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# CONVERSE OF LAGRANGE'S THEOREM (CLT) NUMBERS UNDER 1000

JEAN B. NGANOU

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Abstract. A positive integer $n$ is called a CLT number if every group of order $n$ satisfies the converse of Lagrange's Theorem. In this note, we find all CLT and supersolvable numbers up to 1000. We also formulate some questions about the distribution of these numbers.

## 1. Introduction

Undoubtedly, Lagrange's Theorem is the simplest, yet one of the most important results in finite group theory. It states that the size of any subgroup of a finite group is a divisor of the order of the group. The converse of Lagrange's Theorem (CLT), that is every divisor of the order of a group is the size of a subgroup is well known to be false. The first counter-example, the alternating group $A_4$ on four objects dates back to 1799 and was discovered by Paolo Ruffini. However, this converse holds for several classes of groups such as Abelian groups, $p$-groups, supersolvable groups and more. In addition, it is also known that for any group, this converse holds for various classes of divisors (Sylow's Theorems). The converse of Lagrange's Theorem has been and continues to be of interest to Mathematicians.

A group that satisfies the converse of the Lagrange Theorem is called a CLT group. A positive integer $n$ is called a CLT number (resp. supersolvable number) if every group of order $n$ is a CLT group (resp. supersolvable). In [7], [8], R. Struik studied CLT numbers of the form $p^m q^n$ and Berger completely characterized CLT numbers using the prime factorization [1]. In [6], Pazderski characterized completely the supersolvable numbers. The goal of this article is to find all the CLT numbers and also all the supersolvable numbers up to 1000. The most direct approach would consist in checking the numbers one by one using Berger's criterion, but this could take a considerable amount of time. Our approach

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consists in quickly identifying some classes of non-CLT numbers and also some classes of CLT numbers. This will narrow considerably the amount of numbers to be tested.

We start with the class of groups with no subgroups of index 2.

## 2. Groups with no subgroups of index 2

For any group $G$, we denote by $G^2$ the subgroup of $G$ generated by squares of elements in $G$, that is $G^2 = \langle \{x^2 : x \in G\} \rangle$. It follows from the fact that the set of squares of $G$ is closed under conjugation that $G^2$ is normal in $G$.

We say that $G$ is generated by squares if $G = G^2$. We denote by $I_2(G)$ the number of subgroups of index 2 in $G$.

The following result gives a formula for the function $I_2$.

**Theorem 2.1.** [4] *Let $G$ be a finite group. Then*

$$I_2(G) = [G : G^2] - 1$$

For instance, one can apply this result to the dihedral groups and obtain that $I_2(D_n) = 1$ if $n$ is odd and $I_2(D_n) = 3$ if $n$ is even.

**Corollary 2.2.** *A group has no subgroups of index 2 if and only if it is generated by squares.*

We also have the following result whose proof is straightforward and is left to the reader.

**Lemma 2.3.** *For every groups $G_1$ and $G_2$,*

$$(G_1 \bigoplus G_2)^2 = G_1^2 \bigoplus G_2^2$$

Combining Corollary 2.2 and Lemma 2.3, we get:

**Corollary 2.4.** *If $G_1$ and $G_2$ are groups with no subgroups of index 2, then $G_1 \bigoplus G_2$ has no subgroups of index 2.*

We start with an excel spreadsheet containing all numbers from 1 to 1000. We know that 12 and 24 are non-CLT numbers as they are respectively the orders of $A_4$ and $SL(2, 3)$ which are both famous for not having subgroups of index 2 [3]. By Corollary 2.4, for every odd integer $k$, the groups $A_4 \bigoplus \mathbb{Z}_k$ and $SL(2, 3) \bigoplus \mathbb{Z}_k$ have no subgroups of index 2. This generates non-CLT numbers $12k$ and $24k$ with $k$ odd. We color all these numbers red and applying Corollary 2.4, the product of two reds is again a red. This elementary tool produces more than half of all the non-CLT numbers less than 1000. These are:
$12, 24, 36, 60, 72, 84, 108, 120, 132, 144, 156, 168, 180, 204, 216, 228, 252, 264, 276, 288, 300,$
$312, 324, 348, 360, 372, 396, 408, 420, 432, 444, 456, 468, 492, 504, 516, 540, 552, 564, 576,$
$588, 600, 612, 636, 648, 660, 684, 696, 708, 720, 732, 744, 756, 780, 792, 804, 828, 840, 852,$
$864, 876, 888, 900, 924, 936, 948, 972, 984, 996.$

The following proposition which follows easily from [1, Prop. 3.5] will produce the next class of non-CLT numbers, which we color orange if they are not already red.

**Proposition 2.5.** *Every natural number $x$ of each of the following form is a non-CLT number.*
*(i) $x = a3^m2^n$ with $m \geq 1$, $n \geq 2$, $a \geq 1$ and $(a, 6) = 1$.*
*(ii) $x = a3^m5^n$ with $m \geq 1$, $n \geq 2$, $a \geq 1$ and $(a, 15) = 1$.*
*(iii) $x = a5^m2^n$ with $m \geq 1$, $n \geq 4$, $a \geq 1$ and $(a, 10) = 1$.*
*(iv) $x = a7^m2^n$ with $m \geq 1$, $n \geq 3$, $n \neq 5$, $a \geq 1$ and $(a, 14) = 1$.*
*(v) $x = a13^m3^n$ with $m \geq 1$, $n \geq 3$, $a \geq 1$ and $(a, 39) = 1$.*
*(vi) $x = a3^m11^n$ with $m \geq 1$, $n \geq 2$, $a \geq 1$ and $(a, 33) = 1$.*

Besides some that are already in the list above, this result generates the following numbers: 48, 56, 75, 80, 96, 112, 150, 160, 192, 225, 240, 280, 320, 336, 363, 375, 384, 392, 400, 448, 450, 480, 525, 528, 560, 616, 624, 640, 672, 675, 702, 726, 728, 750, 768, 784, 800, 816, 825, 880, 896, 912, 952, 960, 975.

## 3. Popular CLT numbers

In this section, we summarize some of the well known classes of numbers that are CLT numbers. It is established in [6], [7] and [8], that each of the following class of integers is made of CLT numbers:

(i) Square-free integers
(ii) Prime power integers.
(iii) Integers of the form $p^m q$ where $p, q$ are primes with $q|p-1$.
(iv) Integers of the form $p^m q^2$ where $p, q$ are primes with $q^2|p-1$.

In addition, we can easily verify that integers of the form $p^m q$ where $p, q$ are distinct primes and the exponent $e$ of $p$ modulo $q$ is greater than $m$ are also CLT numbers.
In fact, if $G$ is a group of such order, then $G$ has a unique $q$-Sylow subgroup $N$ which is therefore normal in $G$. In addition $G$ also has a subgroup $H_i$ of order $p^i$ for every $i = 0, 1, \ldots, m$. Thus for every $i$, $NH_i$ is a subgroup of $G$ and $|NH_i| = \dfrac{|N||H_i|}{|N \cap H_i|} = p^i q$.
In fact all the numbers described above are supersolvable numbers.
Our next step is to color green all the numbers described from the preceding classes.

## 4. A test for CLT numbers

In [1], the author defines a class of integers called good integers and he proves that these are exactly the CLT numbers. We want to use his criterion to test for the remaining numbers (color-free numbers), the stubborn numbers. By now, we have narrowed our list down to 80 integers and for some integers, the test could be performed on up to five integers at the same time. The non-CLT numbers detected this way are coloured dark red, and the CLT numbers coloured light green. For the convenience of the reader, we will summarize here the main ideas of the test and indicate through few examples how the test can be executed. This section could make a good project for students taking an Introductory course on Abstract Algebra.

We begin by setting some notations.

For every natural number $d > 1$, we define

$\mathcal{J}(d) = \{1, 2, \ldots, d-1, 2d-1\}$ and $\mathcal{J}'(d) = \{1, 2, \ldots, d-1\}$.

Given a natural number $m$ and a prime number $q$, we define a subset $\mathcal{S}(m, q)$ of $\mathbb{N}$ as follows:

$(i)$ If $p$ is a prime, $p \neq q$, $d$ is the exponent of $q(mod p)$, and $d > 1$, then $\mathcal{S}(p, q) = \mathcal{J}(d)$ if $d$ is odd and $\mathcal{S}(p, q) = \mathcal{J}'(d)$ if $d$ is even.

$(ii)$ If $p$ is prime, $p|q-1$, and $p^2 \nmid q-1$, then $\mathcal{S}(p^2, q) = \mathcal{J}(p)$ if $p > 2$ and $\mathcal{S}(4, q) = \mathcal{J}'(2)$.

$(iii)$ If $p$ is prime, $p|q-1$, then $\mathcal{S}(p^3, q) = \mathcal{J}(p)$ if $p > 2$ and $\mathcal{S}(8, q) = \mathcal{J}'(2)$.

$(iv)$ If $\mathcal{S}(m, q)$ is not defined by $(i), (ii)$ or $(iii)$, set $\mathcal{S}(m, q) = \mathbb{N}$.

Next, we define another set $\mathcal{S}(r, p^u, q)$ where $r, p, q$ are primes and $u$ is a natural number as follows:

$(a)$ If $rp|q-1$, $r \neq p$, and $p$ has odd exponent $u(mod\ r)$, then $\mathcal{S}(r, p^u, q) = \mathcal{J}(r)$ when $r > 2$ and $\mathcal{S}(2, p^u, q) = \mathcal{J}'(2)$ and,

(b) If $\mathcal{S}(r, p^u, q)$ is not defined by $(a)$, set $\mathcal{S}(r, p^u, q) = \mathbb{N}$.

Suppose that $n > 1$ is a natural number and $q$ is a prime divisor of $n$. We define

$$\mathcal{E}(q) = \mathcal{E}_n(q) = (\cap \mathcal{S}(m, q)) \bigcap (\cap \mathcal{S}(r, p^u, q))$$

Where the first intersection ranges over positive divisor $m$ of $n$ and the last intersection ranges over primes $r, p$ such that $rp^u$ divides $n$.

Let $n = \prod_q q^{e(q)}$ be the prime factorization of $n$.

**Definition 4.1.** [1] *A natural number $n$ is called a good number if for every prime divisor $q$ of $n$, $e(q) \in \mathcal{E}(q)$*

The following result is the main tool used for testing CLT numbers.

**Theorem 4.2.** [1] *A natural number is a CLT number if and only if it is a good number.*

Because of this result, we will refer to CLT numbers as good numbers and to non-CLT numbers as bad numbers.

Observe that each of the sets $\mathcal{S}(m, q), \mathcal{S}(r, p^u, q)$ always contains 1, thus $1 \in \mathcal{E}_n(q)$ for every $n$ and every prime divisor $q$ of $n$. Therefore, a natural number $n$ is a CLT number if and only if for every prime divisor $q$ of $n$ such that $e(q) > 1$, then $e(q) \in \mathcal{E}(q)$. In addition $\mathcal{E}(q)$ can be refined as follows:

$$\mathcal{E}(q) = A(q) \cap B(q)$$

Where

$$A(q) = A_n(q) = \bigcap_{p^t|n, p \neq q} \mathcal{S}(p^t, q)$$

and $B(q) = B_n(q)$ is defined using the following guide.

Are there prime divisors $r \neq p$ of $n$ such that $rp|q-1$ and the exponent $u$ of $p(mod\ r)$ is odd? If no, $B(q) = \mathbb{N}$. If yes, $B(q)$ is the intersection of the sets $\mathcal{S}(r, p^u, q)$ over such primes with $rp^u|n$.

For illustrative purpose, we test few numbers that represent all the different types of behaviour encountered.

**Example 4.3.** (1) $90 = 2.3^2.5$

*The only prime to test here is* $q = 3$ *with* $e(3) = 2$. $A(3) = \mathcal{S}(2,3) \cap \mathcal{S}(5,3) = \mathcal{S}(5,3) = \mathcal{J}'(4) = \{1,2,3\}$. *In addition, since there are no primes* $r \neq p$ *with* $rp|3-1$, *then* $B(3) = \mathbb{N}$. *Hence,* $\mathcal{E}(3) = \{1,2,3\}$, *therefore* $e(3) \in \mathcal{E}(3)$. *Thus* 90 *is a CLT number. Note that these same calculations show that* $270 = 2.3^3.5$ *is also a CLT number while* $810 = 2.3^4.5$ *is not a CLT number.*

(2) $200 = 2^3.5^2$

*For* $q = 5$, $A(5) = \mathcal{S}(2,5) \cap \mathcal{S}(2^2,5) \cap \mathcal{S}(2^3,5) = \mathbb{N} \cap \mathbb{N} \cap \mathcal{J}'(2) = \{1\}$. *Hence* $\mathcal{E}(5) = \{1\}$, *so* $e(5) \notin \mathcal{E}(5)$.

*Therefore* 200 *is not a CLT number. Note that it also follows from the above that* $2^4.5^2, 2^5.5^2, 2^3.5^3$ *are not CLT numbers.*

(3) $294 = 2.3.7^2$

*The only prime to test here is* $q = 7$. *Note that* $2.3|7-1$, *and the exponent of* $3(mod\ 2)$ *is* $u = 1$, *which is odd. So,* $\mathcal{E}(7) \subseteq B(7) \subseteq \mathcal{S}(2,3^1,7) = \mathcal{J}'(2) = \{1\}$. *Hence,* $\mathcal{E}(7) = \{1\}$, *and* $e(7) \notin \mathcal{E}(7)$. *Thus* 294 *is not a CLT number.*

(4) $441 = 3^2.7^2$

*We shall test both primes in this case.*

*For* $q = 3$, $A(3) = \mathcal{S}(7,3) \cap \mathcal{S}(7^2,3) = \mathcal{J}'(6) \cap \mathbb{N} = \{1,2,3,4,5\}$. *On the other hand,* $B(3) = \mathbb{N}$, *hence* $\mathcal{E}(3) = \{1,2,3,4,5\}$. *Therefore,* $e(3) \in \mathcal{E}(3)$.

*For* $q = 7$, $A(7) = \mathcal{S}(3,7) \cap \mathcal{S}(3^2,7) = \mathbb{N} \cap \mathcal{J}(3) = \{1,2,5\}$. *On the other hand,* $B(7) = \mathbb{N}$, *hence* $\mathcal{E}(7) = \{1,2,5\}$. *Therefore* $e(7) \in \mathcal{E}(7)$.

*Thus,* 441 *is a CLT number.*

*The last round of bad numbers detected through these tests are:*
196, 200, 294, 405, 484, 810, 867, 882, 968, 980, 992, 1000.

## 5. **Summarizing**

The following numbers (which comes from either red, dark red, or orange) are all the non-CLT numbers less than 1000.

12, 24, 36, 48, 56, 60, 72, 75, 80, 84, 96, 108, 112, 120, 132, 144, 150, 156, 160, 168, 180, 192,
196, 200, 204, 216, 225, 228, 240, 252, 264, 276, 280, 288, 294, 300, 312, 320, 324, 336, 348, 351,
360, 363, 372, 375, 384, 392, 396, 400, 405, 408, 420, 432, 444, 448, 450, 456, 468, 480, 484,
492, 504, 516, 525, 528, 540, 552, 560, 564, 576, 588, 600, 612, 616, 624, 636, 640, 648, 660,
672, 675, 684, 696, 702, 708, 720, 726, 728, 732, 744, 750, 756, 768, 780, 784, 792, 800, 804,
810, 816, 825, 828, 840, 852, 864, 867, 876, 880, 882, 888, 896, 900, 912, 924, 936, 948, 952,
960, 968, 972, 975, 980, 984, 992, 996, 1000.

Recall [6] that a group is supersolvable if and only if all its subgroups are CLT groups. It follows that every supersolvable number is a CLT number, but the converse is false [7, Prop. 4.3]. If $n$ is a CLT number that is not supersolvable, then there exists a CLT group $G$ of order $n$ that is not supersolvable. Hence, $G$ has a subgroup $H$ that is not a CLT group, thus $|H|$ is a bad number. Therefore a CLT number that is not supersolvable is a multiple of a non-CLT number. A quick calculation reveals that 224 is the only CLT number under 1000, multiple of a number from the list of non-CLT numbers above ($224 = 4 \cdot 56$). In addition, applying [7, Prop. 4.3(b)] with $p = 7, q = 2, m = 1, f = 3$, it follows that 224 is not supersolvable. Therefore the only non supersolvable numbers under 1000 are the ones from the list of bad numbers above and 224. In particular 224 is the smallest CLT number that is not supersolvable and the only one under 1000.

## 6. Final Remarks

Given the connection between groups and field extensions, one can use the present work to study degree of fields extensions that do not admit sub-extensions of all degree. The case of degree two was investigated in [5].

On the other hand, our analysis of the CLT numbers up to 1000 suggests the following question:
**Question:** If we call two bad numbers twin when they differ by 2. The only twin bad numbers under 1000 are $(448, 450)$, $(726, 728)$ and $(880, 882)$. Are there infinitely many twin bad numbers?

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**Jean B. Nganou**

Department of Mathematics, University of Oregon, Eugene, Oregon 97403, USA

`Email: nganou@uoregon.edu`