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NONNILPOTENT SUBSETS IN THE SUZUKI GROUPS

MOHAMMAD ZARRIN

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Abstract. Let G be a group and $\mathcal{N}$ be the class of all nilpotent groups. A subset A of G is said to be nonnilpotent if for any two distinct elements a and b in A, $\langle a, b\rangle \notin \mathcal{N}$. If, for any other nonnilpotent subset B in G, $|A| \geq |B|$, then A is said to be a maximal nonnilpotent subset and the cardinality of this subset (if it exists) is denoted by $\omega(\mathcal{N}_G)$. In this paper, among other results, we obtain $\omega(\mathcal{N}_{Suz(q)})$ and $\omega(\mathcal{N}_{PGL(2,q)})$, where $Suz(q)$ is the Suzuki simple group over the field with q elements and $PGL(2, q)$ is the projective general linear group of degree 2 over the finite field with q elements, respectively.

1. Introduction and results

One can associate a graph to a group in many different ways (see for example [1, 2, 3, 5]). Let G be a group. Following [3], we shall use the notation $\mathcal{N}_G$ to denote the nonnilpotent graph, as follows: take G as the vertex set and two vertices are adjacent if they generate a nonnilpotent subgroup. Note that if G is weakly nilpotent (i.e., every two generated subgroup of G is nilpotent), $\mathcal{N}_G$ has no edge. It follows that the nonnilpotent graphs of weakly nilpotent groups with the same cardinality are isomorphic. So we must be interested in non weakly nilpotent groups. A set C of vertices of a graph Λ whose induced subgraph is a complete subgraph is called a clique and the maximal size (if it exists) of a clique in a graph is called the clique number of the graph and it is denoted by $\omega(\Lambda)$. A subset A of G is said to be nonnilpotent if for any two distinct elements a and b in A, $\langle a, b\rangle \notin \mathcal{N}$ (we call a pair of elements $\{a, b\}$ nonnilpotent). If, for any other nonnilpotent subset B in G, $|A| \geq |B|$, then A is said to be a maximal nonnilpotent subset. Thus $\omega(\mathcal{N}_G)$ is simply the cardinality of maximal nonnilpotent subset in the group G. One of our motivations for associating with a group such kind of graph is a problem posed by Erdös: For a group G, consider a graph $\mathcal{A}_G$ whose vertex set is G and join two

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distinct elements if they do not commute. Then he asked: Is there a finite bound for the cardinalities of cliques in $\mathcal{A}_G$, if $\mathcal{A}_G$ has no infinite clique? Neumann [9] answered positively Erdös problem by proving that such groups are exactly the center-by-finite groups and the index of the center can be considered as the requested bound in the problem.

This results suggests that the clique number of a graph of a group not only has some influence on the structure of a group but also finding it, it is important and interesting. Recently Abdollahi and Zarrin in [3], have studied the influence of $\omega(\mathcal{N}_G)$ on the structure of a group. Then Azad in [4], obtained a lower bound for $\omega(\mathcal{N}_{GL(n,q)})$ and he determined $\omega(\mathcal{N}_{PSL(2,q)})$. Also the author in [13], shortly prove, determined $\omega(\mathcal{N}_{PSL(2,q)})$ (see [13, Proposition 4.2]). Clearly, $\mathcal{N}_G$ is a subgraph of $\mathcal{A}_G$ and so

$$\omega(\mathcal{N}_G) \leq \omega(\mathcal{A}_G). \qquad (\star)$$

A group G is an AC-group if $C_G(g)$ is abelian for all $g \in G \setminus Z(G)$, where $C_G(g)$ is the centralizer of the element g in G.

Remark 1.1. *Let G be a centerless AC-group. Then it follows from Lemma 4.1 of [13], that $\omega(\mathcal{N}_G) = \omega(\mathcal{A}_G)$ (note that, as G is an AC-group, for every $a, b \in G$ we have either $C_G(a) \cap C_G(b) = Z(G)$ or $ab = ba$).*

Definition 1.2. *We say that a group G has n nilpotentizers (or that G is an $\mathcal{N}(n)$-group) if $|nilp(G)| = n$, where $nilp(G) = \{nil_G(g) \mid g \in G\}$ and $nil_G(g) = \{x \in G \mid \langle g, x\rangle$ is nilpotent$\}$.*

In this paper we give some properties of $\mathcal{N}(n)$-groups (in fact, by using this class of groups we give an upper bound for $\omega(\mathcal{N}_G)$ in terms of n). Also we determine $\omega(\mathcal{N}_{Suz(q)})$ and $\omega(\mathcal{N}_{PGL(2,q)})$. Finally, by using these results, we characterize all nonabelian finite semisimple groups G with $\omega(\mathcal{N}_G) \leq 72$, in fact we will generalize [4, Theorem 4.5].

Our main results are:

Theorem 1.3. *Let $G = Suz(q)$ ($q = 2^{2m+1}$ and $m > 0$). Then*

$$\omega(\mathcal{N}_G) = (q^2+1) + \frac{q^2(q^2+1)}{2} + \frac{q^2(q^2+1)(q-1)}{4(q+2r+1)} + \frac{q^2(q^2+1)(q-1)}{4(q-2r+1)},$$

where $r = \sqrt{\frac{q}{2}}$.

Theorem 1.4. *We have*

(1) *If $q = 4$ or $q > 5$, then $\omega(\mathcal{N}_{PGL(2,q)}) = \omega(\mathcal{N}_{GL(2,q)}) = q^2 + q + 1$.*
(2) $1015 \leq \omega(\mathcal{N}_{PSL(3,3)}) = \omega(\mathcal{N}_{SL(3,3)})$.

In particular, we conjecture that $\omega(\mathcal{N}_{PSL(3,3)}) = \omega(\mathcal{N}_{PGL(3,3)}) = 1015$.

Theorem 1.5. *Let G be a nonabelian finite semisimple group. Then $\omega(\mathcal{N}_G) \leq 72$ if and only if $G \cong A_5, PSL(2,7), S_5$ or $PGL(2,7)$.*

From Theorem 1.5 we obtain a nice characterization for A_5 and $PSL(2,7)$ (see Corollary 4.3, below).

Throughout this paper, unless or otherwise stated, all groups are finite and we will use the usual notation, for example $A_n, S_n, SL(n,q),\ GL(n,q),\ PSL(n,q)$,
$PGL(n,q)$ and $Suz(q)$ ($q = 2^{2m+1}$ and $m > 0$), respectively, denote the alternating groups on n letters, symmetric group on n letters, special linear group of degree n over the finite field with q elements, general linear group of degree n over the finite field with q elements, projective special linear group of degree n over the finite field with q elements, projective general linear group of degree n over the finite field with q elements and the Suzuki group over the field with q elements.

2. $\mathcal{N}(n)$-groups

Let G be a group. Recall that the centralizer of an element $a \in G$ can be defined by $C_G(a) = \{b \in G \mid \langle a, b\rangle$ is abelian$\}$ and it is a subgroup of G. If, in the above definition, we replace the word abelian with the word nilpotent we get a subset of G, called the *nilpotentizer* of an element $a \in G$ (see also [3]). In fact, this subset is an extension of the centralizer.

Also for a nonempty subset S of G, we define the nilpotentizer of S in G, to be

$$nil_G(S) = \bigcap_{x\in S} nil_G(x).$$

When $S = G$; we call $nil_G(G)$ the nilpotentizer of G, and it will be denoted by $nil(G)$. Thus

$$nil(G) = \{x \in G \mid \langle x, y\rangle \text{ is nilpotent for all } y \in G\}.$$

It is not known whether the subset $nil(G)$ is a subgroup of G, but in many important cases it is a subgroup. In particular, $nil(G)$ is equal to $Z^*(G)$ whenever G satisfies the maximal condition on its subgroups or G is a finitely generated solvable group (see [3, Proposition 2.1]), where $Z^*(G)$ is the hypercenter of G. Also note that in general for an element x of a group G, $nil_G(x)$ is not a subgroup of G. For example, in the group $G = S_4$, clearly the element $u = (13)(24)$ belongs to $O_2(S_4)$, so $nil_G(u)$ contains the union of all Sylow 2-subgroups. The only other elements of S_4 are 3-cycles. Since none of these centralize u, $nil_G(u)$ is exactly the union of all three Sylow 2-subgroups, that is $nil_G(u)$ is not subgroup (see also [3, Lemma 3.3]).

It is clear that a group is an $\mathcal{N}(1)$-group if and only if it is weakly nilpotent. One of our motivations for the above definition is the following Proposition. (In fact, for a nonweakly nilpotent group G, we give some interesting relations between $\omega(\mathcal{N}_G)$ and $|nilp(G)|$.)

Proposition 2.1. *Let G be a non weakly nilpotent group. Then we have*
(1) $\omega(\mathcal{N}_G) + 1 \leq |nilp(G)|$.
(2) If every nilpotentizer of G is a nilpotent subgroup (or G is an $\mathcal{N}n$-group, see [4]), then $\omega(\mathcal{N}_G)+1 = |nilp(G)|$.

Proof. (1) Assume that $X = \{a_1, \dots, a_t\}$ be an arbitrary clique for the graph $\mathcal{N}_G$. It follows that $nil_G(a_i) \neq nil_G(a_j)$ for every $1 \leq i < j \leq t$. From which it follows that $|X| + 1 \leq |nilp(G)|$, as $nil_G(e) = G$ where e is the trivial element of G, and so $\omega(\mathcal{N}_G) + 1 \leq |nilp(G)|$.

(2) Let $A = \{a_1, \dots, a_n\}$ be a maximal nonnilpotent subset of G and $x \in G$ such that $nil_G(x) \neq G$. Then there exists a $1 \leq i \leq n$ such that $x \in nil_G(a_i)$ by the maximality of A. Since $nil_G(a_i)$ is a nilpotent subgroup, $nil_G(a_i) \leq nil_G(x)$. On the other hand $a_i \in nil_G(x)$ gives $nil_G(x) \leq nil_G(a_i)$. Therefore $nil_G(x) = nil_G(a_i)$ and so $|nilp(G)| = |\{G, nil_G(a_i) \mid 1 \leq i \leq n\}|$. This completes the proof. □

Here, we give some properties about $\mathcal{N}(n)$-groups.

Proposition 2.2. *Let G be an $\mathcal{N}(n)$-group. Then*
(1) G is nilpotent if and only if $n < 5$. The group S_3 is an $\mathcal{N}(5)$-group;
(2) If $n < 22$, then G is solvable. The group A_5 is an $\mathcal{N}(22)$-group.

Proof. The result follows from part (1) of Proposition 2.1 and the main result of [7]. □

Corollary 2.3. *There is no finite $\mathcal{N}(n)$-group with $n \in \{2, 3, 4\}$.*

Here (for infinite groups) we prove that every arbitrary $\mathcal{N}(n)$-group with $n = 4$ is an Engel group. Recall that a group G is an Engel group if for each ordered pair (x, y) of elements in G there is a positive integer $n = n(x, y)$ such that $[x,_n y] = 1$, where $[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y$ and $[x,_{m+1} y] = [[x,_m y], y]$ for all positive integers m.

Theorem 2.4. *Let G be an infinite $\mathcal{N}(n)$-group with $n = 4$. Then it is an Engel group.*

Proof. Suppose that G is an $\mathcal{N}(n)$-group with $n = 4$. We claim that for every two arbitrary $x, y \in G$, the subgroup $\langle x, x^y \rangle$ is nilpotent.
To see that, suppose, a contrary, that there exist $x, y \in G$ such that $\langle x, x^y \rangle$ is not nilpotent. Now we consider the set

$$N = \{nil_G(x), nil_G(y), nil_G(xy), nil_G(x^y)\}.$$

As $|nilp(G)| = 4$, it follows that there exist at least two elements $nil_G(a)$, $nil_G(b) \in N$ such that $nil_G(a) = nil_G(b)$. It implies that $\langle x, y \rangle$ or $\langle x, x^y \rangle$ is a nilpotent group. Now since $\langle x, x^y \rangle \leq \langle x, y \rangle$, it follows that $\langle x, x^y \rangle$ is nilpotent, a contrary. Hence for every two arbitrary $x, y \in G$, $\langle x, x^y \rangle$ is nilpotent and so $\langle x, x^{-1y} \rangle$ is nilpotent. It follows that $[y^{-1}x^{-1}y,_t x] = 1$ for some $t \in \mathbb{N}$. Now by using the relation $[ab, c] = [a, c]^b[b, c]$ and $[ab,_t b] = [a,_t b]^b$ we get

$$[y,_{t+1} x] = [[y, x],_t x] = [(x^{-1})^y x,_t x] = [(x^{-1})^y,_t x]^x = 1.$$

That is, G is an Engel group, as required. □

Lemma 2.5. *Suppose that G is a group and $H \leq G$. Then $|nilp(H)| \leq |nilp(G)|$.*

Proof. It follows easily from

$$nil_H(h) = nil_G(h) \cap H$$

for every element $h \in H$. □

Lemma 2.6. *Suppose that G_i is a finite group $(i = 1, \ldots, t)$.* Then

$$|nilp(G_1 \times G_2 \times \cdots \times G_m)| = \prod_{i=1}^{m} |nilp(G_i)|.$$

Proof. Put $H = \prod_{i=1}^{m} G_i$. Applying [13, Proposition 3.1], we can show that $nil_H(x_1, \ldots, x_t) = nil_{G_1}(x_1) \times \cdots \times nil_{G_t}(x_t)$, for all $(x_1, \ldots, x_t) \in H$. It follows that for every $1 \leq i \leq t$ we have

$$nil_H(x_1, \ldots, x_t) = nil_H(y_1, y_2, \ldots, y_t)$$

if and only if $nil_{G_i}(x_i) = nil_{G_i}(y_i)$ and the result follows. □

Lemma 2.7. *Let G be a group. Then $|nilp(G)| \geq |nilp(\frac{G}{Z^*(G)})|$.*

Proof. It is clear that for all $x \in G$, $nil_{\frac{G}{Z^*(G)}}(xZ^*(G)) = \frac{nil_G(x)}{Z^*(G)}$, where $\frac{nil_G(x)}{Z^*(G)} = \{yZ^*(G) \mid y \in nil_G(x)\}$. It follows that if $nil_{\frac{G}{Z^*(G)}}(xZ^*(G)) \neq nil_{\frac{G}{Z^*(G)}}(yZ^*(G))$, then $nil_G(x) \neq nil_G(y)$. This completes this proof. □

3. Proofs of Theorem 1.3 and Theorem 1.4

To prove Theorem 1.3 we need the following lemmas.

Lemma 3.1. [3, Lemma 3.7] *Let G be a group and H a nilpotent subgroup of G in which $C_G(x) \leq H$ for every $x \in H\backslash\{1\}$. Then $nil_G(x) = H$ for every $x \in H\backslash\{1\}$.*

An element g of a group G is called a right Engel element if for each x in G there is a positive integer $n = n(g, x)$ such that $[g,_n x] = 1$ and we denote by $R(G)$, the set of all right Engel elements of G.

Proposition 3.2. [3, Proposition 2.1] *Let G be any group. Then*

(1) *$Z^*(G) \subseteq nil(G) \subseteq R(G)$, where $R(G)$ is the set of right Engel elements of G.*
(2) *If G satisfies the maximal condition on its subgroups or G is finitely generated solvable group then $Z^*(G) = nil(G) = R(G)$.*

Lemma 3.3. *Let G be a group, $H \leq G$ and let $M_1, \ldots, M_n$ be non trivial proper subgroups of H such that $H = \bigcup_{i=1}^{n} M_i$ and $M_i \bigcap M_j = nil(H)$ for $i \neq j$. If $nil_H(g) \subseteq M_i$ for all $g \in M_i \setminus nil(H)$, then*

$$\omega(\mathcal{N}_H) = \sum_{i=1}^{n} \omega(\mathcal{N}_{M_i}).$$

Proof. If N is any clique of $\mathcal{N}_H$ then $N = \bigcup_{i=1}^{n} N_i$ where $N_i \subseteq M_i \setminus nil(H)$ for each $i \in \{1, \ldots, n\}$. It follows that $|N| \leq \sum_{i=1}^{n} \omega(\mathcal{N}_{M_i})$. Now let W_i be a maximal clique of $\mathcal{N}_{M_i}$ for each $i \in \{1, \ldots, n\}$. We claim that $W = \bigcup_{i=1}^{n} W_i$ is a maximal clique for $\mathcal{N}_H$. Suppose for a contradiction that there exist two distinct elements a and b in $\bigcup_{i=1}^{n} W_i$ such that $\langle a, b\rangle$ is a nilpotent group. Thus there exist $i \neq j$ such that $a \in M_i$ and $b \in M_j$. Therefore $a \in nil_H(b) \subseteq M_j$. Thus $a \in M_i \bigcap M_j = nil(H)$. It follows that $H = nil_H(a) \subseteq M_i$ and so $H = M_i$, a contradiction. Now as $|W| = \sum_{i=1}^{n} \omega(\mathcal{N}_{M_i})$, the result follows. $\square$

A set $\mathcal{P} = \{H_1, \ldots, H_n\}$ of subgroups H_i $(i = 1, \ldots, n)$ is said to be a partition of G if every non-identity element $x \in G$ belongs to one and only one subgroup $H_i \in \mathcal{P}$.

Proof of Theorem 1.3. The Suzuki group G contains subgroups F, A, B and C such that $|F| = q^2, |A| = q-1, |B| = q-2r+1$ and $|C| = q+2r+1$ (see [8, Chapter XI, Theorems 3.10 and 3.11]). Also by [8, pp. 192-193, Theorems 3.10 and 3.11], the conjugates of A, B, C and F in G form a partition, say $\mathcal{P}$, for G, and A, B, C are cyclic and F is a Sylow 2-subgroup of G and also for every $M \in \mathcal{P}$ we have $C_G(b) \leq M$ for all $b \in M$.

Assume that $a \in G \setminus \{1\}$. Since $\mathcal{P}$ is a partition of G, $a \in M$ for some $M \in \mathcal{P}$. It follows from Lemma 3.1 that $nil_G(a) = M$ for all $a \in M$ $(\star\star)$. From [8, Chapter XI, Theorems 3.10 and 3.11] implies that the number of conjugates of C, B, A and F in G are respectively, $k = \frac{q^2(q-1)(q^2+1)}{4(q+2r+1)}$, $n = \frac{q^2(q-1)(q^2+1)}{4(q-2r+1)}$, $s = \frac{q^2(q^2+1)}{2}$ and $t = q^2+1$. Since G is a finite simple group, Proposition 3.2 follows that $nil(G) = Z^*(G) = 1$. Now as $\mathcal{P}$ is a partition for G and by $(\star\star)$, we implies, by Lemma 3.3, that $\omega(\mathcal{N}_G)$ is equal to size of the set $\mathcal{P}$ (note that, for a non trivial nilpotent group H, we define $\omega(\mathcal{N}_H) = 1$). Thus $\omega(\mathcal{N}_G) = k + n + s + t$. This completes the proof.

Now in view of the proof of Theorem 1.3, one can see that every nilpotentizer of $Suz(q)$ is a nilpotent subgroup (and hence $Suz(q)$ is an $\mathcal{N}n$-group, see [4]). Therefore, by theorem 1.3 and by Part (2) of Proposition 2.1, we give the following interesting result.

Corollary 3.4. *Let $G = Suz(q)$ ($q = 2^{2m+1}$ and $m > 0$). Then*

$$|nilp(G)| = (q^2+2) + \frac{q^2(q^2+1)}{2} + \frac{q^2(q^2+1)(q-1)}{4(q+2r+1)} + \frac{q^2(q^2+1)(q-1)}{4(q-2r+1)},$$

where $r = \sqrt{\frac{q}{2}}$.

To prove Theorem 1.4 we need the following Lemma.

Lemma 3.5. *Let G be a finite group. Then:*
(1) For any subgroup H of G, $\omega(\mathcal{N}_H) \leq \omega(\mathcal{N}_G)$;
(2) For any nonabelian factor group $\frac{G}{M}$ of G, $\omega(\mathcal{N}_{\frac{G}{M}}) \leq \omega(\mathcal{N}_G)$;
(3) $\omega(\mathcal{N}_{\frac{G}{K}}) = \omega(\mathcal{N}_G)$, where K is a normal subgroup of G with $K \leq Z^*(G)$.

Proof. (1) Clearly.
(2) This is straightforward.
(3) For proof it is enough to note that if $\frac{\langle a,b\rangle}{Z^*(G)\cap\langle a,b\rangle}$ is nilpotent, then $\langle a,b\rangle$ is nilpotent, for all $a, b \in G$. □

Corollary 3.6.

$$\omega(\mathcal{N}_{PGL(n,q)}) = \omega(\mathcal{N}_{GL(n,q)}),$$

and

$$\omega(\mathcal{N}_{PSL(n,q)}) = \omega(\mathcal{N}_{SL(n,q)}).$$

Proof of Theorem 1.4. (1) Since $\omega(\mathcal{N}_{PGL(n,q)}) \leq \omega(\mathcal{A}_{PGL(n,q)})$ and $PSL(n,q) \cong \frac{ZSL(n,q)}{Z} \leq PGL(n,q)$ where Z is the center of $GL(n,q)$, it follows, by parts (1) and (3) of Lemma 3.5, that

$$\omega(\mathcal{N}_{PSL(2,q)}) \leq \omega(\mathcal{N}_{PGL(2,q)}) = \omega(\mathcal{N}_{GL(2,q)}) \leq \omega(\mathcal{A}_{PGL(2,q)}) \leq \omega(\mathcal{A}_{GL(2,q)}).$$

Now the result follows from [2, Proposition 4.3] and [13, Proposition 4.2].

(2) We have used the following function written with GAP [10] program to prove this part of the theorem. The input of the function is a group G and an element $t \in G$ and the output is $nil_G(t)$.

```
f:= function(G,t) local r;
r:=Set(Filtered(G,i→ IsNilpotent(Group(i,t))=true));
return r; end;
```

Let $G = PSL(3,3)$. Therefore it is easy to see, by using the above function and also GAP, that the set of order elements of $nilp(G)$ is

$$\{6, 13, 16, 27, 32, 162, 192, 5616\}$$

and if $A = \{nil_G(g) \mid g \in G, |nil_G(g)| = 6\}$, $B = \{nil_G(g) \mid g \in G, |nil_G(g)| = 16\}$, $C = \{nil_G(g) \mid g \in G, |nil_G(g)| = 13\}$ and $D = \{nil_G(g) \mid g \in G, |nil_G(g)| = 27\}$, then $|A| = 468$, $|B| = 351$, $|C| = 144$ and $|D| = 52$. Thus there exist elements $a_i, b_j, c_k, d_l \in G$ such that $|nil_G(a_i)| = 6$ for $1 \leq i \leq 468$, $|nil_G(b_j)| = 16$ for $1 \leq j \leq 351$, $|nil_G(c_k)| = 13$ for $1 \leq k \leq 144$ and $|nil_G(d_l)| = 27$ for $1 \leq l \leq 52$. Set $X = \{a_1, \ldots, a_{468}, b_1, \ldots, b_{351}, c_1, \ldots, c_{144}, d_1, \ldots, d_{52}\}$. Also we can show that, by GAP [10], that each element in $A \cup B \cup C \cup D$ is a nilpotent subgroup of G.

Now we claim that X is a nonnilpotent subset of G. Let $a, b \in X$ such that $\langle a, b\rangle$ is nilpotent. Therefore $a \in nil_G(b)$. Since $nil_G(b)$ is a nilpotent subgroup, it follows that $nil_G(b) \leq nil_G(a)$. Similarly, $nil_G(a) \leq nil_G(b)$. Hence $nil_G(a) = nil_G(b)$, which is a contradiction. Thus X is a nonnilpotent subset of G and so $1015 = |X| \leq \omega(\mathcal{N}_G)$. This completes the proof.

Note that by [4, Theorem 4.1], we get $52 \leq \omega(\mathcal{N}_{GL(3,3)})$. But by Theorem 1.4 and as $\omega(\mathcal{N}_{PSL(n,q)}) \leq \omega(\mathcal{N}_{PGL(n,q)})$, we can obtain that $1015 \leq \omega(\mathcal{N}_{GL(3,3)}) = \omega(\mathcal{N}_{PGL(3,3)})$. Finally, in view of these results, we state the following conjecture:

Conjecture 3.7. $\omega(\mathcal{N}_{PSL(3,3)}) = \omega(\mathcal{N}_{PGL(3,3)}) = 1015.$

4. Proof of Theorem 1.5

To prove Theorem 1.5 we need the following lemma.

Lemma 4.1. *If G is a nonabelian finite simple group with $\omega(\mathcal{N}_G) \leq 72$. Then G is isomorphic to one of the following groups:*
(1) $G \cong PSL(2,5)$;
(2) $G \cong PSL(2,7)$.

Proof. Suppose that the result is false, and let M be a minimal counterexample. Thus, every proper non-abelian simple section of M is isomorphic to $PSL(2,5)$ or $PSL(2,7)$. By [6, Proposition 2], M is isomorphic to one of the following:

(1) $PSL(2,2^m)$ ($m = 4$ or is a prime);
(2) $PSL(2,3^p), PSL(2,5^p), PSL(2,7^p)$ (p is a prime);
(3) $PSL(2,p), 5 < p$ is a prime;
(4) $PSL(3,3),\ PSL(3,5),\ PSL(3,7)$;
(5) $PSU(3,3),\ PSU(3,5),\ PSU(3,7)$ (the projective special unitary groups of degree three over the finite fields of orders 3, 4 and 7, respectively);
(6) $Sz(2^p), p$ is an odd prime.

In the sequel, the proof follows easily from [4, Corollary 3.4] and by an argument similar to the proof of Theorem 4.5 of [4] (note that, by Theorem 1.3, for the Suzuki group $Suz(2^{2m+1})$ we have $\omega(\mathcal{N}_{Suz(2^{2m+1})}) > 73$ and also it is easy to see (e.g., by GAP [10]), that the number of Sylow 43-subgroups of the projective special unitary group of degree three over the finite fields of order 7, $PSU(3,7)$ is more than 73). □

Recall that a group G is semisimple if G has no non-trivial normal abelian subgroups. If G is a finite group then we call the product of all minimal normal nonabelian subgroups of G the centerless CR-radical of G; it is a direct product of nonabelian simple groups [11, see page 88].

Lemma 4.2. *Suppose that G_i is a finite group* ($i = 1, \ldots, t$). Then

$$\omega(\mathcal{N}_{G_1 \times G_2 \times \cdots \times G_m}) = \prod_{i=1}^{m} \omega(\mathcal{N}_{G_i}).$$

Proof. This follows from [13, Proposition 3.1]. □

Proof of Theorem 1.5. Assume that R is the centerless CR-Radical of G. Then R is a direct product of a finite number, t, of finite nonabelian simple groups, say $R = S_1 \times \cdots \times S_t$. Since $\omega(\mathcal{N}_G) \leq 72$ and $S_i \leq G$, it follows that $\omega(\mathcal{N}_{S_i}) \leq 72$ ($\omega(\mathcal{N}_R) \leq 72$). Therefore Lemma 4.1, follows that $S_i \cong PSL(2,5)$ or $PSL(2,7)$ for $i \in \{1, \ldots, t\}$.

On the other hand by Lemma 4.2, and also since $\omega(\mathcal{N}_R) \leq 72$, we have $t = 1$. Therefore $R \cong PSL(2,5)$ or $PSL(2,7)$. But we know that $C_G(R) = 1$ because of G is semisimple and G is embedded into $Aut(R)$. Hence $G \cong PSL(2,5)$, $PSL(2,7), S_5$ or $PGL(2,7)$. But for these groups we have $\omega(\mathcal{N}_G) \leq \omega(\mathcal{A}_G) \leq 57 \leq 72$ and the result follows.

By Theorem 1.5, we have a nice characterization for the following simple groups.

Corollary 4.3. *Let G be a finite simple group. Then we have*

$G \cong A_5$ *if and only if* $\omega(\mathcal{N}_G) = 21$;

$G \cong PSL(2,7)$ *if and only if* $\omega(\mathcal{N}_G) = 57$.

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References

[1] A. Abdollahi, Engel graph associated with a group, *J. Algebra* , **318** (2007) 680–691.

[2] A. Abdollahi, A. Akbari and H. R. Maimani, Non-commuting graph of a group, *J. Algebra*, **298** (2006) 468–492.

[3] A. Abdollahi and M. Zarrin, Non-nilpotent graph of a group, *Comm. Algebra*, **38** (2010) 4390–4403.

[4] A. Azad, On nonnilpotent subsets in general liner groups, *Bull. Austral. Math. Soc.*, **83** (2011) 369–375.

[5] A. Ballester-Bolinches, J. Cossey and R. Esteban-Romero, On a graph related to permutability in finite groups, *Ann. Mat. Pura Appl. (4)*, **189** (2010) 567–570.

[6] R. D. Blyth and D. J. S. Robinson, Semisimple groups with the rewriting property Q_5, *Comm. Algebra*, **23** (1995) 2171–2180.

[7] G. Endimioni, Groupes finis satisfaisant la condition $(\mathcal{N}, n)$, *C. R. Acad. Sci. Paris Sér. I Math.*, **319** (1994) 1245–1247.

[8] B. Huppert and N. Blackburn, Finite groups, III, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **243**, Springer-Verlag, Berlin-New York, 1982.

[9] B. H. Neumann, A problem of Paul Erdös on groups, *J. Austral. Math. Soc. Ser. A*, **21** (1976) 467–472.

[10] The GAP Group, GAP-Groups, Algorithms, and Programming, Version 4.4; 2005, http://www.gap-system.org.

[11] D. J. S. Robinson, A course in the theory of groups, Second edition. Graduate Texts in Mathematics, **80**, Springer-Verlag, New York, 1996.

[12] J. G. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable (Part I), *Bull. Amer. Math. Soc.*, **74** (1968) 383–437.

[13] M. Zarrin, Ensuring a group is weakly nilpotent, *Comm. Algebra*, **40** (2012) 4739–4752.

Mohammad Zarrin

Department of Mathematics, University of Kurdistan, P.O. Box: 416 Sanandaj, Iran and,

Institute for Studies in Theoretical Physics and Mathematics (IPM), Tehran, Iran.

Email: M.zarrin@uok.ac.ir, Zarrin@ipm.ir