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INDUCED OPERATORS ON SYMMETRY CLASSES OF POLYNOMIALS

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Abstract. In this paper, we give a necessary and sufficient condition for the equality of two symmetrized decomposable polynomials. Then, we study some algebraic and geometric properties of the induced operators over symmetry classes of polynomials in the case of linear characters.

1. Introduction

Symmetry classes of polynomials are introduced in [12], by Shahryari. In [11], Rodtes studied symmetry classes of polynomials associated with the irreducible characters of the semidihedral group. In [1], Babaei and Zamani studied symmetry classes of polynomials with respect to irreducible characters of the direct product of permutation groups. In [2, 13, 14], they computed the dimensions of symmetry classes of polynomials with respect to irreducible characters of dihedral, dicyclic and cyclic groups, respectively. Also, they discussed the existence of o-basis for these classes. In [3], Babaei, Zamani and Shahryari investigated an embedding of the symmetry classes of polynomials into the symmetry classes of tensors and used it to obtain some results concerning non-vanishing of the symmetry classes of polynomials. They also computed the dimensions of the symmetry classes of polynomials with respect to the irreducible characters of the symmetric group S_m and the alternating group A_m. In this paper we first give a necessary and sufficient condition for the equality of two symmetrized decomposable polynomials. Then, we study some properties of the induced operators on symmetry classes of polynomials. We first give a review of symmetry classes of polynomials.

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Let $H_d[x_1, \ldots, x_m]$ be the complex space of homogeneous polynomials of degree d with independent commuting variables $x_1, \ldots, x_m$. Let $\Gamma^+_{m,d}$ be the set of all m- tuples of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_m)$, such that $\sum_{i=1}^m \alpha_i = d$. For any $\alpha \in \Gamma^+_{m,d}$, let

$$X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_m^{\alpha_m}.$$

Then the set $\{X^\alpha \mid \alpha \in \Gamma^+_{m,d}\}$ is a basis of $H_d[x_1, \ldots, x_m]$. An inner product on $H_d[x_1, \ldots, x_m]$ is defined by

$$< X^\alpha, X^\beta > = \alpha! \; \delta_{\alpha,\beta}, \tag{1.1}$$

where $\alpha! = \prod_{i=1}^m \alpha_i!$.

Suppose G is a subgroup of the symmetric group S_m. Then G acts on $H_d[x_1, \ldots, x_m]$ by

$$q^\sigma(x_1, \ldots, x_m) = q(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(m)}),$$

and this action is extended linearly to the group algebra $\mathbb{C}G$. Let χ be an irreducible complex character of G. Consider the idempotent (the symmetrizer corresponding to χ)

$$S_\chi = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma,$$

in the group algebra $\mathbb{C}G$. The image of $H_d[x_1, \ldots, x_m]$ under the map S_χ is called *the symmetry class of polynomials of degree d with respect to G and χ*, and it is denoted by $H_d(G, \chi)$. For any $q \in H_d[x_1, \ldots, x_m]$,

$$q^* = S_\chi(q)$$

is called a *symmetrized polynomial* with respect to G and χ. If χ is a linear character of G, then $H_d(G, \chi)$ is the set of all $q \in H_d[x_1, \ldots, x_m]$, such that for any $\sigma \in G$, we have $q^\sigma = \chi(\sigma^{-1})q$. If $\alpha \in \Gamma^+_{m,d}$, then we denote the symmetrized monomial $(X^\alpha)^*$ by $X^{\alpha,*}$. Clearly, we have

$$H_d(G, \chi) = \langle X^{\alpha,*} | \; \alpha \in \Gamma^+_{m,d} \rangle.$$

The group G also acts on $\Gamma^+_{m,d}$ by

$$\alpha\sigma = (\alpha\sigma(1), \ldots, \alpha\sigma(m)).$$

Let Δ be a set of representatives of orbits of $\Gamma^+_{m,d}$ under the action of G. For any $\alpha \in \Gamma^+_{m,d}$, we have

$$\|X^{\alpha,*}\|^2 = \chi(1)\alpha! \; \frac{[\chi, 1]_{G_\alpha}}{|G : G_\alpha|}, \tag{1.2}$$

where G_α is the stabilizer subgroup of α under the action of G and $[\ ,\ ]_G$ denotes the inner product of characters (see [8]). Hence $X^{\alpha,*} \neq 0$ if and only if $[\chi, 1]_{G_\alpha} \neq 0$. Let Ω be the set of all $\alpha \in \Gamma^+_{m,d}$ with $[\chi, 1]_{G_\alpha} \neq 0$ and suppose $\bar{\Delta} = \Delta \cap \Omega$. It is proved that if χ is linear, then the set $\{X^{\alpha,*} | \; \alpha \in \bar{\Delta}\}$ is an orthogonal basis of $H_d(G, \; \chi)$.

Let $\Gamma_{d,m}$ be the set of all $\alpha = (\alpha_1, \ldots, \alpha_d)$ such that for any $1 \leq i \leq d$, $1 \leq \alpha_i \leq m$. The group G acts on $\Gamma_{d,m}$ by

$$\sigma\alpha = (\sigma(\alpha_1), \ldots, \sigma(\alpha_d)),$$

where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \Gamma_{d,m}$ and $\sigma \in G$. Let $G_{d,m}$ be the subset of $\Gamma_{d,m}$ consisting of all nondecreasing sequences. We define the weight of $\alpha \in G_{d,m}$ by

$$\omega(\alpha) = \prod_{i=1}^{d} x_{\alpha(i)}.$$

Then, the set $\mathfrak{B} = \{\omega(\alpha) | \alpha \in G_{d,m}\}$ is a rearrangment of basis $\{X^\alpha |\ \alpha \in \Gamma^+_{m,d}\}$ of $H_d[x_1, \ldots, x_m]$. For any $\alpha \in G_{d,m}$, denote by $m_t(\alpha) = |\alpha^{-1}(t)|$, the multiplicity of the integer t in the sequence α. If $\alpha \in G_{d,m}$, then

$$m(\alpha) = (m_1(\alpha), \ldots, m_m(\alpha)) \in \Gamma^+_{m,d}$$

and it is called the *multiplicity vector of* α. For any $\alpha \in \Gamma^+_{m,d}$, we define the sequence $\widetilde{\alpha} \in G_{d,m}$ by

$$\widetilde{\alpha} = (\overbrace{1, \ldots, 1}^{\alpha_1 times}, \overbrace{2, \ldots, 2}^{\alpha_2 times}, \ldots, \overbrace{m, \ldots, m}^{\alpha_m times}).$$

Then, the map $\alpha \mapsto \widetilde{\alpha}$ is a one to one correspondence between $G_{d,m}$ and $\Gamma^+_{m,d}$. By lexicographic ordering on $\Gamma^+_{m,d}$ and $G_{d,m}$, it is easy to see that for any $\alpha,\ \beta \in \Gamma^+_{m,d}$, we have $\alpha \le \beta$ iff $\widetilde{\beta} \le \widetilde{\alpha}$.
If $\alpha, \beta \in G_{d,m}$, then $\alpha = \beta$ iff $m(\alpha) = m(\beta)$. Hence

$$(\omega(\alpha), \omega(\beta)) = (X^{m(\alpha)}, X^{m(\beta)}) = \delta_{\alpha,\beta}\ \nu(\alpha),$$

where $\nu(\alpha) = \prod_{i=1}^{m} m_i(\alpha)!$.
Since, the set $\{X^{\alpha,*} |\ \alpha \in \bar{\Delta}\}$ is a basis of $H_d(G, \chi)$ (if χ is linear) and the map $\alpha \mapsto \widetilde{\alpha}$ is a one to one function from $\Gamma^+_{m,d}$ onto $G_{d,m}$, so

$$\mathfrak{B}^* = \{\omega(\widetilde{\alpha})^* |\ \alpha \in \bar{\Delta}\}$$

is a rearrangment of basis $\{X^{\alpha,*} |\ \alpha \in \bar{\Delta}\}$.

Let $f_i = \sum_{j=1}^{m} a_{ij} x_j$, $1 \le i \le d$. Clearly $f_1 f_2 \ldots f_d \in H_d[x_1, \ldots, x_m]$. The polynomial

$$S_\chi(f_1 f_2 \ldots f_d) \in H_d(G,\ \chi)$$

is called *symmetrized decomposable polynomial* and is denoted by $f_1 * f_2 * \ldots * f_d$. The study of symmetrized decomposable polynomials is an important topic in Multilinear Algebra since the knowledge of their properties is necessary for the understanding of symmetry classes. In this paper, we give a necessary and sufficient condition for the equality of two symmetrized decomposable polynomials.

Let V be the complex vector space of homogeneous linear polynomials in the variables $x_1, \ldots, x_m$. For any linear operator $T \in L(V)$, there is a linear operator $P(T)$ (see [15]) acting on $H_d[x_1, \ldots, x_m]$ by

$$P(T) q(x_1, \ldots, x_m) = q(Tx_1, \ldots, Tx_m).$$

Here $L(V)$ is the space of linear operators on V. It is easy to see that $P(T)S_\chi = S_\chi P(T)$. So $H_d(G, \chi)$ is an invariant subspace of $P(T)$. Denote by $K_\chi(T)$ the restriction of $P(T)$ to $H_d(G, \chi)$.

Then $K_\chi(T)$ is called the *induced operator* associated with G and χ. The induced operator $K_\chi(T)$ acts on symmetrized decomposable polynomials by

$$K_\chi(T)(f_1 * f_2 * \ldots * f_d) = Tf_1 * Tf_2 * \ldots * Tf_d.$$

In particular, for any $\alpha \in \Gamma^+_{m,d}$, we have

$$K_\chi(T)\omega(\widetilde{\alpha})^* = Tx_{\widetilde{\alpha}(1)} * Tx_{\widetilde{\alpha}(2)} * \ldots * Tx_{\widetilde{\alpha}(d)}.$$

The properties of the induced operator $P(T)$ are studied in [15]. In this paper, we study some algebraic and geometric properties of the induced operator $K_\chi(T)$, when the symmetrizer is associated with a linear character of G.

2. Equality of symmetrized decomposable polynomials

The vanishing and equality problems for symmetrized decomposable tensors are studied extensively (see [6, 7, 10]). In this section, we study these problems for symmetrized decomposable polynomials.

Let $M_m(\mathbb{C})$ be the set of all $m\times m$ matrices over $\mathbb{C}$, and define the permanent of $A = (a_{ij}) \in M_m(\mathbb{C})$ as follows:

$$perA = \sum_{\sigma\in S_m} \prod_{i=1}^{m} a_{i\sigma(i)}.$$

We first prove that the symmetrizer S_χ is a Hermitian linear operator on $H_d[x_1, \ldots, x_m]$.

Lemma 2.1. *If χ is an irreducible character of a subgroup G of S_m, then the symmetrizer S_χ is a Hermitian linear operator on $H_d[x_1, \ldots, x_m]$.*

Proof. For any $\sigma \in G$, $\alpha, \beta \in G_{d,m}$, we have

$$\begin{aligned}(\omega(\alpha)^\sigma, \omega(\beta)) &= (\omega(\sigma^{-1}\alpha), \omega(\beta))\\ &= (\omega(\sigma^{-1}\alpha\tau), \omega(\beta)), \quad (\tau \in S_d, \sigma^{-1}\alpha\tau \in G_{d,m})\\ &= \delta_{\sigma^{-1}\alpha\tau,\beta}\ \nu(\beta).\end{aligned}$$

The last expression is equal to zero or $\nu(\beta)$, where in the second case we have $\alpha\tau = \sigma\beta$. Thus

$$\begin{aligned}(\omega(\alpha), \omega(\beta)^{\sigma^{-1}}) &= (\omega(\alpha), \omega(\sigma\beta))\\ &= (\omega(\alpha), \omega(\alpha\tau))\\ &= (\omega(\alpha), \omega(\alpha))\\ &= \nu(\alpha)\\ &= \nu(\beta).\end{aligned}$$

So $(\omega(\alpha)^{\sigma}, \omega(\beta)) = (\omega(\alpha), \omega(\beta)^{\sigma^{-1}})$. Therefore

$$\begin{aligned}(S_{\chi}\omega(\alpha), \omega(\beta)) &= \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)(\omega(\alpha)^{\sigma}, \omega(\beta)) \\ &= \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)(\omega(\alpha), \omega(\beta)^{\sigma^{-1}}) \\ &= (\omega(\alpha), \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma^{-1})\omega(\beta)^{\sigma^{-1}}) \\ &= (\omega(\alpha), S_{\chi}\omega(\beta)),\end{aligned}$$

so the result holds. □

Theorem 2.2. *Let χ be an irreducible character of a subgroup G of S_m. For any $1 \leq i \leq d$, suppose $f_i = \sum_{j=1}^{m} a_{ij}x_j$, $g_i = \sum_{j=1}^{m} b_{ij}x_j$. Let $f = f_1 \dots f_d$ and $g = g_1 \dots g_d$. Then*

(i) $(f, g) = per(AB^*)$,

(ii) $(f^*, g^*) = \dfrac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)per(AP(\sigma)B^*)$,

(iii) $f^* = 0$ *iff* $\sum_{\sigma\in G}\chi(\sigma)per(AP(\sigma)A^*) = 0$,

where $A = (a_{ij})$, $B = (b_{ij})$ are $d \times m$ matrices over $\mathbb{C}$ and $P(\sigma) = (\delta_{i\sigma(j)})_{m\times m}$ is the permutation matrix corresponding to σ.

Proof. (i) We have

$$\begin{aligned}f &= f_1 \dots f_d \\ &= \sum_{j_1=1}^{m} a_{1j_1}x_{j_1} \dots \sum_{j_d=1}^{m} a_{1j_d}x_{j_d} \\ &= \sum_{\alpha\in\Gamma_{d,m}} (\prod_{i=1}^{d} a_{i,\alpha(i)})\omega(\alpha) \\ &= \sum_{\alpha\in G_{d,m}} \frac{1}{\nu(\alpha)}\sum_{\tau\in S_d}(\prod_{i=1}^{d} a_{i,\alpha\tau(i)})\omega(\alpha) \\ &= \sum_{\alpha\in G_{d,m}} \big(\frac{1}{\nu(\alpha)} perA[1, 2, \dots, d|\alpha]\big)\omega(\alpha). \qquad (2.1)\end{aligned}$$

Similarly,

$$g = \sum_{\beta\in G_{d,m}} \big(\frac{1}{\nu(\beta)} perB[1, 2, \dots, d|\beta]\big)\omega(\beta).$$

Then

$$\begin{aligned}(f,g) &= \sum_{\alpha,\beta\in G_{d,m}} \frac{1}{\nu(\alpha)\nu(\beta)} perA[1,2,\ldots,d|\alpha]\ \overline{perB[1,2,\ldots,d|\beta]}(\omega(\alpha),\omega(\beta)) \\ &= \sum_{\alpha,\beta\in G_{d,m}} \frac{1}{\nu(\alpha)\nu(\beta)} perA[1,2,\ldots,d|\alpha]\ perB^*[\beta|1,2,\ldots,d]\nu(\beta)\delta_{\alpha,\beta} \\ &= \sum_{\alpha\in G_{d,m}} \frac{1}{\nu(\alpha)} perA[1,2,\ldots,d|\alpha]\ perB^*[\alpha|1,2,\ldots,d] \\ &= per(AB^*).\end{aligned}$$

Note that, the last expression is obtained from Cauchy-Binet Theorem for permanents (see [9, 10]).
(ii) By Lemma 2.1, S_χ is Hermitian. Thus

$$\begin{aligned}(f^*,g^*) &= (S_\chi f, S_\chi g) \\ &= (f, S_\chi^* S_\chi g) \\ &= (f, {S_\chi}^2 g) \\ &= (f, S_\chi g) \\ &= \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\overline{\chi(\sigma)}(f,g^\sigma).\end{aligned}$$

Let $P(\sigma) = (\delta_{i\sigma(j)})_{m\times m}$ be the permutation matrix corresponding to $\sigma \in G$. Then

$$g_i^\sigma = \sum_{j=1}^m b_{ij}x_{\sigma^{-1}(j)} = \sum_{j=1}^m b_{i\sigma(j)}x_j = \sum_{j=1}^m (BP(\sigma))_{ij}x_j.$$

Applying part (i), we get

$$\begin{aligned}(f^*,g^*) &= \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\overline{\chi(\sigma)}per(A(BP(\sigma))^*) \\ &= \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma^{-1})per(AP(\sigma^{-1})B^*) \\ &= \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)per(AP(\sigma)B^*)\end{aligned}$$

(iii) By part (ii), it is clear.

□

By Theorem 2.2, we have

$$\|f\|^2 = per(AA^*),\ \ \|g\|^2 = per(BB^*),$$

$$\|f^*\|^2 = \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)per(AP(\sigma)A^*),$$

$$\|g^*\|^2 = \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)per(BP(\sigma)B^*).$$

Therefore using Cauchy-Schwartz inequality, we obtain the following result. Let $A=(a_{ij})$ and $B=(b_{ij})$ be $d\times m$ complex matrices. Then

(i)

$$|\sum_{\sigma\in G}\chi(\sigma)per(AP(\sigma)B^*)|^2\leq\sum_{\sigma\in G}\chi(\sigma)per(AP(\sigma)A^*)$$
$$\sum_{\tau\in G}\chi(\tau)per(BP(\tau)B^*),$$

(ii)

$$|per(AB^*)|^2\leq per(AA^*)per(BB^*).$$

Notice that, part (ii) can be deduced from part (i) by replacing $G=\{1\}$ and $\chi=1$.

Now, we give a necessary and sufficient condition for the equality of two symmetrized decomposable polynomials.

Theorem 2.3. *Let χ be a linear character of a subgroup G of S_m. Let $f_i=\sum_{j=1}^m a_{ij}x_j$, $g_i=\sum_{j=1}^m b_{ij}x_j$, $1\leq i\leq d$. Suppose $f=f_1\dots f_d$, $g=g_1\dots g_d$, A=$(a_{ij})_{d\times m}$ and $B=(b_{ij})_{d\times m}$. Then*

(i) *$f^*=0$ iff for every $\alpha\in\bar{\Delta}$,*

$$\sum_{\sigma\in G}\chi(\sigma^{-1})perA[1,2,\dots,d|\widetilde{\alpha\sigma}]=0.$$

(ii) *$f^*=g^*$ iff for all $\alpha\in\bar{\Delta}$,*

$$\sum_{\sigma\in G}\chi(\sigma^{-1})perA[1,2,\dots,d|\widetilde{\alpha\sigma}]=\sum_{\sigma\in G}\chi(\sigma^{-1})perB[1,2,\dots,d|\widetilde{\alpha\sigma}].$$

Proof. (i) Since χ is a linear character, so for any $\alpha\in\bar{\Delta}$ and $\sigma\in G$, $\omega(\widetilde{\alpha\sigma})^*=\chi(\sigma^{-1})\omega(\widetilde{\alpha})^*$. By (2.1), we have

$$\begin{aligned}f^*&=\sum_{\alpha\in G_{d,m}}\frac{1}{\nu(\alpha)}perA[1,2,\dots,d|\alpha]\omega(\alpha)^*\\&=\sum_{\alpha\in\bar{\Delta}}\frac{1}{|G_\alpha|}\sum_{\sigma\in G}\frac{1}{\nu(\widetilde{\alpha\sigma})}perA[1,2,\dots,d|\widetilde{\alpha\sigma}]\omega(\widetilde{\alpha\sigma})^*\\&=\sum_{\alpha\in\bar{\Delta}}\frac{1}{\alpha!|G_\alpha|}\sum_{\sigma\in G}\chi(\sigma^{-1})perA[1,2,\dots,d|\widetilde{\alpha\sigma}]\omega(\widetilde{\alpha})^*\end{aligned}$$

Since $\mathfrak{B}=\{\omega(\widetilde{\alpha})^*|\ \alpha\in\bar{\Delta}\}$ is a basis of $H_d(G,\chi)$, so $f^*=0$ iff for $\alpha\in\bar{\Delta}$,

$$\sum_{\sigma\in G}\chi(\sigma^{-1})perA[1,2,\dots,d|\widetilde{\alpha\sigma}]=0.$$

(ii) Applying the proof of part (i), we have

$$f^*=\sum_{\alpha\in\bar{\Delta}}\frac{1}{\alpha!|G_\alpha|}\sum_{\sigma\in G}\chi(\sigma^{-1})perA[1,2,\dots,d|\widetilde{\alpha\sigma}]\omega(\widetilde{\alpha})^*,$$

$$g^*=\sum_{\alpha\in\bar{\Delta}}\frac{1}{\alpha!|G_\alpha|}\sum_{\sigma\in G}\chi(\sigma^{-1})perB[1,2,\dots,d|\widetilde{\alpha\sigma}]\omega(\widetilde{\alpha})^*,$$

hence $f^* = g^*$ iff for all $\alpha \in \bar{\Delta}$,

$$\sum_{\sigma\in G}\chi(\sigma^{-1}) perA[1,2,\ldots,d|\widetilde{\alpha\sigma}] = \sum_{\sigma\in G}\chi(\sigma^{-1}) perB[1,2,\ldots,d|\widetilde{\alpha\sigma}].$$

□

3. The algebraic properties of the induced operator $K_\chi(T)$

Let V be the complex space of homogeneous linear polynomials with complex variables $x_1,\ldots,x_m$. Suppose $T \in L(V)$. In [15], the authors studied many algebraic properties of $P(T)$. In this section, we study some algebraic properties of the induced operator $K_\chi(T)$.

Theorem 3.1. *Let χ be a linear character of the subgroup G of S_m. Suppose T is a linear operator on V. If $A = (a_{ij})$ is the matrix representation of T with respect to ordered basis $\mathbb{E} = \{x_1,\ldots,x_m\}$, then for any $\alpha,\beta \in \bar{\Delta}$, the $(\widetilde{\alpha},\widetilde{\beta})$ entry of the matrix representation of $K_\chi(T)$ with respect to ordered basis*

$$\mathfrak{B}^* = \{\omega(\widetilde{\alpha})^* |\ \alpha \in \bar{\Delta}\}$$

is

$$\frac{1}{\alpha!|G_\alpha|}\sum_{\sigma\in G}\chi(\sigma^{-1}) perA[\widetilde{\alpha\sigma}|\widetilde{\beta}].$$

Proof. Since χ is linear, so for all $\alpha \in \bar{\Delta}$ and $\sigma \in G$, we have

$$\omega(\widetilde{\alpha\sigma})^* = X^{\alpha\sigma,*} = \chi(\sigma^{-1})X^{\alpha,*} = \chi(\sigma^{-1})\omega(\widetilde{\alpha})^*.$$

Hence for any $\beta \in \bar{\Delta}$,

$$\begin{aligned}
K_\chi(T)\omega(\widetilde{\beta})^* &= Tx_{\widetilde{\beta}(1)} * Tx_{\widetilde{\beta}(2)} * \ldots * Tx_{\widetilde{\beta}(d)} \\
&= \sum_{i_1=1}^{m} a_{i_1\widetilde{\beta}(1)}x_{i_1} * \ldots * \sum_{i_d=1}^{m} a_{i_d\widetilde{\beta}(d)}x_{i_d} \\
&= \sum_{\alpha\in\Gamma_{d,m}}\prod_{i=1}^{d} a_{\alpha(i),\widetilde{\beta}(i)}\prod_{i=1}^{d} x_{\alpha(i)} \\
&= \sum_{\alpha\in G_{d,m}}\frac{1}{\nu(\alpha)}\sum_{\tau\in S_d}\prod_{i=1}^{d} a_{\alpha(i),\widetilde{\beta}\tau(i)}\omega(\alpha)^* \\
&= \sum_{\alpha\in\bar{\Delta}}\frac{1}{|G_\alpha|}\sum_{\sigma\in G}\frac{1}{\nu(\widetilde{\alpha\sigma})}\sum_{\tau\in S_d}\prod_{i=1}^{d} a_{\widetilde{\alpha\sigma}(i),\widetilde{\beta}\tau(i)}\omega(\widetilde{\alpha\sigma})^* \\
&= \sum_{\alpha\in\bar{\Delta}}\Big(\frac{1}{\alpha!|G_\alpha|}\sum_{\sigma\in G}\chi(\sigma^{-1}) perA[\widetilde{\alpha\sigma}|\widetilde{\beta}]\Big)\omega(\widetilde{\alpha})^*,
\end{aligned}$$

so the result holds. □

Notice that if $G = \{1\}$ and $\chi = 1$, the principal character of G, then $H_d(G, \chi) = H_d[x_1, \ldots, x_m]$, and for any $\alpha, \beta \in \Gamma^+_{m,d}$,

$$([K_\chi(T)]_{\mathfrak{B}^*})_{\widetilde{\alpha},\widetilde{\beta}} = \frac{1}{\nu(\widetilde{\alpha})} perA[\widetilde{\alpha}|\widetilde{\beta}] = (P(T))_{\widetilde{\alpha},\widetilde{\beta}},$$

hence $K_\chi(T) = P(T)$.

Definition 3.2. *Let χ be a linear character of the subgroup G of S_m. Let $A \in M_m(\mathbb{C})$. Then the* induced matrix *defined by A is the matrix whose the $(\widetilde{\alpha}, \widetilde{\beta})$ entry is*

$$\frac{1}{\alpha!|G_\alpha|} \sum_{\sigma \in G} \chi(\sigma^{-1}) perA[\widetilde{\alpha\sigma}|\widetilde{\beta}], \ \alpha, \beta \in \bar{\Delta}.$$

This matrix is denoted by $K_\chi(A)$. If A is the matrix representation of $T \in L(V)$ with respect to the orthonormal basis $\mathbb{E}$ of V, then $K_\chi(A)$ is the matrix representation of $K_\chi(T)$ with respect to basis $\mathfrak{B}^*$.

Proposition 3.3. *Let χ be a linear character of the subgroup G of S_m. Suppose $T \in L(V)$. If $\mathbb{E} = \{x_1, \ldots, x_m\}$ is a triangular basis for T, then $\mathfrak{B}^* = \{\omega(\widetilde{\alpha})^* | \ \alpha \in \bar{\Delta}\}$ is a triangular basis for $K_\chi(T)$.*

Proof. Suppose $\mathbb{E}$ is an upper triangular basis for T. Then

$$Tx_1 = \lambda_1 x_1, \ Tx_i = \lambda_i x_i + u_i; \ u_i \in \langle x_1, x_2, \ldots, x_{i-1} \rangle, 1 < i \le m.$$

Set $u_1 = 0$. Suppose α is an arbitrary element of $\bar{\Delta}$. Then

$$\begin{aligned} K_\chi(T)\omega(\widetilde{\alpha})^* &= Tx_{\widetilde{\alpha}(1)} * Tx_{\widetilde{\alpha}(2)} * \ldots * Tx_{\widetilde{\alpha}(d)} \\ &= (\lambda_{\widetilde{\alpha}(1)} x_{\widetilde{\alpha}(1)} + u_{\widetilde{\alpha}(1)}) * \ldots * (\lambda_{\widetilde{\alpha}(d)} x_{\widetilde{\alpha}(d)} + u_{\widetilde{\alpha}(d)}) \\ &= (\prod_{i=1}^{d} \lambda_{\widetilde{\alpha}(i)})\omega(\widetilde{\alpha})^* + \sum_{\widetilde{\gamma}} c_{\widetilde{\gamma}} \omega(\widetilde{\gamma})^*, \end{aligned}$$

where the sum is over those $\gamma \in \Omega$ such that $\widetilde{\gamma}(i) \le \widetilde{\alpha}(i)$, $1 \le i \le d$ with at least one strict inequality. Fix $\gamma \in \Omega$ such that $\widetilde{\gamma}(i) \le \widetilde{\alpha}(i)$, $1 \le i \le d$. Consider $\tau \in G$ such that $\gamma\tau \in \bar{\Delta}$. Then

$$\omega(\widetilde{\gamma\tau})^* = \chi(\tau^{-1})\omega(\widetilde{\gamma})^*.$$

Therefore $\omega(\widetilde{\gamma})^*$ is a linear combination of symmetrized decomposable polynomials that come strictly before $\omega(\widetilde{\alpha})^*$ in the ordered basis $\mathfrak{B}^*$. This completes the proof. □

Corollary 3.4. *Let χ be a linear character of the subgroup G of S_m. Let T be a linear operator on V. Suppose T has eigenvalues $\lambda_1, \ldots, \lambda_m$ (multiplicities included). Then the eigenvalues of $K_\chi(T)$ are*

$$\lambda_{\widetilde{\alpha}} = \prod_{i=1}^{d} \lambda_{\widetilde{\alpha}(i)}, \ \alpha \in \bar{\Delta}.$$

It follows that from Corollary 3.4 , $tr K_\chi(T) = \sum_{\alpha\in\bar{\Delta}} \prod_{i=1}^{d} \lambda_{\widetilde{\alpha}(i)}$ and

$$\det K_\chi(T) = \prod_{\alpha\in\bar{\Delta}} \prod_{i=1}^{d} \lambda_{\widetilde{\alpha}(i)} = \prod_{\alpha\in\bar{\Delta}} \prod_{i=1}^{m} \lambda_i^{m_i(\widetilde{\alpha})} = \prod_{i=1}^{m} \lambda_i^{e_i(\chi)},\ e_i(\chi) = \sum_{\alpha\in\bar{\Delta}} m_i(\widetilde{\alpha}).$$

Proposition 3.5. [9, p. 149] *Let $S : U \longrightarrow U$ be an idempotent Hermitian linear operator and suppose $T \in L(V)$ such that $TS = ST$. If T is normal, Hermitian, positive definite, positive semidefinite, unitary, skew Hermitian (d is odd), then $T_1 = T|_{Im\ S}$ has the corresponding property.*

Theorem 3.6. *Suppose $T \in L(V)$. If T has one of the properties normal, Hermitian, positive definite, positive semidefinite, unitary, skew Hermitian (d is odd), then $K_\chi(T)$ has the corresponding property.*

Proof. Let $P(T)$ be the induced operator on $H_d[x_1,\ldots,x_m]$. Suppose T has one of the properties normal, Hermitian, positive definite, positive semidefinite, unitary, skew Hermitian (d is odd), then $P(T)$ has the corresponding property (see [15]). By Lemma 2.1, S_χ is Hermitian. But S_χ is idempotent and $K_\chi(T) = P(T)|_{Im\ S_\chi}$, so the result is obtained from Proposition 3.5. □

Theorem 3.7. *Let $G \le S_m$ and χ be an irreducible character of G. Suppose $T, S \in L(V)$. If $H_d(G,\chi) \neq 0$ then*
(i) $K_\chi(ST) = K_\chi(S)K_\chi(T)$.
(ii) T is nonsingular iff $K_\chi(T)$ is nonsingular.
(iii) $K_\chi(T)^* = K_\chi(T^*)$.
(iv) If $S,T \ge 0$, then $K_\chi(S+T) \ge K_\chi(S) + K_\chi(T)$. In particular

$$K_\chi(T) \ge K_\chi(S), \quad \text{whenever } T \ge S.$$

Proof. (i) Since $H_d(G,\chi)$ is an invariant subspace of $P(T)$ and $P(S)$, furthermore $P(ST) = P(S)P(T)$, so the result is obtained by restricting of the both sides of the previous equality to $H_d(G,\chi)$.
(ii) If T is invertible then by part (i), $K_\chi(T)$ is invertible.
Conversely, suppose $K_\chi(T)$ is invertible and $Tx_1 = 0$. Since $H_d(G,\chi) \neq 0$, there is $\alpha \in \bar{\Delta}$ such that $\omega(\widetilde{\alpha})^* \neq 0$. So $\omega(\widetilde{\alpha})^*$ is an element of basis of $H_d(G,\chi)$. Consider $1 \in \text{Im}\ \widetilde{\alpha}$, then

$$K_\chi(T)\omega(\widetilde{\alpha})^* = Tx_{\widetilde{\alpha}(1)} * \ldots * Tx_{\widetilde{\alpha}(d)} = 0,$$

a contradiction.

(iii) By [15], $P(T)^* = P(T^*)$. So

$$K_\chi(T)^* = (P(T)|_{Im\ S_\chi})^* = P(T)^*|_{Im\ S_\chi} = P(T^*)|_{Im\ S_\chi} = K_\chi(T^*).$$

(iv) By [15], $P(T+S) \ge P(T) + P(S)$ and if $T \ge S$, then $P(T) \ge P(S)$. The result is obtained by restricting of the both sides of the above inequality to $H_d(G,\chi)$. □

Proposition 3.8. *Let χ be a linear character of $G \le S_m$. Suppose $T \in L(V)$ with $rank(T) = r$. Then $rank(K_\chi(T)) = |\Gamma^+_{r,d} \cap \bar{\Delta}|$. In particular, if $G = S_m$ and $\chi = 1$, then $rank(K_\chi(T)) = |G^+_{r,d}|$.*

Proof. Let $\{x_1,\dots,x_r,x_{r+1},\dots,x_m\}$ be a basis of V so chosen that $\{Tx_i=y_i|\ i=1,\dots,r\}$ is a basis of Im T and $\{x_{r+1},\dots,x_m\}$ is a basis of Ker T. Let $\alpha\in\Gamma^+_{m,d}$ such that $\alpha(t)\neq 0$, for some $t>r$. Then $\widetilde{\alpha}(\alpha_1+\cdots+\alpha_{t-1}+1)>r$, so $Tx_{\widetilde{\alpha}(\alpha_1+\cdots+\alpha_{t-1}+1)}=0$. Thus

$$K_\chi(T)\omega(\widetilde{\alpha})^*=Tx_{\widetilde{\alpha}(1)}*\dots*Tx_{\widetilde{\alpha}(d)}=0.$$

On the other hand, since $\{y_1,\dots,y_r\}$ is a part of a basis of V, so, the polynomials

$$\omega(\widetilde{\alpha};\ y_1,\dots,y_r)^*=(\prod_{i=1}^{r}y_{\widetilde{\alpha}(i)})^*\neq 0,\quad \alpha\in\Gamma^+_{r,d}\cap\bar{\Delta}$$

are part of a basis of $H_d(G,\chi)$. If $G=S_m$ and $\chi=1$, then $\bar{\Delta}=\Delta=G^+_{m,d}$, so the result holds. □

Proposition 3.9. *Let χ be a linear character of the subgroup G of S_m. If $A\in M_m(\mathbb{C})$ is a diagonal matrix, then the induced matrix $K_\chi(A)$ is diagonal.*

Proof. Let A=diag$(\lambda_1,\dots,\lambda_m)$. Then, for any $\alpha,\beta\in\bar{\Delta}$, we have

$$\begin{aligned}
K_\chi(A)_{\widetilde{\alpha},\widetilde{\beta}}&=\frac{1}{\alpha!|G_\alpha|}\sum_{\sigma\in G}\chi(\sigma^{-1})perA[\widetilde{\alpha\sigma}|\widetilde{\beta}]\\
&=\frac{1}{\alpha!|G_\alpha|}\sum_{\sigma\in G}\chi(\sigma^{-1})\sum_{\tau\in S_d}\prod_{i=1}^{d}a_{\widetilde{\alpha\sigma}(i),\widetilde{\beta}\tau(i)}\\
&=\frac{1}{\alpha!|G_\alpha|}\sum_{\sigma\in G}\chi(\sigma^{-1})\sum_{\tau\in S_d}\prod_{i=1}^{d}\lambda_{\widetilde{\beta}\tau(i)}\delta_{\widetilde{\alpha\sigma}(i),\widetilde{\beta}\tau(i)}\\
&=\frac{1}{\alpha!|G_\alpha|}\sum_{\sigma\in G}\chi(\sigma^{-1})\sum_{\tau\in (S_d)_{\widetilde{\beta}}}\prod_{i=1}^{d}\lambda_{\widetilde{\beta}(i)}\delta_{\widetilde{\alpha\sigma}(i),\widetilde{\beta}(i)}\\
&=\frac{\prod_{i=1}^{d}\lambda_{\widetilde{\beta}(i)}}{\alpha!|G_\alpha|}\sum_{\sigma\in G}\chi(\sigma^{-1})\sum_{\tau\in (S_d)_{\widetilde{\beta}}}\delta_{\widetilde{\alpha\sigma},\widetilde{\beta}}.
\end{aligned}$$

Since

$$\widetilde{\beta}=\widetilde{\alpha\sigma}\ \Leftrightarrow\ m(\widetilde{\alpha\sigma})=m(\widetilde{\beta})\ \Leftrightarrow\ \alpha\sigma=\beta\ \Leftrightarrow\ \sigma\in G_\alpha,$$

and for any $\alpha\in\bar{\Delta}$, $\sigma\in G_\alpha$, $\chi(\sigma)=1$, so, we have

$$\begin{aligned}
K_\chi(A)_{\widetilde{\alpha},\widetilde{\beta}}&=\frac{\prod_{i=1}^{d}\lambda_{\widetilde{\beta}(i)}|(S_d)_{\widetilde{\beta}}|}{\alpha!|G_\alpha|}\sum_{\sigma\in G_\alpha}\chi(\sigma^{-1})\delta_{\alpha,\beta}\\
&=\lambda_{\widetilde{\beta}}\frac{\beta!}{\alpha!}\delta_{\alpha,\beta}.
\end{aligned}$$

Hence $K_\chi(A)$ is diagonal. □

4. The geometric properties of the induced operator $K_\chi(T)$

Let X and Y be two complex inner product spaces. The norm of an element T of $L(X)$ is defined as

$$\|T\| = sup\{\|Tx\| : \ x \in X, \ \|x\| = 1\}.$$

The spectral radius of T is defined by

$$\rho(T) = max\{|\lambda| : \ \lambda \ is \ an \ eigenvalue \ of \ T\}.$$

It is proved that [4] if T is normal, then $\|T\| = \rho(T)$.
Let $f : L(X) \longrightarrow L(Y)$ be a map. The derivative of f at $T \in L(X)$ is a linear map $Df(T)$ from $L(X)$ into $L(Y)$. Recall that

$$Df(T)(S) = \frac{d}{dt}\Big|_{t=0} f(T + tS).$$

For brevity we say that $T \in L(X)$ is *positive* if it is positive semidefinite. A linear map $\phi : L(X) \longrightarrow L(Y)$ is called *positive* if it maps positive elements of $L(X)$ into positive elements of $L(Y)$. It is said to be unital if $\phi(I) = I$. Positive linear maps ϕ enjoy a very special property: $\|\phi\| = \|\phi(I)\|$. This is a consequence of the well-known Russo-Dye Theorem [4].

Again let V be the complex space of homogeneous linear polynomials with complex variables $x_1, \ldots, x_m$. Suppose $T \in L(V)$. In [15], the authors obtained the spectral radius and norm of the induced operator $P(T)$ in terms of $\rho(T)$ and $\|T\|$, respectively. They also computed the norm of the derivative of the map $T \mapsto P(T)$.
Let $K_\chi : L(V) \longrightarrow L(H_d(G,\chi))$ be the induced map. For brevity let $DK_\chi(A,B) = DK_\chi(A)(B)$, the image of B under the derivative $DK_\chi(A)$. In the following, we compute the spectral radius and norm of the induced operator $K_\chi(T)$. Also we obtain the norm of the derivative of the induced map K_χ. The main idea have taken from [5].

Theorem 4.1. *Let G be a subgroup of the symmetric group S_m and χ be a linear character of G. Let T be a linear operator on V. Let $\lambda_1, \ldots, \lambda_m$ and $\nu_1, \ldots, \nu_m$ be the eigenvalues and singularvalues of T, respectively. Then the spectral radius and the norm of the induced operator $K_\chi(T)$ are*

$$\rho(K_\chi(T)) = \max\{|\lambda_{\widetilde{\alpha}}| : \ \alpha \in \bar{\Delta}\},$$

$$\|K_\chi(T)\| = \sqrt[2]{\max\{|\nu_{\widetilde{\alpha}}| : \ \alpha \in \bar{\Delta}\}}.$$

Proof. The assertion follows from definition of spectral radius, Theorem 3.6 and the relation

$$\|K_\chi(T)\|^2 = \|K_\chi(T)^* K_\chi(T)\| = \|K_\chi(T^*T)\| = \rho(K_\chi(T^*T)).$$

□

Theorem 4.2. *Let χ be a linear character of a subgroup G of S_m. Let T be a positive linear operator on V and suppose that the matrix representation of T with respect to the basis $\mathbb{E} = \{x_1, \ldots, x_m\}$ is the diagonal matrix $A = diag(\nu_1, \nu_2, \ldots, \nu_m)$. Then $DK_\chi(T, I)$ is diagonal with respect to the basis $\mathfrak{B}^* = \{\omega(\widetilde{\alpha})^* |\ \alpha \in \bar{\Delta}\}$ and its $(\widetilde{\alpha}, \widetilde{\alpha})$ entry is*

$$DK_\chi(A, I)_{\widetilde{\alpha},\widetilde{\alpha}} = \sum_{j=1}^{d} \prod_{i \neq j} \nu_{\widetilde{\alpha}(i)},\ \ \alpha, \beta \in \bar{\Delta}.$$

Proof. For brevity we set $DP(T, I) = DP(T)(I)$. By [15], the operator $DP(T, I)$ with respect to the basis $\mathfrak{B} = \{\omega(\widetilde{\alpha}) |\ \alpha \in \Gamma^+_{m,d}\}$ is diagonal and for any $\alpha \in \Gamma^+_{m,d}$, its $(\widetilde{\alpha}, \widetilde{\alpha})$ entry is

$$DP(A, I)_{\widetilde{\alpha},\widetilde{\alpha}} = \sum_{j=1}^{d} \prod_{i \neq j} \nu_{\widetilde{\alpha}(i)}.$$

Now, for any $\alpha \in \bar{\Delta}$, we have

$$\begin{aligned}
DK_\chi(T, I)\omega(\widetilde{\alpha})^* &= \frac{d}{dt}|_{t=0} K_\chi(T + tI)\omega(\widetilde{\alpha})^* \\
&= \frac{d}{dt}|_{t=0} P(T + tI)S_\chi\omega(\widetilde{\alpha}) \\
&= \frac{d}{dt}|_{t=0} S_\chi P(T + tI)\omega(\widetilde{\alpha}) \\
&= S_\chi(\frac{d}{dt}|_{t=0} P(T + tI))\omega(\widetilde{\alpha}) \\
&= S_\chi DP(T, I)\omega(\widetilde{\alpha}) \\
&= S_\chi(\sum_{j=1}^{d} \prod_{i \neq j} \nu_{\widetilde{\alpha}(i)})\omega(\widetilde{\alpha}) \\
&= (\sum_{j=1}^{d} \prod_{i \neq j} \nu_{\widetilde{\alpha}(i)})S_\chi(\omega(\widetilde{\alpha})) \\
&= (\sum_{j=1}^{d} \prod_{i \neq j} \nu_{\widetilde{\alpha}(i)})\omega(\widetilde{\alpha})^*,
\end{aligned}$$

so the assertion holds. □

Theorem 4.3. *Let χ be a linear character of a subgroup G of S_m. Let $T \mapsto K_\chi(T)$ be the map that associates to each element T of $L(V)$, the induced operator $K_\chi(T)$ of $L(H_d(G, \chi))$. Then the norm of the derivative of this map at T is given by*

$$\|DK_\chi(T)\| = sup\{\sum_{j=1}^{d} \prod_{i \neq j}^{d} \nu_{\widetilde{\alpha}(i)} |\ \alpha \in \bar{\Delta}\},$$

where $\nu_1, \nu_2, \ldots, \nu_m$ are the singular values of T.

Proof. Suppose that A is the matrix representation of T with respect to the ordered basis $\mathbb{E} = \{x_1, \ldots, x_m\}$. Let $K_\chi(A)$ be the induced matrix of A. Let $A = U_1A^+U_2$ be the singular value decomposition of A, where U_1 and U_2 are unitary matrices and A^+ is the diagonal matrix whose diagonal entries are the singular values of A. By Theorem 3.6, $R^* = K_\chi(U_1^*)$ and $Q^* = K_\chi(U_2^*)$ are unitary and for any $B \in M_m(\mathbb{C})$, we have

$$\begin{aligned}
\|DK_\chi(A)(B)\| &= \|R^*\big(DK_\chi(A)(B)\big)Q^*\| \\
&= \|K_\chi(U_1^*)\big(DK_\chi(A)(B)\big)K_\chi(U_2^*)\| \\
&= \|K_\chi(U_1^*)\big(\frac{d}{dt}|_{t=0}K_\chi(A+tB)\big)K_\chi(U_2^*)\| \\
&= \|\frac{d}{dt}|_{t=0}K_\chi(U_1^*)K_\chi(A+tB)K_\chi(U_2^*)\| \\
&= \|\frac{d}{dt}|_{t=0}K_\chi(U_1^*AU_2^* + tU_1^*BU_2^*)\| \\
&= \|DK_\chi(A^+)(U_1^*BU_2^*)\|.
\end{aligned}$$

Thus

$$\begin{aligned}
\|DK_\chi(A)\| &= sup\{\|DK_\chi(A)(B)\| : \ \|B\| = 1\} \\
&= sup\{\|DK_\chi(A^+)(U_1^*BU_2^*)\| : \ \|U_1^*BU_2^*\| = \|B\| = 1\} \\
&= \|DK_\chi(A^+)\|.
\end{aligned}$$

So, we may replace A by the positive semidefinite diagonal matrix A^+ and observers that $DK_\chi(A^+)$ is a positive linear map. By Russo-Dye Theorem, such a map attains its norm at the identity I, therefore

$$\|DK_\chi(A)\| = \|DK_\chi(A^+, I)\|$$

By Theorem 4.2, $DK_\chi(A^+, I)$ is diagonal and its $(\widetilde{\alpha}, \widetilde{\alpha})$ entry is

$$DK_\chi(A^+, I)_{\alpha,\alpha} = \sum_{j=1}^{d} \prod_{i \neq j} \nu_{\widetilde{\alpha}(i)}, \quad \alpha \in \bar{\Delta}.$$

Therefore

$$\|DK_\chi(T)\| = sup\{\sum_{j=1}^{d} \prod_{i \neq j}^{d} \nu_{\widetilde{\alpha}(i)} : \ \alpha \in \bar{\Delta}\}.$$

□

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