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# UNITS IN $\mathbb{Z}_2(C_2 \times D_\infty)$

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Abstract. In this paper we consider the group algebra $R(C_2 \times D_\infty)$. It is shown that $R(C_2 \times D_\infty)$ can be represented by a $4 \times 4$ block circulant matrix. It is also shown that $\mathcal{U}(\mathbb{Z}_2(C_2 \times D_\infty))$ is infinitely generated.

## 1. Introduction

Let $RG$ denote the group ring of a group $G$ over a ring $R$, and $\mathcal{U}(RG)$ denote the unit group of $RG$. A lot is known about the unit group of group rings of finite groups. For details see [9], [1] and [2]. In this paper we deal with units in the group algebra $R(C_2 \times D_\infty)$ over a commutative integral domain $R$, by representing it by $4 \times 4$ block circulant matrix. The idea that the group ring $RD_{2n}$ can be written as a block matrix was introduced by Hurley in [6]. Additionally this method was also used by Gildea in [[4], [5]] to establish the structure of certain unit groups of group algebras. Maciez Mirowicz in [7] studied the group of units $\mathcal{U}(RD_\infty)$ of the group ring of the infinite dihedral group $D_\infty$ over a commutative integral domain $R$. He obtained the structure of $\mathcal{U}(\mathbb{Z}_2 D_\infty)$. In this paper, we extend his results and obtain some subgroups of the unit group $\mathcal{U}(\mathbb{Z}_2(C_2 \times D_\infty))$. We have shown that $\mathcal{U}(\mathbb{Z}_2(C_2 \times D_\infty))$ is not finitely generated.

Let $R$ be a commutative domain with unity. The infinite dihedral group is a two generator group with a known presentation as:

$$D_\infty = \langle t, x | x^2 = 1, xt = t^{-1}x \rangle.$$

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$C_2$ is the cyclic group of order 2 generated by $y$, that is, $C_2 = \langle y \rangle$. Since the canonical form of elements of $D_\infty$ is $t^i x^j$ for some $i \in \mathbb{Z}$ and $0 \leq k \leq 1$ and $y$ commutes with $t$ and $x$ we can write any element $\alpha \in R(C_2 \times D_\infty)$ in the form: $\alpha = (a + bx) + (c + dx)y$, where $a,\ b,\ c,\ d \in RC_\infty$, where $C_\infty$ denotes an infinite cyclic group.

Let $C_\infty = \langle t \rangle$ be an infinite cyclic group generated by $t$ and let $* : RC_\infty \to RC_\infty$ be the involution map of the group ring $RC_\infty$ which comes from the non-trivial automorphism of the group $C_\infty$, that is, $\left(\sum\limits_{i\in\mathbb{Z}} a_i t^i\right)^* := \sum\limits_{i\in\mathbb{Z}} a_i t^{-i}$. We can easily get that for any $a \in RC_\infty$ the relation $xa = a^* x$ holds.

## 2. Units in $\mathbb{Z}_2(C_2 \times D_\infty)$

In this section we obtain some infinitely generated subgroups of the unit group of $\mathbb{Z}_2(C_2 \times D_\infty)$. First, we prove some lemmas.

**Lemma 2.1.** *Let $\theta : R(C_2 \times D_\infty) \to M_4(RC_\infty)$ defined by*

$$\theta((a + bx) + (c + dx)y) = \begin{pmatrix} a & b & c & d \\ b^* & a^* & d^* & c^* \\ c & d & a & b \\ d^* & c^* & b^* & a^* \end{pmatrix} = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

*where $A = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}$ and $B = \begin{pmatrix} c & d \\ d^* & c^* \end{pmatrix}$. Then $\theta$ is a monomorphism.*

*Proof.* Let $\alpha = (a_1 + b_1 x) + (c_1 + d_1 x)y$ and $\beta = (a_2 + b_2 x) + (c_2 + d_2 x)y$. Then $\alpha\beta = (p + qx) + (r + sx)y$, where $p = a_1 a_2 + b_1 b_2^* + c_1 c_2 + d_1 d_2^*$, $q = a_1 b_2 + a_2^* b_1 + c_1 d_2 + c_2^* d_1$, $r = a_1 c_2 + a_2 c_1 + b_1 d_2^* + b_2^* d_1$ and $s = a_1 d_2 + a_2^* d_1 + b_1 c_2^* + b_2 c_1$.

$$\text{Now, } \theta(\alpha\beta) = \begin{pmatrix} p & q & r & s \\ q^* & p^* & s^* & r^* \\ r & s & p & q \\ s^* & r^* & q^* & p^* \end{pmatrix} = \theta(\alpha)\theta(\beta)$$

Hence $\theta$ is a homomorphism.

As $\theta((a + bx) + (c + dx)y) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow a = b = c = d = 0$, thus $\theta$ is one-one. Hence the lemma. □

**Lemma 2.2.** *If $R$ is a commutative domain then $\alpha \in \mathcal{U}(RG) \Leftrightarrow det(\alpha) \in \mathcal{U}(R)$, where $G$ is $(C_2 \times D_\infty)$.*

*Proof.* Let $\alpha = \begin{pmatrix} a & b & c & d \\ b^* & a^* & d^* & c^* \\ c & d & a & b \\ d^* & c^* & b^* & a^* \end{pmatrix}$.

Then, $det(\alpha) = (aa^* - bb^*)^2 + (cc^* - dd^*)^2 + 2(abc^*d^* + a^*b^*cd + ab^*c^*d + a^*bcd^*) - 2(aa^*dd^* + bb^*cc^*) - (a^2c^{*2} + a^{*2}c^2 + b^2d^{*2} + b^{*2}d^2)$.

Thus we get, $(det(\alpha))^* = det(\alpha)$.

Suppose $\alpha \in \mathcal{U}(RG)$ then there exists $\beta \in RG$ such that $\alpha\beta = 1 \Rightarrow det(\alpha)det(\beta) = det(I) = 1$. Thus $det(\alpha) \in \mathcal{U}(RC_\infty)$. But $\mathcal{U}(RC_\infty) = \{rt^i \mid i \in \mathbb{Z},\ r \in R\}$. Thus we have $det(\alpha) = rt^i$ for some $i$. But as $(det(\alpha))^* = det(\alpha)$, we get $(rt^i)^* = rt^i$. This gives $rt^{-i} = rt^i \Rightarrow i = 0$. Hence $det(\alpha) = r \in R$ □

For $0 \neq a = \sum_{i \in Z} \alpha_i t^i \in RC_\infty$ we fix:

$$\begin{aligned} max\ a &:= max\{i \mid \alpha_i \neq 0\} \\ min\ a &:= min\{i \mid \alpha_i \neq 0\} \\ deg\ a &:= max\ a - min\ a\ =\ max\ aa^* \end{aligned}$$

If $\alpha\ =\ a + bxy \in R(C_2 \times D_\infty) \cong \begin{pmatrix} a & o & o & b \\ 0 & a^* & b^* & 0 \\ 0 & b & a & 0 \\ b^* & 0 & 0 & a^* \end{pmatrix}$ is a non-trivial unit then $a \neq 0,\ b \neq 0$.

Thus $det(\alpha) = (aa^* - bb^*)^2 \in \mathcal{U}(R)$ from Lemma 2.2. Now $(aa^* - bb^*) \in RC_\infty$. But $\mathcal{U}(RC_\infty) = \{rt^i \mid i \in \mathbb{Z},\ r \in R\}$. Thus we have $aa^* - bb^* = rt^i$ for some $i$. But as $(aa^* - bb^*)^* = (aa^* - bb^*)$, we get $(rt^i)^* = rt^i$. This gives $rt^{-i} = rt^i \Rightarrow i = 0$. Therefore, $aa^* - bb^* \in \mathcal{U}(R)$.

Hence we define $deg\ \alpha = \max aa^* = \max bb^*\ =\ deg\ b > 0$. For trivial units, we extend this definition by setting $deg\ \alpha\ :=\ 0$.

Let $\mathrm{sgn}(i)$ denotes the sign of $i$. We consider special non-trivial nilpotent elements in the group ring $R(C_2 \times D_\infty)$:

$$\begin{aligned} \eta_{ij} &= (1 + \mathrm{sgn}(i)t^j xy)t^{|i|}(1 - \mathrm{sgn}(i)t^j xy) \\ &= (-t^{-|i|} + t^{|i|}) + \mathrm{sgn}(i)t^j(t^{-|i|} - t^{|i|})xy \text{ for } i(\neq 0),\ j \in \mathbb{Z}. \end{aligned}$$

Also $\eta_{ij}^2 = 0$ as

$$\begin{aligned} (\eta_{ij})^2 &= (1 \pm t^j xy)t^{|i|}(1 \mp t^j xy)(1 \pm t^j xy)t^{|i|}(1 \mp t^j xy) \\ &= (1 \pm t^j xy)t^{|i|}(1 - (t^j xy)^2)t^{|i|}(1 \mp t^j xy) \\ &= 0 \text{ because } (t^j xy)^2 = 1. \end{aligned}$$

For any $r \in R$, $i,\ j \in \mathbb{Z}$, the element $1 + r\eta_{ij}$ is a unit in $R(C_2 \times D_\infty)$. Also inverse of $1 + r\eta_{ij}$ is $1 - r\eta_{ij}$ because

$$(1 + r\eta_{ij})(1 - r\eta_{ij})\ =\ 1 - r^2(\eta_{ij})^2 = 1.$$

All the units of the above form generate a subgroup of the unit group of $R(C_2 \times D_\infty)$, so let

$$U = \langle 1 + r\eta_{ij} \rangle_{i,j \in \mathbb{Z}, r \in R}$$

For all $k > 0,\ j \in \mathbb{Z}$:

$$V_j^k \ = \ \langle 1 + r\eta_{ij} \rangle_{0 < i \leq k, r \in R}$$

Obviously, the groups $\{V_j^k\}$ form an ascending system. We set:

$$V_j \ = \ \varinjlim_k V_j^k.$$

Natural inclusions induce homomorphisms from the free products:

$$\phi_k : \ *_j V_j^k \ \to U \text{ for} k > 0 \text{ and}$$
$$\phi \ = \ \varinjlim_k \phi_k : *_j V_j \to U.$$

Now we describe the groups $V_j^k$. Without loss of generality, we can take sgn$(i)$ and sgn$(l)$ to be +ve. Thus

$$\eta_{ij} \cdot \eta_{lj} \ = \ (1 + t^j xy)t^i(1 - t^j xy)(1 + t^j xy)t^l(1 - t^j xy)$$
$$= (1 + t^j xy)t^i \cdot 0 \cdot t^l(1 - t^j xy) \ = \ 0,$$

therefore the function $\sigma : \ R^k \to V_j^k$ given by:

$$\sigma(r_1, \ldots, r_k) = 1 + r_1\eta_{1j} + \cdots + r_k\eta_{kj}$$
$$= \left(1 - \sum_{i=0}^{k} r_i t^{-i} + \sum_{i=0}^{k} r_i t^i\right) + t^j \left(\sum_{i=0}^{k} r_i t^{-i} - \sum_{i=0}^{k} r_i t^i\right) xy$$

$\sigma$ is an isomorphism from the additive group of $R^k$ onto the multiplicative group $V_j^k$. Therefore we obtain isomorphisms $V_j^k \to R^k$ and $V_j \cong \oplus_{i>0} R$

**Lemma 2.3.** *Let $k > 0$ and let $w \in *_{j \in \mathbb{Z}} V_j^k$ be a non-empty reduced word with the last letter $g$ (i.e., $l(wg^{-1}) < l(w)$), where $l$ denotes the length of the word. If $\phi_k(w) \ = \ a + bxy \in U \subseteq \mathcal{U}(\mathbb{Z}_2(C_2 \times D_\infty))$, then:*

(i) *deg $\phi_k w > 0$(in particular $\phi_k$ is a monomorphism)*
(ii) *$g \in V_j^k \ \Leftrightarrow \ max\ (t^{-j}b + a) < max\{max\ a, max\ t^{-j}b\}$ or $min\ (t^{-j}b + a) > min\{min\ a, min\ t^{-j}b\}$.*

*Proof.* We will prove the result by induction on the length of the word $w$. Let $l(w) = 1$. So, $w \in V_j^k$ for some $j$ and so we can write $w$ as:

$$w = \left(1 + \sum_{i=0}^{k} r_i t^{-i} + \sum_{i=0}^{k} r_i t^i\right) + t^j \left(\sum_{i=0}^{k} r_i t^{-i} + \sum_{i=0}^{k} r_i t^i\right) xy.$$

Now, $\phi_k(w) = w \neq 1$ as $w$ is non-empty, hence we have $r_i \neq 0$ for some $1 \leq i \leq k$. Thus, *deg* $\phi_k(w) \geq 2i > 0$.

Also, when $l(w) = 1$ then $g = w \in V_j^k$ for some fix $j$. Then $a = \left(1 + \sum_{i=0}^{k} r_i t^{-i} + \sum_{i=0}^{k} r_i t^i\right)$ and $b = t^j \left(\sum_{i=0}^{k} r_i t^{-i} + \sum_{i=0}^{k} r_i t^i\right)$. This implies that

$$max\ (t^{-j}b + a) \ = \ 0 < max\{max\ a,\ max\ t^{-j}b\}.$$

Hence the result is true for length 1.

For $c + dxy \in V_j^k$ we obtain some observations as follows:

(1) $c = c^*$, because
$c = \sum_{i=1}^{k} r_i t^{-i} + 1 + \sum_{i=1}^{k} r_i t^i$ and $c^* = \sum_{i=1}^{k} r_i t^i + 1 + \sum_{i=1}^{k} r_i t^{-i}$,
thus we have, $c + c^* = 2$ which implies $c = c^*$ as char of $R = \mathbb{Z}_2$ is 2.

(2) $d^* = dt^{-2j}$, because
$d = \left(\sum_{i=1}^{k} r_i t^{-i} + \sum_{i=1}^{k} r_i t^i\right) t^j$ and $d^* = \left(\sum_{i=1}^{k} r_i t^i + \sum_{i=1}^{k} r_i t^{-i}\right) t^{-j}$,
thus we have, $dt^{-j} = d^* t^j$ which implies $d^* = dt^{-2j}$.

(3) $c + t^{-j}d = 1$.

Now let us assume that the lemma holds for words of length $n \geq 1$. Suppose $l(w) = n + 1$ implies $w = v \cdot g$ where $l(v) = n$ and $g \in V_j^k$.

Let $\phi_k(v) = p + qxy$ and $g = c + dxy$. So

$\phi_k(w) = (p + qxy)(c + dxy) = (pc + qd^*) + (pd + qc^*)xy = a + bxy$(say). Last word of $v$ does not belong to $V_j^k$ and by induction, we obtain the following inequalities:

$$max\ (t^{-j}q + p) \geq max\{max\ p, max\ t^{-j}q\} \qquad \ldots .(1)$$

$$min\ (t^{-j}q + p) \leq min\{min\ p, min\ t^{-j}q\} \qquad \ldots .(2)$$

We get

$$a = pc + qd^* = pc + t^{-2j}qd = pc - t^{-j}q(1 + c) = c(p + t^{-j}q) + t^{-j}q$$

From (1) it follows that

$$\begin{aligned} max\ (c(p + t^{-j}q)) &= max\ c + max(p + t^{-j}q) \geq max\ c + max\ t^{-j}q \\ &> max\ t^{-j}q \end{aligned}$$

which implies that

$$\begin{aligned} max\ a = max(c(p + t^{-j}q) + t^{-j}q) &= max(c(p + t^{-j}q)) \\ &> max\ t^{-j}q \qquad \ldots .(3) \end{aligned}$$

Similarly, using (1) we get

$$max\ a = max(c(p + t^{-j}q)) > max\ p \qquad \ldots .(4)$$

By applying similar calculations and replacing $max$ by $min$, we can obtain $min\ a < min\ p$. Thus $deg\ \phi_k(w) = deg\ a = max\ a - min\ a > max\ p - min\ p = deg\ \phi_k(v) > 0$. Which completes the induction for $(i)$.

Now, we will prove $(ii)$ part. Let $g \in V_j^k$, then by using the above mentioned observations we have,

$$\begin{aligned} t^{-j}b + a &= t^{-j}(pd + qc^*) + (pc + qd^*) = t^{-j}pd + t^{-j}qc + pc + t^{-2j}qd \\ &= (p + t^{-j}q)(c + t^{-j}d) = p + t^{-j}q \text{ since } c + t^{-j}d = 1. \end{aligned}$$

Therefore, $max\ (t^{-j}b + a) = max(p - t^{-j}q) \leq max\{max\ p,\ max\ t^{-j}q\}$. But $max\ p < max\ a$. Thus, we get $max\ t^{-j}b = max(p + t^{-j}q + a) > max\ t^{-j}q$ by using (3) and (4). So

$$max(t^{-j}b + a) \leq max\{max\ p,\ max\ t^{-j}q\} < max\{max\ a,\ max\ t^{-j}b\}.$$

By replacing $max$ by $min$ and applying the same type of calculations we can easily get that:

$$g \in V_j^k \;\Rightarrow\; min(t^{-j}b + a) \;>\; min\{min\ a,\ min\ t^{-j}b\}.$$

**Converse** of $(ii)$. Suppose $max(t^{-j}b + a) \;<\; max\{max\ a,\ max\ t^{-j}b\}$ or $min(t^{-j}b + a) \;>\; min\{min\ a,\ min\ t^{-j}b\}$.

To show $g \in V_j^k$.

Suppose $g \in V_l^k$ then if $max(t^{-l}b + a) \;<\; max\{max\ a,\ max\ t^{-l}b\}$ for $j,\ l \in \mathbb{Z}$ or $min(t^{-l}b + a) > min\{min\, a,\ min\, t^{-l}b\}$. Also $max(t^{-j}b{+}a) \;<\; max\{max\, a,\ max\, t^{-j}b\}$ then $max\ a \;=\; max\ t^{-j}b$ and hence $max(t^{-j}b + a) \;<\; max\ t^{-j}b \;=\; j \;+\; max\ b$. Therefore if $max(t^{-j}b + a) \;<\; max\{max\ a,\ max\ t^{-j}b\}$ and $max(t^{-l}b + a) \;<\; max\{max\ a,\ max\ t^{-l}b\}$ then

$$\begin{aligned} max(t^{-j}b + a + t^{-l}b + a) &\le max\{max(a + t^{-j}b), max\ (a + t^{-l}b)\} \\ &< max\{max\ b - j,\ max\ b - l\}. \end{aligned}$$

But $max(t^{-j}b + a + t^{-l}b + a) \;=\; max(t^{-j}b + t^{-l}b) \;=\; max\ b \;+\; max(t^{-j} + t^{-l})$.
So when $t^{-j} - t^{-l} \;\neq\; 0$, we have $max(t^{-j}b + t^{-l}b) \;=\; max\{max\ b - j,\ max\ b - l\}$ which gives a contradiction. Therefore $t^{-j} - t^{-l} \;=\; 0$, i.e., $j = l$ and hence $g \;=\; a + bx \in V_j^k$ for some $j$. Similarly we can prove the converse if $min(t^{-l}b + a) \;>\; min\{min\ a,\ min\ t^{-l}b\}$ and $min(t^{-l}b + a) \;>\; min\{min\ a,\ min\ t^{-l}b\}$. □

**Theorem 2.4.** *Let $G \;=\; \langle U, D\rangle$, where $U = \langle 1 + r\eta_{ij}\rangle_{i,j\in\mathbb{Z},r\in R}$ and $D \cong (C_2 \times D_\infty) \times \mathcal{U}(\mathbb{Z}_2)$, i.e., $D$ denote the group of trivial units of $\mathbb{Z}_2(C_2 \times D_\infty)$. Then:*

(i) *$U \;\cong\; *_{j\in\mathbb{Z}}V_j \;\cong\; *_{\mathbb{Z}} \oplus_{\mathbb{N}} R^+$, where $R^+$ denotes the additive group of the ring $R = \mathbb{Z}_2$.*
(ii) *$G \;=\; UD$.*

*Proof.* $(i)$ To prove $(i)$, we have to show that the homomorphism $\phi \;=\; \varinjlim_k \phi_k : *_jV_j \to U$ is an isomorphism. The mapping $\phi$ is on to because each generator $1 + r\eta_{ij}$ lies in the image of $\phi$. Further, $\phi$ is one-one because for $1 \neq w \in *_jV_j$ there exists $k \in \mathbb{N}$ such that $w \in *_jV_j^k$ and by above lemma $\phi(w) \;=\; \phi_k(w) \neq 1$. This proves part$(i)$.
$(ii)$ In order to prove(ii) it is enough to show that:

1. $U \cap D \;=\; \{1\}$ 2. $U$ is a normal subgroup of $G$

1. If $1 \neq \alpha \in U$ then by previous lemma deg $\alpha > 0$ so $\alpha \notin D$. Thus $U \cap D \;=\; \{1\}$.

2. $D \cong (C_2 \times D_\infty) \times U(\mathbb{Z}_2)$, where $C_2 \;=\; \langle y\rangle$. $y,\ U(\mathbb{Z}_2)$ are contained in center of $\mathbb{Z}_2(C_2 \times D_\infty)$, therefore it is sufficient to show that $tUt^{-1} \subseteq U$ and $xUx^{-1} \subseteq U$.
Since in $\mathbb{Z}_2(C_2 \times D_\infty)$, $\eta_{ij} = (t^{-i} + t^i) + t^j(t^{-i} + t^i)xy$.
Thus $t\eta_{ij}t^{-1} = (t^{-i} + t^i) + t^{j+2}(t^{-i} + t^i)xy$.
Similarly, $x\eta_{ij}x^{-1} = (t^{-i} + t^i) + t^{-j}(t^{-i} + t^i)xy$ and hence

$$tUt^{-1} \;=\; t(1 + rt\eta_{ij}t^{-1} \;=\; 1 + r\eta_{i(j+2)} \in U$$

$$xUx^{-1} \;=\; x(1 + rt\eta_{ij})x^{-1} \;=\; 1 + r\eta_{i(-j)} \in U.$$

This completes the proof of the theorem.

□

**Remark 2.5.** *By Lemma 2.3,* $1 \neq \alpha \in im\ \phi_k$ *it implies that* $deg\ \alpha > 0$ *so* $im\ \phi_k \cap D = \{1\}$. *Also by the above Theorem 2.4,* $im\ \phi_k$ *is a normal subgroup of* $\langle im\ \phi_k, D\rangle$. *Thus* $\langle im\ \phi_k, D\rangle = im\ \phi_k D$.

**Proposition 2.6.** *U and G are infinitely generated subgroups of unit group of* $\mathbb{Z}_2(C_2 \times D_\infty)$.

*Proof.* If $\alpha_1, \alpha_2, ..., \alpha_n \in U$ then there exists $k \in \mathbb{N}$ such that $\alpha_1\alpha_2...\alpha_n \in \phi(*_j V_j^k)$. But $1 + \eta_{(k+1)j} \notin im\ \phi_k$ because $\phi_{k+1}$ is a monomorphism. Therefore, $\langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle \neq U$. Similarly, if $\beta_1, \beta_2, \ldots, \beta_n \in G$ then by above theorem there exists $k \in \mathbb{N}$ such that $\beta_1\beta_2 \cdots \beta_n \in \phi_k D$. But by above remark, $1 + \eta_{(k+1)j} \notin im\ \phi_k D$ because $\phi_{k+1}$ is a monomorphism. Therefore, $\langle \beta_1, \beta_2, \ldots, \beta_n\rangle \neq G$. □

**Theorem 2.7.** *Any* $\alpha = a + bxy \in \mathcal{U}(\mathbb{Z}_2(C_2 \times D_\infty))$ *is in G (as above defined).*

*Proof.* $\alpha = a + bxy$. If $deg\ \alpha = 0$then $\alpha$ is a trivial unit. Hence we assume that $deg\ \alpha > 0$.

Let $j = max\ b - max\ a$, and $k = min\{min(a + t^{-j}b) - min\ a, max\ a - max(a + t^{-j}b)\}$.

Observe that $aa^* - bb^* = 1$ this implies that $aa^* \neq bb^*$ and hence $a \neq t^{-j}b$ since if $a = t^{-j}b$ then $aa^* = bb^*$. Also $k \geq 1$ because $max\ aa^* = max\ bb^*$ and $deg\ \alpha = deg\ a = deg\ b$ this gives $max\ a - min\ a = max\ b - min\ b$ and thus $j = max\ a - max\ b = min\ a - min\ b$. Hence $min(t^{-j}b) = min\ a - min\ b + min\ b = min\ a$, so $min(a + t^{-j}b) > min\ a > 0$. Similarly, $max\ a > max(a + t^{-j}b)$.

$$\begin{aligned}\alpha(1 + \eta_{kj}) &= (a + bxy)[(1 + t^k + t^{-k}) + t^j(t^k + t^{-k})xy] \\ &= [a + t^k(a + t^{-j}b) + t^{-k}(a + t^{-j}b)] \\ &\quad +[b(1 + t^k + t^{-k}) + a(t^j(t^k + t^{-k})]xy.\end{aligned}$$

Let $h = t^k(a + t^{-j}b) + t^{-k}(a + t^{-j}b)$.

$$\begin{aligned}max\ h &= k + max(a + t^{-j}b)(\text{as } k \geq 1) \\ &\leq max\ a - max(a + t^{-j}b) + max(a + t^{-j}b) \\ &\leq max\ a \text{ (by using definition of k).}\end{aligned}$$

Similarly, we can show that $min\ h \geq min\ a$.

By definition of $k$ either $max\ h = max\ a$ or $min\ h = min\ a$.

**Case 1** If $max\ h = max\ a$. Then $max(a + h) < max\ a$.

$$\begin{aligned}deg(\alpha(1 + \eta_{kj})) &= deg(a + h) = max(a + h) - min(a + h) \\ &< max\ a - min\ a = deg\ \alpha.\end{aligned}$$

Similarly, if $min\ h = min\ a$, we can get $deg(\alpha(1 + \eta_{kj})) < min\ a$.

Hence by induction we can show that $\alpha \in G$.

□

**Corollary 2.8.** *Every unit in* $\mathcal{U}(\mathbb{Z}_2(C_2 \times D_\infty))$ *of the form* $ax + by = (a + bxy)x \in G$.

If $\alpha = a + bx \in R(C_2 \times D_\infty) \cong \begin{pmatrix} a & b & o & 0 \\ b^* & a^* & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b^* & a^* \end{pmatrix}$ is a non-trivial unit then $a \neq 0,\ b \neq 0$.

Thus $det(\alpha) = (aa^* - bb^*)^2 \in \mathcal{U}(R)$ from Lemma 2.2. Now $(aa^* - bb^*) \in RC_\infty$. Therefore, $aa^* - bb^* \in \mathcal{U}(R)$ as we have shown earlier.

Hence $deg\ \alpha = \max\ aa^* = \max\ bb^* =\ deg\ b > 0$.

By considering special non-trivial nilpotent elements in the group ring $R(C_2 \times D_\infty)$ of the form :

$$\begin{aligned} \delta_{ij} &= (1+\mathrm{sgn}(i)t^j x)t^{|i|}(1-\mathrm{sgn}(i)t^j x) \\ &= (-t^{-|i|}+t^{|i|})+\mathrm{sgn}(i)t^j(t^{-|i|}-t^{|i|})x \text{ for } i(\neq 0),\ j\in\mathbb{Z}. \end{aligned}$$

For any $r \in R$, $i,\ j \in \mathbb{Z}$, the element $1 + r\delta_{ij}$ is a unit in $R(C_2 \times D_\infty)$. Also inverse of $1 + r\delta_{ij}$ is $1 - r\delta_{ij}$ because

$$(1+r\delta_{ij})(1-r\delta_{ij}) = 1-r^2(\delta_{ij})^2 = 1.$$

All the units of the above form generate a subgroup of the unit group of $R(C_2 \times D_\infty)$. Let

$$V = \langle 1+r\delta_{ij}\rangle_{i,j\in\mathbb{Z},r\in R}$$

For all $k > 0,\ j \in \mathbb{Z}$, define

$$V_j^k = \langle 1+r\delta_{ij}\rangle_{i<k,r\in R}$$

Obviously, the groups $V_j^1 \subseteq V_j^2 \subseteq \cdots$ is an ascending chain. We set:

$$V_j = \varinjlim_k V_j^k$$

Thus, we can get following results, that follows from [7].

**Theorem 2.9** ([7])**.** *Let $G' = \langle V, D\rangle$, where $V = \langle 1+r\delta_{ij}\rangle_{i,j\in\mathbb{Z},r\in R}$ and $D \cong (C_2 \times D_\infty) \times \mathcal{U}(\mathbb{Z}_2)$, i.e., $D$ denotes the group of trivial units of $\mathbb{Z}_2(C_2 \times D_\infty)$. Then:*

(i) $V \cong *_{j\in\mathbb{Z}} V_j \cong *_{\mathbb{Z}} \oplus_{\mathbb{N}} R^+$, *where $R^+$ denotes the additive group of the ring $R$.*

(ii) $G' = VD$.

**Corollary 2.10.** *$V$ and $G'$ are infinitely generated subgroups of $\mathcal{U}(\mathbb{Z}_2(C_2 \times D_\infty))$.*

**Theorem 2.11.** *Any $\alpha = a + bx \in \mathcal{U}(\mathbb{Z}_2(C_2 \times D_\infty))$ is in $G'$ (as above defined).*

**Corollary 2.12.** *Every unit in $\mathcal{U}(\mathbb{Z}_2(C_2 \times D_\infty))$ of the form $ay + bx = (ay + bx)y \in G'$.*

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