ such that the equality $H \cap H^x \cap H^y = O_\pi(G)$ holds (see also [7]).

In connection with these results, in the *Kourovka Notebook* [8] the author formulated the following Problem 17.55:

Does there exist an absolute constant k such that for any prefrattini subgroup H in any finite soluble group G there exist k conjugates of H whose intersection is $\Phi(G)$?

The main goal of this paper is to give an affirmative answer to this question.

Theorem 1.1. *Let H be a prefrattini subgroup of a soluble group G. Then there exist elements $x, y \in G$ such that the equality $H \cap H^x \cap H^y = \Phi(G)$ holds.*

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2. Preliminary results

Recall that if H is a subgroup and A/B is a normal factor of G, then:

1) H is a *complement for* A/B *in* G if $HA = G$ and $H \cap A = B$ (in this case A/B is called a *complemented factor*);

2) H *covers* the factor A/B if $HB \supseteq A$;

3) H *avoids* the factor A/B if $H \cap A \subseteq B$.

First of all we shall appeal to the concept of a crown of a soluble group G. It was introduced by Gaschütz in [9]. Considering a complemented chief factor H/K of a soluble group G as a G-module, Gaschütz proved that G has a normal section that is a completely reducible G-module whose composition components are G-isomorphic to H/K, and its composition length m is equal to the number of complemented and G-isomorphic to H/K factors of a chief series of G. That section is denoted by $\mathrm{Cr}_G(H/K)$ and called a *crown of* G *corresponding to* H/K. A constructive definition of a crown of a soluble group G corresponding to a complemented chief factor H/K is given as follows:

$$\mathrm{Cr}_G(H/K) = C_G(H/K)/R,$$

where R is the intersection of the cores of maximal subgroups complementing H/K.

A crown of a soluble group G is a crown corresponding to a complemented chief factor of G. The set of all crowns of G is denoted by $\mathrm{Cr}(G)$. From the Jordan-Hölder theorem it follows that for the construction of $\mathrm{Cr}(G)$ it is enough to consider some chief series of G and to choose in it the maximal system $H_1/K_1, \ldots, H_t/K_t$ pairwise non-G-isomorphic complemented chief factors. Then we have $\mathrm{Cr}(G) = \{\mathrm{Cr}_G(H_1/K_1), ..., \mathrm{Cr}_G(H_t/K_t)\}$.

Let G be a soluble group, and $\mathrm{Cr}(G) = \{\mathrm{Cr}_G(H_1/K_1), ..., \mathrm{Cr}_G(H_t/K_t)\}$. Let G_i be a complement of $\mathrm{Cr}_G(H_i/K_i)$ in G, where $i \in I = \{1, 2, ..., t\}$. Then the subgroup $\bigcap_{i \in I} G_i$ is called a *prefrattini subgroup of* G.

By the definition, every soluble group has at least one prefrattini subgroup.

The following theorem gives basic properties of prefrattini subgroups.

Theorem 2.1. [9] *Let G be a soluble group and N be a normal subgroup of G. If H is a prefrattini subgroup of G, then the following conditions hold:*

a) HN/N is a prefrattini subgroup of G/N;

b) H covers all Frattini chief factors of G and avoids all complemented chief factors of G;

c) $Core_G(H) = \Phi(G)$;

d) $|H|$ is the product of the orders of all Frattini chief factors in a chief series of G;

e) H^x is a prefrattini subgroup of G for any element $x \in G$;

f) any two prefrattini subgroups of G are conjugate in G.

A maximal subgroup M of G is called *critical* if $MF(G) = G$.

Lemma 2.2. [1, Chapter V, Corollary 5.16] *Let G be a soluble group. If W is a prefrattini subgroup of a critical maximal subgroup of G, then $W\Phi(G)$ is a prefrattini subgroup of G.*

Lemma 2.3. [1, Chapter V, Lemma 5.13] *Let N and S be subgroups of a soluble group $G = NS$ with $N \trianglelefteq G$ and $N \cap S = 1$. Assume that $N \subseteq Soc(G)$. Then $\Phi(G) = C_{\Phi(S)}(N)$.*

In the proof of Theorem 1.1, we are going to use the following result of S. Dolfi and E. P. Vdovin.

Theorem 2.4. [6, 7] *Let π be a set of primes, G a π-soluble group and S a Hall π-subgroup of G. Then there exist elements $x, y \in G$ such that $S \cap S^x \cap S^y = O_\pi(G)$.*

3. Proof of Theorem 1.1

Let G be a counter example to the statement of the theorem of minimal order. Assume first that $\Phi(G) \neq 1$, and consider the group $G/\Phi(G)$. Clearly, $G/\Phi(G)$ satisfies the statement of the theorem. Thus since by Theorem 2.1 $H\Phi(G)/\Phi(G)$ is a prefrattini subgroup of $G/\Phi(G)$, there exist elements $x, y \in G$ such that

$$H\Phi(G)/\Phi(G) \cap H^x\Phi(G)/\Phi(G) \cap H^y\Phi(G)/\Phi(G) = \Phi(G/\Phi(G)).$$

By Theorem 2.1, $\Phi(G) \subseteq H$, $\Phi(G) \subseteq H^x$ and $\Phi(G) \subseteq H^y$. Moreover, $\Phi(G/\Phi(G)) = 1$. Therefore the equality $H \cap H^x \cap H^y = \Phi(G)$ holds. A contradiction with the fact that G is a minimal counter example. Hence $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G. Since G is soluble, there exists a maximal subgroup M of G such that $G = MN$ and $M \cap N = 1$. Let W is a prefrattini subgroup of M. Then, by Lemma 2.2, $W\Phi(G)$ is a prefrattini subgroup of G. Clearly, $W\Phi(G) \subseteq M$. By Theorem 2.1, $H = (W\Phi(G))^w \subseteq M^w$ for some $w \in G$. Moreover, $G = M^wN$ and $M^w \cap N = 1$. Thus, without loss of generality, we can assume that $H \subseteq M$.

By Theorem 2.1, there exist elements $x, y \in G$ such that

$$HN/N \cap H^xN/N \cap H^yN/N = \Phi(G/N).$$

Since $G = MN$ and $N \triangleleft G$, we can assume that $x, y \in M$. Therefore $H \cap H^x \cap H^y \subseteq M$. Moreover, by Theorem 2.1, the subgroups H, H^x and H^y are prefrattini subgroups of M. Hence $\Phi(M) \subseteq H \cap H^x \cap H^y$.

From the standard isomorphism $M \to MN/N$ we have $\Phi(G/N) = \Phi(M)N/N$. Thus

$$\Phi(M)N \subseteq (H \cap H^x \cap H^y)N \subseteq HN \cap H^xN \cap H^yN = \Phi(M)N,$$

and consequently $\Phi(M) = H \cap H^x \cap H^y$. Moreover, the Frattini argument implies that the equalities $H \cap \Phi(M)N = H^x \cap \Phi(M)N = H^y \cap \Phi(M)N = \Phi(M)$ hold.

Let p the prime divisor of $|N|$. Assume that $N \subset O_p(\Phi(M)N)$. By the Frattini argument

$$O_p(\Phi(M)N) = O_p(\Phi(M)N) \cap \Phi(M)N = (O_p(\Phi(M)N) \cap \Phi(M))N.$$

Thus we get that $O_p(\Phi(M)N) \cap \Phi(M) \neq 1$. Since $\Phi(M)N \triangleleft G$, it follows that

$$C_G(N) \supseteq F(G) \supseteq F(\Phi(M)N) \supseteq O_p(\Phi(M)N) \cap \Phi(M).$$

Therefore, by Lemma 2.3, $\Phi(G) \neq 1$. We obtain the contradiction to $\Phi(G) = 1$. This contradiction shows that $N = O_p(\Phi(M)N)$. Since $\Phi(G/N)$ is nilpotent, N is a Sylow p-subgroup of $\Phi(M)N$ and $\Phi(M)$ is its Hall p'-subgroup. Hence, by Theorem 2.4, there exist elements $u, v \in O_p(\Phi(M)N) = N$ such that

$$\Phi(M) \cap (\Phi(M))^u \cap (\Phi(M))^v =$$

$$= (H \cap \Phi(M)N) \cap (H^x \cap \Phi(M)N)^u \cap (H^y \cap \Phi(M)N)^v = O_{p'}(\Phi(M)N).$$

Consider $D = H \cap H^{xu} \cap H^{yv}$. Since $D \subseteq H$, we have $DN \subseteq HN$. Since $D \subseteq H^{xu}$ and $u \in N$, we get $DN \subseteq H^{xu}N = H^xN$. By analogy $DN \subseteq H^{yv}N = H^yN$. Thus

$$DN \subseteq HN \cap H^xN \cap H^yN = \Phi(M)N.$$

Consequently $D \subseteq \Phi(M)N$.

Now we have

$$D = D \cap \Phi(M)N = H \cap H^{xu} \cap H^{yv} \cap \Phi(M)N =$$

$$= (H \cap \Phi(M)N) \cap (H^{xu} \cap \Phi(M)N) \cap (H^{yv} \cap \Phi(M)N) =$$

$$= (H \cap \Phi(M)N) \cap (H^x \cap \Phi(M)N)^u \cap (H^y \cap \Phi(M)N)^v =$$

$$= O_{p'}(\Phi(M)N) \subseteq \Phi(M).$$

Hence $[N, D] = 1$. Thus by Lemma 2.3 $D \subseteq C_{\Phi(M)}(N) = \Phi(G)$. If $D \neq 1$, then $\Phi(G) \neq 1$, a contradiction to $\Phi(G) = 1$. Consequently $D = 1$. But then $D = H \cap H^{xu} \cap H^{yv} = 1 = \Phi(G)$. A contradiction with the fact that G is a minimal counter example. The proof is complete.

4. **Corollaries**

Let $\varepsilon : G \to G/\Phi(G)$ be the natural homomorphism. Let x and y are as in Theorem 1.1. Using the elementary equality

$$|A \cdot B| = \frac{|A| \cdot |B|}{|A \cap B|},$$

where A and B are subgroups of G, we obtain (under Theorem 1.1) that for a prefrattini subgroup H the series of inequalities

$$|\varepsilon(G)| \geq |\varepsilon(H) \cdot \varepsilon(H^x)| = \frac{|\varepsilon(H)| \cdot |\varepsilon(H^x)|}{|\varepsilon(H) \cap \varepsilon(H^x)|} =$$

$$= \frac{|\varepsilon(H)| \cdot |\varepsilon(H^x)| \cdot |\varepsilon(H^y)|}{|\varepsilon(H) \cap \varepsilon(H^x) \cap \varepsilon(H^y)| \cdot |(\varepsilon(H) \cap \varepsilon(H^x)) \cdot \varepsilon(H^y)|} \geq$$

$$\geq \frac{|\varepsilon(H)| \cdot |\varepsilon(H^x)| \cdot |\varepsilon(H^y)|}{|\varepsilon(H) \cap \varepsilon(H^x) \cap \varepsilon(H^y)| \cdot |(\varepsilon(G)|} = \frac{|\varepsilon(H)|^3}{|\varepsilon(G)|}$$

hold.

Thus $|H/\Phi(G)|^3 \leq |G/\Phi(G)|^2$ and $|H| \leq \sqrt[3]{|G|^2 \cdot |\Phi(G)|}$. On the other hand, by $|H/\Phi(G)|^3 \leq |G/\Phi(G)|^2$ we have that $|H/\Phi(G)| \leq |G : H|^2$.

Corollary 4.1. *Let H be a prefrattini subgroup of a soluble group G. Then the inequalities*

$$|H| \leq \sqrt[3]{|G|^2 \cdot |\Phi(G)|}$$

and

$$|H/\Phi(G)| \leq |G:H|^2$$

hold.

Corollary 4.2. *Let H be a prefrattini subgroup of a soluble group G. If $\Phi(G) = 1$, then there exist elements $x, y \in G$ such that the equality $H \cap H^x \cap H^y = 1$ holds.*

Corollary 4.3. *Let H be a prefrattini subgroup of a soluble group G. If $\Phi(G) = 1$, then the inequalities $|H| \leq \sqrt[3]{|G|^2}$ and $|H| \leq |G:H|^2$ hold.*

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