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FACTORIZATION NUMBERS OF FINITE ABELIAN GROUPS

M. FARROKHI D. G.

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Abstract. The number of factorizations of a finite abelian group as the product of two subgroups is computed in two different ways and a combinatorial identity involving Gaussian binomial coefficients is presented.

1. Introduction

A group G is *factorized* if $G = AB$ for some subgroups A and B of G and such an expression is called a *factorization* of G. The factorization of groups has a very long history in the theory of finite and infinite groups in such a way that how the structure of subgroups in the factorization influences the structure of the whole group. Also, it is important to know what groups have a nontrivial factorization by proper subgroups and to determine all factorizations of a given finite or infinite group (see [13] for details on factorizations of finite simple groups).

Counting the number of factorizations of groups with a finite number of factorizations is of some interesting combinatorial flavor, for if we are able to compute the factorization number of a group in two different ways, then we may obtain combinatorial identities, which is of independent interest. The number of factorizations of a group, *the factorization number*, also can be applied to compute the subgroup permutability degree of finite groups recently defined by Tărnăuceanu in [22]. If $F_2(G)$ denotes the factorization number of a group G, then the subgroup permutability degree of G is

$$spd(G) = \frac{1}{|L(G)|^2} \sum_{H \leq G} F_2(H),$$

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where $L(G)$ is the lattice of all subgroups of G. Recently, the author and Saeedi [17, 18] computed the factorization numbers of subgroups of $PSL(2,p^n)$ and used them to obtain the subgroup permutability degree of $PSL(2,p^n)$.

The aim of this paper is to obtain the factorization numbers of finite abelian groups in two different ways, from which we get also an identity involving Gaussian binomial coefficients. Since $F_2(H\times K)=F_2(H)F_2(K)$ for finite groups H and K of coprime orders, the problem of computing the factorization number of finite abelian groups reduces to just finite abelian p-groups. Hence, throughout this paper we choose a fixed prime p and all groups will be finite abelian p-groups.

In what follows, we set the following notations for a given non-increasing sequence of natural numbers $\mathcal{A}=(a_1,\ldots,a_n)$:

(i) $\mathcal{A}_i=(a_1,\ldots,a_i)$, for each $i=1,\ldots,n$;
(ii) $G_p(\mathcal{A})=\mathbb{Z}_{p^{a_1}}\times\cdots\times\mathbb{Z}_{p^{a_n}}$ is an abelian p-group of type $\mathcal{A}$;
(iii) $S(\mathcal{A})=\{(b_1,\ldots,b_m):m\leq n, b_m\leq\cdots\leq b_1 \text{ and } b_i\leq a_i, \text{ for } i=1,\ldots,m\}$, and
(iv) if $\mathcal{B}=(b_1,\ldots,b_m)\in S(\mathcal{A})$, then $T_p(\mathcal{A}:\mathcal{B})$ is the set of all m-tuples $(x_1,\ldots,x_m)$ of elements of $G_p(\mathcal{A})$ such that $\langle x_1,\ldots,x_m\rangle\cong G_p(\mathcal{B})$ and $|x_i|=p^{b_i}$, for each $i=1,\ldots,m$.

For a p-group G, the subgroup generated by all elements of order at most p^i is denoted by $\Omega_i(G)$ and the subgroup generated by all nth power of elements of G is denoted by G^n. In other words, $\Omega_i(G)=\langle x\in G:x^{p^i}=1\rangle$ and $G^n=\langle x^n:x\in G\rangle$. Also, if G is a group, then $d(G)$ stand for the minimum number of generators of G.

2. Main Results

To begin computing the factorization number of a finite abelian p-group, we first need to obtain some principal lemmas about special subsets of the group.

Lemma 2.1. *If $\mathcal{A}=(a_1,\ldots,a_n)$ is a non-increasing sequence of natural numbers and $\mathcal{B}=(b_1,\ldots,b_m)\in S(\mathcal{A})$, then*

$$|T_p(\mathcal{A}:\mathcal{B})|=\prod_{i=1}^{m}\left(p^{\mu_{b_i}(\mathcal{A})}-\frac{p^{\mu_{b_i-1}(\mathcal{A})+\mu_{b_i}(\mathcal{B}_{i-1})}}{p^{\mu_{b_i-1}(\mathcal{B}_{i-1})}}\right),$$

where $\mu_i(\mathcal{C})=i\max\{j:c_j\geq i\}+c_{\max\{j:c_j\geq i\}+1}+\cdots+c_k$ for every $\mathcal{C}=(c_1,\ldots,c_k)\in S(\mathcal{A})$.

Proof. Let $G=G_p(\mathcal{A})$ and $H=G_p(\mathcal{B})$. If $x_1,\ldots,x_m\in G$ such that $|x_i|=p^{b_i}$ and $\langle x_1,\ldots,x_m\rangle\cong H$, then we may choose x_1 to be any element of $\Omega_{b_1}(G)\setminus\Omega_{b_1-1}(G)$ and inductively x_i to be any element of $\Omega_{b_i}(G)\setminus\Omega_{b_i-1}(G)\langle x_1,\ldots,x_{i-1}\rangle$, for $i=2,\ldots,m$. Hence the number of such m-tuples $(x_1,\ldots,x_m)$ is

$$\prod_{i=1}^{m}|\Omega_{b_i}(G)\setminus\Omega_{b_i-1}(G)\langle x_1,\ldots,x_{i-1}\rangle|.$$

On the other hand,

$$\Omega_{b_i}(G)\setminus\Omega_{b_i-1}(G)\langle x_1,\ldots,x_{i-1}\rangle=\Omega_{b_i}(G)\setminus\Omega_{b_i-1}(G)\Omega_{b_i}(\langle x_1,\ldots,x_{i-1}\rangle)$$

and

$$|\Omega_{b_i-1}(G)\Omega_{b_i}(\langle x_1,\ldots,x_{i-1}\rangle)| = \frac{|\Omega_{b_i-1}(G)||\Omega_{b_i}(\langle x_1,\ldots,x_{i-1}\rangle)|}{|\Omega_{b_i-1}(\langle x_1,\ldots,x_{i-1}\rangle)|}$$

so that the number of m-tuples $(x_1,\ldots,x_m)$ is

$$\prod_{i=1}^{m}\left(|\Omega_{b_i}(G)| - \frac{|\Omega_{b_i-1}(G)||\Omega_{b_i}(\langle x_1,\ldots,x_{i-1}\rangle)|}{|\Omega_{b_i-1}(\langle x_1,\ldots,x_{i-1}\rangle)|}\right).$$

Hence the problem reduces to computing the order of $\Omega_t(K)$ for a given abelian p-group K of type $\mathcal{C}=(c_1,\ldots,c_k)$ and a positive integer t.

If $\alpha_k(\mathcal{C}) = \max\{i : c_i \geq k\}$, then $\Omega_t(K)$ is of type $(p^t,\ldots,p^t,p^{c_{\alpha_t(\mathcal{C})+1}},\ldots,p^{c_k})$ and consequently $|\Omega_t(K)| = p^{t\alpha_t(\mathcal{C})+c_{\alpha_t(\mathcal{C})+1}+\cdots+c_k} = p^{\mu_t(\mathcal{C})}$. Now, by assumption $|\langle x_1,\ldots,x_{i-1}\rangle| = p^{b_1+\cdots+b_{i-1}}$ and the number of m-tuples $(x_1,\ldots,x_m)$ is

$$\prod_{i=1}^{m}\left(p^{\mu_{b_i}(\mathcal{A})} - \frac{p^{\mu_{b_i-1}(\mathcal{A})+\mu_{b_i}(\mathcal{B}_{i-1})}}{p^{\mu_{b_i-1}(\mathcal{B}_{i-1})}}\right),$$

as required. □

The above lemma has an interesting application in the case where $\mathcal{A}=\mathcal{B}$.

Corollary 2.2. *Let G be a finite abelian p-group of type $\mathcal{A}$. Then*

(i) $|\mathrm{Aut}(G)| = |T_p(\mathcal{A}:\mathcal{A})|$, *and*

(ii) *if $\mathcal{A}=(a_1,\ldots,a_1,\ldots,a_m,\ldots,a_m)=(b_1,\ldots,b_n)$, where the number of a_i is k_i and $a_i > a_{i+1}$, then*

$$|\mathrm{Aut}(G)| = \prod_{i=1}^{m}\prod_{j=N_{i-1}+1}^{N_i}\left(p^{b_jN_i+b_{N_i+1}+\cdots+b_m} - p^{(b_j-1)N_{i'}+b_{N_{i'}+1}+\cdots+b_m+j-1}\right)$$

and in particular, the size of the Sylow p-subgroup of $\mathrm{Aut}(G)$ *is*

$$|\mathrm{Aut}(G)|_p = \prod_{i=1}^{m}\prod_{j=N_{i-1}+1}^{N_i} p^{(b_j-1)N_{i'}+b_{N_{i'}+1}+\cdots+b_m+j-1},$$

where $i' = i+1-\mathrm{Sign}(a_{i+1}-a_i+1)$ and $N_i = k_1+\cdots+k_i$, for each $i=1,\ldots,m$.

Proof. (i) Let $G=\langle x_1,\ldots,x_n\rangle$ and $|x_i| = p^{a_i}$, for each $i=1,\ldots,n$. Then, the result follows from the fact that the map $\mathrm{Aut}(G)\to T_p(\mathcal{A}:\mathcal{A})$, which sends an automorphism $\varphi\in\mathrm{Aut}(G)$ to $(\varphi(x_1),\ldots,\varphi(x_n))$ is a bijection.

(ii) The result follows by computing the values of $\mu_{b_j}(\mathcal{A})$, $\mu_{b_j-1}(\mathcal{A})$, $\mu_{b_j}(\mathcal{A}_{j-1})$ and $\mu_{b_j-1}(\mathcal{A}_{j-1})$, for each $1\leq j\leq n$. To end this, let $N_{i-1} < j \leq N_i$. Now, an easy observation shows that

$$\begin{aligned}
\mu_{b_j}(\mathcal{A}) &= b_jN_i + b_{N_i+1}+\cdots+b_n,\\
\mu_{b_j-1}(\mathcal{A}) &= (b_j-1)N_{i'} + b_{N_{i'}+1}+\cdots+b_n,\\
\mu_{b_j}(\mathcal{A}_{j-1}) &= (j-1)b_j,\\
\mu_{b_j-1}(\mathcal{A}_{j-1}) &= (j-1)(b_j-1),
\end{aligned}$$

where $i' = i+1-\mathrm{Sign}(a_{i+1}-a_i+1)$, as required. □

Note that, the order of automorphism group of a finite abelian p-group is also obtained alternatively by several authors and we may refer the reader to [3, 7, 8, 9, 19, 20] for more details.

There are various papers, which involved with the computation of certain subgroups of a given finite abelian p-group. Let G be a finite abelian p-group. Miller [14, 15, 16] gives some partial results on the the number of certain subgroups of G, say cyclic subgroups etc, Stehling [21] gives a recursive formula for the number of subgroup of a given order, and Delsarte [6], Kinosita [11] and Yeh [23] give different formulas for the number of subgroups of a given type. Also, Davies [5] computes the number of subgroups of special type with a given special factor group, where the group G has also a special type. Utilizing Lemma 2.1, we obtain an alternative formula for the number of subgroups of a given type in an arbitrary finite abelian p-group. We note that, our proof of the formula is both shorter and simpler than Delsarte's, Kinosita's and Yeh's methods.

Lemma 2.3. *The number of subgroups of type $\mathcal{B}$ of a finite abelian p-group of type $\mathcal{A}$ is*

$$\begin{bmatrix}\mathcal{A}\\ \mathcal{B}\end{bmatrix}_p = \frac{|T_p(\mathcal{A}:\mathcal{B})|}{|T_p(\mathcal{B}:\mathcal{B})|}.$$

Proof. The result is obvious by the definitions. □

Corollary 2.4. *The number of subgroups of a finite abelian p-group G of type $\mathcal{A}$ is*

$$|L(G)| = \sum_{\mathcal{B}\in S(\mathcal{A})} \begin{bmatrix}\mathcal{A}\\ \mathcal{B}\end{bmatrix}_p,$$

where $L(G)$ is the set of all subgroups of G.

Now, we are able to obtain our first formula for the factorization number of a finite abelian p-group.

Theorem 2.5. *If G is a finite abelian p-group of type $\mathcal{A}$, then*

$$F_2(G) = |L(G)|^2 - \sum_{\mathcal{B}\in S(\mathcal{A})\setminus\{\mathcal{A}\}} \begin{bmatrix}\mathcal{A}\\ \mathcal{B}\end{bmatrix}_p F_2(G_p(\mathcal{B})).$$

Proof. Since $AB \leq G$ for all subgroups A and B of G, we have

$$|L(G)|^2 = \sum_{A,B\leq G} 1 = \sum_{\substack{H=AB\\ A,B\leq H\leq G}} 1 = \sum_{H\leq G} F_2(H).$$

Now since $S(\mathcal{A})$ is the set of all types of subgroups of G, we get

$$|L(G)|^2 = \sum_{\mathcal{B}\in S(\mathcal{A})} \begin{bmatrix}\mathcal{A}\\ \mathcal{B}\end{bmatrix}_p F_2(G_p(\mathcal{B})) = F_2(G) + \sum_{\mathcal{B}\in S(\mathcal{A})\setminus\{\mathcal{A}\}} \begin{bmatrix}\mathcal{A}\\ \mathcal{B}\end{bmatrix}_p F_2(G_p(\mathcal{B})),$$

from which the result follows. □

It is worth noting that, we may obtain a recursive formula for the number of factorizations of a finite abelian p-group into k subgroups in the same way as in the proof of Theorem 2.5.

To get the next formula for the factorization number of a given finite abelian p-group G, we use the fact that if $G = CD$, then $G^p = C^pD^p$. In fact, we take an ordered pair of subgroups (A, B) of G^p such that $G^p = AB$ and we shall count the number of ordered pairs of subgroups (C, D) of G with $G = CD$ and $(C^p, D^p) = (A, B)$. In what follows, we set the following notations.

For any real number $q > 0$ and integer n, the numbers $[n]_q$ and $[n]_q!$ denote the *q-number* and *q-factorial* defined by

$$[n]_q = \frac{q^n - 1}{q - 1} \text{ and } [n]_q! = [n]_q[n-1]_q \dots [2]_q[1]_q,$$

respectively. Moreover, the *Gaussian binomial coefficients* are defined in terms of q-factorials by

$$\begin{bmatrix} n \\ i \end{bmatrix}_q = \frac{[n]_q!}{[i]_q![n-i]_q!} = \frac{(q^n - 1)\dots(q-1)}{(q^i - 1)\dots(q-1)(q^{n-i} - 1)\dots(q-1)}$$

and as usual the *Gaussian polynomial coefficients* are defined by

$$\begin{bmatrix} n \\ i_1, \dots, i_k \end{bmatrix}_q = \frac{[n]_q!}{[i_1]_q! \dots [i_k]_q![n - i_1 - \dots - i_k]_q!},$$

where $0 \leq i_1 + \dots + i_k \leq n$. Given a prime power q, the Gaussian binomial coefficient $\begin{bmatrix} n \\ i \end{bmatrix}_q$ is the number of subspaces of dimension i in a vector space of dimension n over the field of order q. We refer the interested reader to [1, 2, 4, 10, 12] for more details on q-numbers and related topics. As we have seen before, Lemma 2.3 generalizes the Gaussian binomial coefficients as the number of subgroups of a given type of a finite abelian p-group.

We begin with two principal lemmas.

Lemma 2.6. *Let G be an elementary abelian p-group and $X \leq G$. Then, the number of subgroups Y of G of order p^n ($n \leq d(G) - d(X)$) such that $X \cap Y = 1$ is*

$$p^{nd(X)} \begin{bmatrix} d(G) - d(X) \\ n \end{bmatrix}_p.$$

Proof. To count the number of n-tuples $(y_1, \dots, y_n)$ of elements of G such that $\langle y_1, \dots, y_n \rangle$ is a subgroup of order p^n intersecting trivially with X, we may choose $y_1 \in G \setminus X$ and inductively $y_i \in G \setminus \langle X, y_1, \dots, y_{i-1} \rangle$, for every $i = 2, \dots, n$. On the other hand, to count the number of n-tuples $(z_1, \dots, z_n)$ generating a given subgroup $Y = \langle y_1, \dots, y_n \rangle$ of order p^n, we may choose $z_1 \in Y \setminus \{1\}$ and inductively $z_i \in Y \setminus \langle z_1, \dots, z_{i-1} \rangle$, for every $i = 2, \dots, n$. Hence, the number of subgroups Y is

$$\begin{aligned} \prod_{i=0}^{n-1} \frac{p^{d(G)} - p^{d(X)+i}}{p^n - p^i} &= \prod_{i=0}^{n-1} \frac{p^{d(X)+i}}{p^i} \cdot \frac{p^{d(G)-d(X)-i} - 1}{p^{n-i} - 1} \\ &= p^{nd(X)} \begin{bmatrix} d(G) - d(X) \\ n \end{bmatrix}_p, \end{aligned}$$

as required. □

Lemma 2.7. *Let G be an elementary abelian p-group and $X \leq Y \leq G$. Then, the number of subgroups Z of G of order $p^{d(G)-d(Y)+n}$ ($n \leq d(Y) - d(X)$) such that $X \cap Z = 1$ and $YZ = G$ is*

$$p^{nd(X)+(d(Y)-n)(d(G)-d(Y))} \begin{bmatrix} d(Y) - d(X) \\ n \end{bmatrix}_p.$$

Proof. Let $N \leq X$ be a subgroup of order p^n. If $Z \leq G$ such that $YZ = G$, then $|Y \cap Z| = p^n$ and the number of such Z equals to the product of the number of subgroups Z_1 of Y such that $|Z_1| = p^n$ and $X \cap Z_1 = 1$ by the number of subgroups Z_2/N of G/N such that $X/N \cdot Z_2/N = G/N$ and $X/N \cap Z_2/N = 1_{G/N}$. By Lemma 2.6, the first and second numbers are

$$p^{nd(X)} \begin{bmatrix} d(Y) - d(X) \\ n \end{bmatrix}_p$$

and

$$p^{(d(G)-d(Y))(d(Y)-n)} \begin{bmatrix} d(G) - d(Y) \\ d(G) - d(Y) \end{bmatrix}_p = p^{(d(Y)-n)(d(G)-d(Y))},$$

respectively. Therefore, the number of subgroups Z is

$$p^{nd(X)+(d(Y)-n)(d(G)-d(Y))} \begin{bmatrix} d(Y) - d(X) \\ n \end{bmatrix}_p$$

and the proof is complete. □

Utilizing the above lemmas, we can obtain our second formula for the factorization number of a finite abelian p-group.

Theorem 2.8. *Let G be a finite abelian p-group. Then*

$$F_2(G) = \sum_{G^p=AB} \left[\frac{|\Omega_1(G^p)|^{2n+d(A)+d(B)}}{|\Omega_1(A)|^{d(A)}|\Omega_1(B)|^{d(B)}} \sum_{0\leq i+j\leq n} \frac{p^{id(A)+jd(B)}}{|\Omega_1(G^p)|^{i+j}} \cdot p^{ij} \begin{bmatrix} n \\ i,j \end{bmatrix}_p \right],$$

where $n = d(\Omega_1(G)) - d(\Omega_1(G^p))$.

Proof. First we note that if $X \leq G^p$, then $X = Y^p$ for some subgroup Y of G. In particular, if $X = \langle x_1 \rangle \times \cdots \times \langle x_m \rangle$, then $Y = \langle y_1 \rangle \times \cdots \times \langle y_m \rangle \times U$, where U is an elementary abelian p-subgroup such that $U \cap G^p = 1$ and $y_i^p = x_i$, for $i = 1, \ldots, m$. Now, if z_i is any element with $z_i^p = x_i$, then $(z_i y_i^{-1})^p = 1$ that is $z_i y_i^{-1} \in \Omega_1(G)$. Let $Z = \langle z_1 \rangle \times \cdots \times \langle z_m \rangle \times V$ be another subgroup of G with $Z^p = X$. Then, we may assume that $z_i = w_i y_i$ for some $w_i \in \Omega_1(G)$. Suppose that $Y = Z$. Then $|U| = |V|$ and $\Omega_1(X)U = \Omega_1(X)V$. Since $y_1 \in Z$, there exist integers $\alpha_1, \ldots, \alpha_m$ and element $v \in V$ such that $y_1 = (w_1 y_1)^{\alpha_1} \ldots (w_m y_m)^{\alpha_m} v$. Thus $x_1 = x_1^{\alpha_1} \ldots x_m^{\alpha_m}$, which implies that $\alpha_1 \equiv 1 \pmod{|x_1|}$ that is $\alpha_1 = 1 + |x_1| t_1$ for some integer t_1. Hence $w_1^{-1} = y_1^{|x_1|t_1} (w_2 y_2)^{\alpha_2} \ldots (w_m y_m)^{\alpha_m} v$ and consequently $|x_i| | \alpha_i$, for each $i = 2, \ldots, m$, say $\alpha_i = |x_i| t_i$. Therefore $w_1 = y_1^{|x_1|t_1} \ldots y_m^{|x_m|t_m} v \in \Omega_1(Y)$ and similarly $w_i \in \Omega_1(Y)$, for each $i = 2, \ldots, m$. Conversely, it can be easily seen that if $\Omega_1(X)U = \Omega_1(X)V$ and $w_i \in \Omega_1(Y)$, then the subgroups Y and Z are equal.

Now let $G^p = AB$ be a factorization of G^p and $n = \alpha(G) - \beta(G)$, where $\alpha(G) = d(\Omega_1(G))$ and $\beta(G) = d(\Omega_1(G^p))$. Then, by Lemmas 2.6 and 2.7, the number of factorizations $G = CD$ with $(C^p, D^p) = (A, B)$ such that $[\Omega_1(C) : \Omega_1(A)] = p^i$ and $[\Omega_1(D) : \Omega_1(B)] = p^j$ $(j = n - i + k)$ is

$$\frac{|\Omega_1(G)|^{d(A)}}{|\Omega_1(C)|^{d(A)}} \cdot \frac{|\Omega_1(G)|^{d(B)}}{|\Omega_1(D)|^{d(B)}} \cdot p^{i\beta(G)} \begin{bmatrix} n \\ i \end{bmatrix}_p \cdot p^{k\beta(G)+(n-i)(\beta(G)+i-k)} \begin{bmatrix} i \\ k \end{bmatrix}_p,$$

which simplifies to

$$\frac{|\Omega_1(G)|^{d(A)+d(B)}|\Omega_1(G^p)|^{i+j}}{|\Omega_1(A)|^{d(A)}|\Omega_1(B)|^{d(B)}} \cdot \frac{p^{(n-i)(n-j)}}{p^{id(A)+jd(B)}} \begin{bmatrix} n \\ i \end{bmatrix}_p \begin{bmatrix} i \\ n-j \end{bmatrix}_p. \tag{2.1}$$

Now, by putting $i' = n-i$ and $j' = n-j$ in (2.1) and much more simplification, it yields

$$\frac{|\Omega_1(G^p)|^{2n+d(A)+d(B)}}{|\Omega_1(A)|^{d(A)}|\Omega_1(B)|^{d(B)}} \cdot \frac{p^{i'd(A)+j'd(B)}}{|\Omega_1(G^p)|^{i'+j'}} \cdot p^{i'j'} \begin{bmatrix} n \\ i', j' \end{bmatrix}_p$$

factorizations $G = CD$ with the given conditions, from which the result follows. □

Corollary 2.9. *If (V, F) is a vector space of dimention n over a finite field F of order q, then the number of factorizations of V as the sum of two subspaces is*

$$F_2(V) = \sum_{0 \leq i+j \leq n} q^{ij} \begin{bmatrix} n \\ i, j \end{bmatrix}_q.$$

Proof. If $q = p$ is a prime, then V can be identified with the elementary abelian p-group of order p^n, and the result follows by Theorem 2.8. Now if q is any prime power, then by substituting groups by vector spaces and subgroups by subspaces in Lemmas 2.6 and 2.7, one can reprove Theorem 2.8 for vector spaces, from which the result follows. □

Once we combine Theorems 2.5 and 2.8 in the special case of elementary abelian p-groups, we will obtain the following combinatorial identity for Gaussian binomial coefficients, which is of independent interest. Note that, the Gaussian binomial coefficients occur in the theory of partitions and counting of symmetric polynomials.

Corollary 2.10. *For each natural number n and prime power q, we have*

$$\left(\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q \right)^2 = \sum_{0 \leq i+j+k \leq n} q^{ij} \begin{bmatrix} n \\ i, j, k \end{bmatrix}_q.$$

Proof. Since the both sides of the given identity are polynomials in q, it is enough to show that the result holds when q is a prime. We know from the proof of Theorem 2.5 that for an elementary abelian p-group of order p^n

$$\left(\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_p \right)^2 = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_p F_2(\mathbb{Z}_p^{n-i}). \tag{2.2}$$

Also, by Corollary 2.9, for an elementary abelian p-group of order p^m, we have

$$F_2(\mathbb{Z}_p^m) = \sum_{0 \leq j+k \leq m} p^{jk} \begin{bmatrix} m \\ j, k \end{bmatrix}_p.$$

Hence

$$\begin{aligned} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_p F_2(\mathbb{Z}_p^{n-i}) &= \sum_{i=0}^{n} \sum_{0 \leq j+k \leq n-i} p^{jk} \begin{bmatrix} n \\ i \end{bmatrix}_p \begin{bmatrix} n-i \\ j, k \end{bmatrix}_p \\ &= \sum_{0 \leq i+j+k \leq n} p^{jk} \begin{bmatrix} n \\ i, j, k \end{bmatrix}_p, \end{aligned}$$

from which the result follows. □

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Mohammad Farrokhi Derakhshandeh Ghouchan

Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O.Box 1159, 91775, Mashhad, Iran

Email: `m.farrokhi.d.g@gmail.com`