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ON THE GROUPS SATISFYING THE CONVERSE OF SCHUR'S THEOREM

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ABSTRACT. A famous theorem of Schur states that for a group G finiteness of $G/Z(G)$ implies the finiteness of G' . The converse of Schur's theorem is an interesting problem which has been considered by some authors. Recently, Podoski and Szegedy proved the truth of the converse of Schur's theorem for capable groups. They also established an explicit bound for the index of the center of such groups. This paper is devoted to determine some families of groups among non-capable groups which satisfy the converse of Schur's theorem and at the same time admit the Podoski and Szegedy's bound as the upper bound for the index of their centers.

1. Introduction

Let G be an arbitrary group. A basic theorem of Schur asserts that if the center of a group G has finite index, then the derived subgroup of G is finite. Moreover, some bounds for the order of the derived subgroup in terms of the index of the center were given by some authors. The best bound was given by Wiegold [14] which shows that if $[G : Z(G)] = n$, then $|G'| \leq n^{\frac{1}{2} \log_2 n}$. A question that can be naturally raised here is the truth of the converse of Schur's theorem. Unfortunately the answer is negative in general, as it can be seen for the infinite extra special p - groups. By knowing this fact Hall [12, Page 423] concentrated on the converse of Schur's theorem by replacing the central factor with the second central factor. In other words, he proved that if $|G'|$ is finite, then $[G : Z_2(G)]$ is finite, where $Z_2(G)$ denotes the third term of the upper central series of G .

Thence both the problems of improving the bound obtained by Hall for $[G : Z_2(G)]$ and finding some conditions under which the converse of Schur's theorem holds, have been interesting for some authors.

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For example, Macdonald [6] gave an explicit bound for $[G : Z_2(G)]$. Also, Podoski and Szegedy [10] improved the Macdonald's bound as $[G : Z_2(G)] \leq |G'|^{2\log_2 |G'|}$. On the other hand, Neumann [8, Corollary 5.41] proved that the converse of Schur's theorem is true for finitely generated groups. Niroomand [9] also showed that the converse of Schur's theorem holds when $G/Z(G)$ considered to be finitely generated and in such case he could obtain a bound for $[G : Z(G)]$ as $[G : Z(G)] \leq |G'|^{d(G/Z(G))}$, where $d(X)$ denotes the minimal number of generators of X . Isaacs [4] also verified the converse of Schur's theorem for finite capable groups and gave a bound for the index of the center in such groups. In fact, he proved that if G is a finite capable group and $|G'| = n$, then $[G : Z(G)]$ is bounded above by some function B of n (by a capable group, we mean a group G which can be occurred as the group of inner automorphisms of some group E). Indeed he defined the relevant function $B(n)$ recursively. But one should note that the definition of B was not explicit. In 2005, Podoski and Szegedy [11] tried to improve the Isaac's result in such a way that with a weaker condition, they could attain a better conclusion. They actually proved

Theorem 1.1. *If G is a group (not necessarily finite) and $[G' : G' \cap Z(G)] = n$, then $[G : Z_2(G)] \leq n^{2\log_2^2}$.*

An interesting result of Theorem 1.1 is as follows.

Corollary 1.2. *If G is a capable group and $|G'| = n$, then $[G : Z(G)] \leq n^{2\log_2^2}$.*

Note that the extra special 3-group of order 27 and exponent 9 is non-capable and satisfies the inequality of Corollary 1.2. So the capability of the group is only a sufficient condition in Corollary 1.2.

The following definition helps us to state some of the statements easier.

Definition 1.3. *We say that a group G has the property \mathfrak{P} if and only if the index of the center $Z(G)$ in G is bounded above by the function of the order of the derived subgroup as $n^{2\log_2^2}$, in which $n = |G'|$.*

As mentioned above, not only the capable groups with finite derived subgroup have the property \mathfrak{P} , but also non-capable groups may have this property.

In this paper, we will determine some families of groups, among non-capable groups, which have the property \mathfrak{P} .

Now to attain our desirable aim, we have to provide some definitions and theorems. The first definition that we need, is the notion of c -capability of groups which was introduced by Moghaddam and the last author [7] and also by Burns and Ellis [2] in 1997 simultaneously.

Definition 1.4. *A group G is said to be c -capable if there exists a group E such that $G \cong E/Z_c(E)$.*

Obviously the 1-capability is the capability and also c -capability implies the 1-capability for a group. The existence of a capable group which is not 2-capable is explained in [2]. But for finitely generated abelian groups, the c -capability and capability coincide. In fact, Burns and Ellis [2] proved that

Theorem 1.5. *Fix $c \geq 1$. A finitely generated abelian group G is c -capable if and only if it is capable.*

The capability of abelian groups has been investigated by various authors. One of the most important theorems in this area is the Baer's Statement [1] which describes all capable abelian groups that are direct sums of cyclic groups. The next theorem explains a result of Baer's Statement for finitely generated abelian groups.

Theorem 1.6. *Let G be a finitely generated abelian group written as*

$$G = \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_k},$$

such that each integer n_{i+1} is divisible by n_i , where $\mathbb{Z}_0 = \mathbb{Z}$, the infinite cyclic group. Then G is capable if and only if $k \geq 2$ and $n_{k-1} = n_k$.

Yadav [15] characterized the structure of central factor group for some capable p -groups as follows.

Theorem 1.7. *Let G be a finite capable p -group of nilpotency class 2 with cyclic commutator subgroup G' . Then $G/Z(G)$ is generated by two elements and $|G/Z(G)| = |G'|^2$.*

The following important equivalence relation was defined by Hekster [3].

Definition 1.8. *Two groups G and H are n -isoclinic if there exist isomorphisms*

$$\alpha : \frac{G}{Z_n(G)} \rightarrow \frac{H}{Z_n(H)} \quad \text{and} \quad \beta : \gamma_{n+1}(G) \rightarrow \gamma_{n+1}(H),$$

such that $\beta[g_1, g_2, \dots, g_{n+1}] = [h_1, h_2, \dots, h_{n+1}]$, where $g_i \in G$, $h_i Z_n(H) = \alpha(g_i Z_n(G))$, for each $1 \leq i \leq n + 1$. In this case, we write $G \approx H$. 1-isoclinic groups G and H are briefly called isoclinic and shown by $G \sim H$.

2. Groups with the property \mathfrak{F}

In this section, we would like to obtain some groups which have the property \mathfrak{F} . According to Corollary 1.2, every finite capable group has this property. In what follows, we intend to determine some groups with the property \mathfrak{F} , among non-capable groups. The first class of such groups is obtained by the following theorem.

Theorem 2.1. *Let G be a group with trivial Frattini subgroup. If the derived subgroup of G is finite, then G has the property \mathfrak{F} .*

Proof. Since the Frattini subgroup of G is trivial, then the intersection of the derived subgroup and the center of G is trivial and so $Z_2(G)$ is equal to $Z(G)$. Now the result follows by Theorem 1.1. \square

One should note that Theorem 2.1, may be used among non-capable groups. For instance, the cyclic p -group \mathbb{Z}_p satisfies the conditions of Theorem 2.1, whereas is not capable.

Now, for providing the other class of groups with the property \mathfrak{F} , we need to use the isoclinism theory and its generalization.

It is clear that if G and H are isoclinic and one of them has the property \mathfrak{F} , then so does the other one. In particular, if $G \sim H$ and H is a capable group with finite derived subgroup, then G will have the property \mathfrak{F} . Furthermore, since the function $h(x) = x^{2 \log_2 x}$ is ascending, then one can easily observe that:

Lemma 2.2. *Let G and H be two groups with the same central factors, that is $G/Z(G) \cong H/Z(H)$. If H has the property \mathfrak{P} and $|H'| \leq |G'|$, then G too, has the property \mathfrak{P} .*

In the following, it will be shown that the property \mathfrak{P} will not be inheritable between two groups just having the same central factors. To show this, we first state a theorem that was proved by Hekster [3]. Recall that a *section* of a group G is a group that is isomorphic to a quotient group of a subgroup of G .

Theorem 2.3. [3] *Let G be a group. The following properties are equivalent.*

- (1) G is n -isoclinic to a finite group.
- (2) $G/Z_n(G)$ is finite.
- (3) G is n -isoclinic to a finite section of itself.

Let E be an extra special 3-group of order 3^{2n+1} and $n > 1$. It is clear that E does not have the property \mathfrak{P} . Combining Theorems 1.5 and 2.3, one can obtain

Theorem 2.4. *Let E be as above. Then there exists a group G with the property \mathfrak{P} such that $G/Z(G) \cong E/Z(E)$.*

Proof. Since E is an extra special 3-group of order 3^{2n+1} , the central factor group of E is an elementary abelian 3-group of rank $2n$. Hence $E/Z(E)$ is a 2-capable group, by Theorem 1.5. Then there exists a group K such that $E/Z(E) \cong K/Z_2(K)$. Using Theorem 2.3, K is 2-isoclinic to a finite section K_0 of itself. Now, one can introduce the group G , as the central factor of K_0 . \square

Now we would like to know under which conditions the property \mathfrak{P} may be transferred between two groups with the same central factors. One of these cases is as follows.

Lemma 2.5. *Let G be a finite group and $H \leq G$. If the central factors of G and H are isomorphic, then G has the property \mathfrak{P} if and only if H has also the property \mathfrak{P} .*

Proof. It is not difficult to show that $G = HZ(G)$ and so $G' = H'$. Now the result follows easily. \square

The next theorem is proved in [3].

Theorem 2.6. *Let \mathcal{C} be an n -isoclinism class of groups. Let $T \in \mathcal{C}$ and $\gamma_n(T)$ be finite. Then the following are equivalent.*

- (1) $Z(T) \cap \gamma_n(T) \leq \gamma_{n+1}(T)$.
- (2) $|\gamma_n(T)| = \min\{|\gamma_n(G)| : G \in \mathcal{C}\}$.

Recall that a group S is an n -stemgroup if it satisfies $Z(S) \leq \gamma_{n+1}(S)$. Using the above theorem we have

Theorem 2.7. *Let G and H be two groups, $G \approx H$ and their central factors be isomorphic. If H is an n -stemgroup and has the property \mathfrak{P} , then G has also the property \mathfrak{P} .*

Proof. Since G and H have the same central factors, then the n -th lower central series of these two factor groups are isomorphic, or equivalently, we have $\gamma_n(G)/(\gamma_n(G) \cap Z(G)) \cong \gamma_n(H)/(\gamma_n(H) \cap Z(H))$. Using Theorem 2.6, it is easy to see that the order of the n -th lower central series of H is less than or equal to $|\gamma_n(G)|$. Then $|\gamma_n(H) \cap Z(H)| \leq |\gamma_n(G) \cap Z(G)|$, and hence $|Z(H)| \leq |\gamma_2(G) \cap Z(G)|$. Since G and H have the same central factors, one can obtain $|\gamma_2(H)| \leq |\gamma_2(G)|$, and thereby concludes the result. \square

In the following, we intend to give an example in order to show that how Theorem 2.7 can be helpful for finding some groups with the property \mathfrak{P} among non-capable groups. In other words, it is shown that there exist groups H and G satisfy the assumptions of Theorem 2.7 but G is not capable.

Example 2.8. *Let p be a prime number greater than 5 and G_1 and G_2 be two groups of order p^5 as follows:*

$$G_1 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid \begin{array}{l} [\alpha_j, \alpha] = \alpha_{j+1} \ (j = 1, 2, 3), \ \alpha^p = \alpha_i^p = 1 \ (i = 1, 2, 3, 4), \\ [a, b] = 1 \ \text{(for all other commutators of generators } a \text{ and } b) \end{array} \rangle,$$

and

$$G_2 = \langle \beta, \beta_1, \beta_2, \beta_3, \beta_4 \mid \begin{array}{l} [\beta_j, \beta] = \beta_{j+1} \ (j = 1, 2, 3), \ \beta^p = \beta_i^p = 1 \ (i = 1, 2, 3, 4), \ [\beta_1, \beta_2] = \beta_4, \\ [a, b] = 1 \ \text{(for all other commutators of generators } a \text{ and } b) \end{array} \rangle.$$

Now, one can check that the structures of lower central series (except the first terms), centers, and even the structures of the central factor groups of these two groups are the same. Moreover, G_1 and G_2 are 3-isoclinic and also 3-stemgroups. In addition, let G_3 and G_4 be as follows.

$$G_3 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \mid \begin{array}{l} [\alpha_j, \alpha] = \alpha_{j+1} \ (j = 1, 2, 3), \ \alpha^p = \alpha_i^p = 1 \ (i = 1, 2, 3, 4, 5), \\ [\alpha_2, \alpha_3] = [\alpha_3, \alpha_1] = [\alpha_4, \alpha_1] = \alpha_5, \\ [a, b] = 1 \ \text{(for all other commutators of generators } a \text{ and } b) \end{array} \rangle,$$

and

$$G_4 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \mid \begin{array}{l} [\alpha_j, \alpha] = \alpha_{j+1} \ (j = 1, 2, 3, 4), \ \alpha^p = \alpha_i^p = 1 \ (i = 1, 2, 3, 4, 5), \\ [\alpha_1, \alpha_2] = \alpha_4 \alpha_5^{-1}, \ [\alpha_1, \alpha_3] = \alpha_5, \\ [a, b] = 1 \ \text{(for all other commutators of generators } a \text{ and } b) \end{array} \rangle.$$

James stated in [5] that these two groups are of order p^6 and also $G_1 \cong \text{Inn}(G_3)$ and $G_2 \cong \text{Inn}(G_4)$, where $\text{Inn}(X)$ denotes the group of inner automorphisms of X . Therefore, G_1 and G_2 are capable. Now let $q \neq p$ be a prime number, $H = G_1$ and $G = G_2 \times \mathbb{Z}_q$. Since a finite nilpotent group is capable if and only if each of its Sylow subgroup is capable, one can conclude that G is not capable. Hence, two given groups H and G satisfy the assumptions of Theorem 2.7 whereas G is not capable.

From now on, we are looking forward for the third class of groups with the property \mathfrak{P} . For this, suppose that H is a capable p -group such that $Z(H) = H'$ is a finite cyclic group. Then by Corollary 1.2, the factor group $H/Z(H)$ is finite and hence H is finite. Now using Theorem 1.7, one can conclude that the factor group $H/Z(H)$ is generated by two elements and also $|H/Z(H)| = |H'|^2$. It is clear that groups such as H have the property \mathfrak{P} and also they might be helpful as an instrument for determining some other groups with this property. This fact is illustrated in the next theorem.

Theorem 2.9. *Let H be as above and G be an arbitrary nilpotent group of class two. If $d(G/Z(G)) \leq d(H)$ and $G' \cong H'$, then G has the property \mathfrak{P} .*

Proof. Let $G' \cong H' \cong \mathbb{Z}_n$. Then, the factor group $H/Z(H)$ is a two generator group of order n^2 . Let $H/Z(H) = \langle \bar{x}_1, \bar{x}_2 \rangle$. Since this group is capable, so $|\bar{x}_1| = |\bar{x}_2| = n$, by Theorem 1.6. Moreover $d(H) = d(H/Z(H))$, as $H' = Z(H)$. Hence $d(G/Z(G)) = 2$. Let $G/Z(G) = \langle \bar{y}_1, \bar{y}_2 \rangle$. The capability of $G/Z(G)$ implies that $|\bar{y}_1| = |\bar{y}_2|$. Now, it is easy to see that $G' = \langle [y_1, y_2] \rangle$ and then $|\bar{y}_1| = n = |\bar{y}_2|$. Therefore, $G/Z(G) \cong H/Z(H)$ and hence the result holds. \square

The following example shows that the assumption $d(G/Z(G)) \leq d(H)$ in Theorem 2.9 is necessary.

Example 2.10. *Let H be an extra special p -group of order p^3 and exponent p and G be an extra special p -group of order p^5 . Then G does not have the property \mathfrak{P} .*

Invoking Theorem 2.9, we can determine some non-capable groups with the property \mathfrak{P} .

Example 2.11. *Let E_1 and E_2 be two extra special p -groups of order p^3 and exponents p and p^2 , respectively. We know from [13] that E_2 can not be a direct summand of a capable nilpotent group. So, if we assume $H = E_1$ and $G = E_2 \times A$, in which A is an arbitrary finite abelian group, then H and G satisfy the assumptions of Theorem 2.9, while G is not capable.*

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