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CHARACTERIZATION OF A_5 AND $PSL(2,7)$ BY SUM OF ELEMENT ORDERS

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Abstract. Let G be a finite group. We denote by $\psi(G)$ the integer $\sum_{g\in G} o(g)$, where $o(g)$ denotes the order of $g \in G$. Here we show that $\psi(A_5) < \psi(G)$ for every non-simple group G of order 60, where A_5 is the alternating group of degree 5. Also we prove that $\psi(PSL(2,7)) < \psi(G)$ for all non-simple groups G of order 168. These two results confirm the conjecture posed in [J. Algebra Appl., **10** No. 2 (2011) 187-190] for simple groups A_5 and $PSL(2,7)$.

1. Introduction

Let G be a finite group. We define the function

$$\psi(G) = \sum_{g\in G} o(g),$$

where $o(g)$ denotes the order of $g \in G$. We propose the following general question:

Question 1.1. *What information about a group G can be obtained from $\psi(G)$ and $|G|$?*

The starting point on studying the function ψ is in [1], where the maximum value of ψ on the groups of the same order is investigated. In fact it is proved that

Theorem 1.2. *Let C be a cyclic group of order n. Then $\psi(G) < \psi(C)$ for all non-cyclic groups of order n.*

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It follows that the cyclic groups are determined by their orders and sum of element orders.

In general, the invariants $|G|$ and $\psi(G)$ do not determine G. For example, there are two non-isomorphic groups G_1 and G_2 of order 27 such that $\psi(G_1) = \psi(G_2)$.

Note that the function ψ is multiplicative, that is if G_1 and G_2 are two groups satisfying $\gcd(|G_1|, |G_2|) = 1$, then $\psi(G_1 \times G_2) = \psi(G_1)\psi(G_2)$.

Following Theorem 1.1, one can ask about the structure of groups having the minimum sum of element orders on all groups of the same order. In [2] it is proved that:

Theorem 1.3. *Let G be a nilpotent group of order n. Then $\psi(G) \leq \psi(H)$ for every nilpotent group H of order n if and only if each Sylow subgroup of G is of prime exponent.*

Theorem 1.4. *Let n be a positive integer such that there exists a non-nilpotent group of order n. Then there exists a non-nilpotent group K of order n with the property that $\psi(K) < \psi(H)$ for every nilpotent group H of order n.*

In other words the minimum value of $\psi(G)$ occurs on a non-nilpotent group, where G varies on all groups of the same order.

Also it is conjectured in [2] that:

Conjecture 1.5. *Let S be a simple group. If G is a non-simple group of order $|S|$, then $\psi(S) < \psi(G)$.*

In other words if n is a natural number such that there is a simple group of order n, then the minimum of ψ, on all groups of order n, occurs in a simple group. Here we confirm Conjecture 1.5 for A_5, the alternating group of degree 5, and $PSL(2, 7)$, the projective special linear group of 2×2 matrices over the field of order 7. It is hoped that the methods be useful for some other simple groups. Note that we determine A_5 and $PSL(2, 7)$ by their orders and sum of element orders.

Most of our notations are standard. If p is a prime, then $n_p = n_p(G)$ denotes the number of Sylow p-subgroups of G and the set of all Sylow p-subgroups of G is denoted by $Syl_p(G)$. If n is a positive integer, then C_n is a cyclic group of order n.

2. Minimum of ψ on all groups of order 60

It is well-known that A_5 has 15 elements of order 2, 20 elements of order 3 and 24 elements of order 5. Therefore $\psi(A_5) = 211$.

Theorem 2.1. *Let G be any group of order 60. Then $\psi(G) \geq 211$ and $\psi(G) = 211$ if and only if $G \cong A_5$.*

Proof. Since the order of G is 60, the number of Sylow 5-subgroups is 1 or 6 and $n_3 = 1$ or 10. If $n_3 = 1$ or $n_5 = 1$, then G contains a cyclic subgroup of order 15. Thus G contains at least 8 elements of order 15. Also G contains at least four elements of order 5 and at least two elements of order 3. So G contains at most 45 elements of order at least 2. Hence

$$\psi(G) \geq 1 + 45(2) + 2(3) + 4(5) + 8(15) = 236.$$

So we may assume that $n_3 = 10$ and $n_5 = 6$. Then G contains 20 elements of order 3 and 24 elements of order 5. If $I = \{x \in G | o(x) = 3 \text{ or } 5\}$, then $|I| = 44$. Therefore if there exits an element of order greater than 2 in $G \setminus I$, then $\psi(G) > 211$. So suppose that each non-identity element of $G \setminus I$ has order 2 which implies that $|C_G(x)| = 4$ for every element of order 2. It yields that the intersection of two distinct Sylow 2-subgroups of G is trivial and so $n_2 = 5$. Hence G is isomorphic to a subgroup of S_5, the symmetric group on 5 letters. It follows that $G \cong A_5$. This completes the proof. □

Corollary 2.2. *If G is a non-simple group of order 60, then $\psi(G) > \psi(A_5)$.*

3. Minimum of ψ on all groups of order 168

It is easy to check that $PSL(2, 7)$ has 21 elements of order 2 and 42 elements of order 4, since $PSL(2, 7)$ has 21 Sylow 2-subgroups isomorphic to D_8. Also $PSL(2, 7)$ contains 56 elements of order 3 and 48 elements of order 7 because $n_3 = 28$ and $n_7 = 8$. Therefore $\psi(PSL(2, 7)) = 715$.

Lemma 3.1. *Let G be a group of order 168. If G contains an element of order 21, then $\psi(G) > 715$.*

Proof. Suppose first that G contains at least two cyclic subgroups of order 21. Then G has at least $2\phi(21) = 24$ elements of order 21 and so G contains at most 144 elements which are not of order 21. Therefore $\psi(G) \geq 24(21) + 143(2) + 1 = 791 > 715$.

Now suppose that G contains unique cyclic subgroup T of order 21. Then $n_7(G) = 1$ and $n_3(G) = 1$. It follows that

$$\frac{G}{C_G(P)} \hookrightarrow Aut(C_7),$$

where P is the Sylow 7-subgroup of G. Therefore $|C_G(P)| = 2^3 \cdot 3 \cdot 7$ or $2^2 \cdot 3 \cdot 7$. In the former case P is a central subgroup of G. It follows from [1, Corollary B] that $\psi(G) = \psi(\frac{G}{P})\psi(P) > 715$. If $|C_G(P)| = 2^2 \cdot 3 \cdot 7$, P is central in $C_G(P)$ and so $\psi(C_G(P)) = 43(24) > 715$, as desired. □

Lemma 3.2. *Let G be a group of order 168. If G contains no element of order 21 and $n_7 = 8$, then either $\psi(G) > 715$ or $G \cong PSL(2, 7)$.*

Proof. By hypothesis, $K = N_G(P) = QP$, where $P \in Syl_7(G)$ and $Q \in Syl_3(G)$. Since $n_3(K) = 7$, $n_3(G) \geq 7$ and so $n_3 = 7$ or 28.

If the intersection of two conjugates K_1 and K_2 of K is trivial, then $|K_1K_2| > 168$, a contradiction. Therefore the intersection of any two conjugates of K has order 3 and since K has eight conjugates, $n_3 = 28$. This implies that $N_G(Q) \cong S_3$ or C_6.

If $N_G(Q) \cong C_6$, then G contains 56 elements of order 6 and so

$$\psi(G) > 48(7) + 56(3) + 56(6) + 1 > 715.$$

If $N_G(Q) \cong S_3$, then $C_G(Q) = Q$ and so the centralizer subgroup of any p-element is a p-subgroup for any prime p. If $n_2 \leq 7$, then G contains at most 49 non-identity 2-elements. Since $|G| = 168$, there exists an element of G which is not a p-element, a contradiction. Hence $n_2 = 21$. If the intersection of any two distinct Sylow 2-subgroups is trivial, then $|G| > 168$. Therefore there exists $x \in T_1 \cap T_2$, where $T_i \in Syl_2(G)$ for $i = 1, 2$. If T_i is abelian then $|C_G(x)| > 8$, which is a contradiction. So T_i is not abelian. It follows that $T_1 \cong D_8$ or Q_8. If $o(x) = 4$, then $C_G(x) = \langle x \rangle$ and so x has 42 conjugates in G. Now G has 48 elements of order 7, 56 elements of order 3 and 42 elements of order 4. This implies that G must have 21 elements of order 2. Thus in the latter case G has no non-trivial minimal normal subgroup and so G is simple. Therefore $G \cong PSL(2, 7)$. This completes the proof. □

Lemma 3.3. *Let G be a group of order 168. If G contains no element of order 21 and $n_7 = 1$, then $\psi(G) > 715$.*

Proof. Suppose that P is the unique Sylow 7-subgroup of G. Then $|C_G(P)| = 2^3 \cdot 7$ or $2^2 \cdot 7$.

If $|C_G(P)| = 2^3 \cdot 7$, then $C_G(P) = P \times T$, where $T \in Syl_2(G)$. It follows that $n_2(G) = 1$, since $C_G(P)$ is normal in G. It yields that $\psi(C_G(P)) \geq 5 \cdot 43 = 645$. Since the order of each element in $G \backslash C_G(P)$ is at least 3 and $|G \backslash C_G(P)| = 112$, we have $\psi(G) \geq 645 + 112(3) > 715$.

Now suppose that $|C_G(P)| = 2^2 \cdot 7$. Then $C_G(P) = D \times P$, where $|D| = 4$. Since $C_G(P)$ is normal in G and D is characteristic in $C_G(P)$, D is normal in G. So D is the intersection of all Sylow 2-subgroups of G.

This is clear that $n_3 = 7$ or 28 and $n_2 = 7$ or 21. Set

$$E = \{x \in G | x \text{ is a non-identity 2-element}\},$$

and

$$S = \{x \in G | x \text{ is a non-identity 3-element}\}.$$

It follows that $|E| = 31$ or 87 and $|S| = 14$ or 56. Note that if $x \in G \backslash (E \cup S \cup P)$, then either $o(x) = 6$ or $o(x) \geq 12$. Also if $o(x) = 6$, then G contains $2n_3(G)$ elements of order 6.

If $D \cong C_4$, then $\psi(C_G(P)) = 473$. Since $|G \setminus C_G(P)| = 140$, $\psi(G) \geq 473 + 140(2) > 715$.

If $D \cong C_2 \times C_2$, then G has 18 elements of order 14. Now we consider four following cases:

1- If $|E| = 31$ and $|S| = 14$, then

$$\psi(G) \geq 31(2) + 14(3) + 6(7) + |G \setminus (E \cup S \cup P)|6 > 715.$$

2- If $|E| = 87$ and $|S| = 14$, then

$$\psi(G) \geq 87(2) + 14(3) + 6(7) + |G \setminus (E \cup S \cup P)|6 > 715.$$

3- If $|E| = 31$ and$|S| = 56$, then

$$\psi(G) \geq 31(2) + 56(3) + 6(7) + |G \setminus (E \cup S \cup P)|6 > 715.$$

4- If $|E| = 87$ and $|S| = 56$, then G has 21 Sylow 2-subgroups. Suppose that $T \in Syl_2(G)$ and $T \cong C_2 \times C_2 \times C_2$. If $x \in D$, then $C_G(x) = TP$, since $|E \cup S \cup P| = 150$ and G has 18 elements of order 14. Therefore $n_2(C_G(x)) = 7$. This is a contradiction because $C_G(x)$ contains all 21 Sylow 2-subgroups of G. Hence T is not isomorphic to $(C_2)^3$. Thus each Sylow 2-subgroup of G contains at least one cyclic subgroup of order 4. Since D is isomorphic to $C_2 \times C_2$, the intersection of any two distinct Sylow 2-subgroups of G does not have any element of order 4. It follows that G contains 21 cyclic subgroups of order 4. Thus

$$\psi(G) \geq 87(2) + 56(3) + 18(14) + 42(4) > 715.$$

This completes the proof. □

Theorem 3.4. *Let G be any group of order 168. Then $\psi(G) \geq 715$.*

Proof. It follows from Lemmas 3.1, 3.2 and 3.3. □

Corollary 3.5. *Let G be any non-simple group of order* 168. *Then $\psi(G) > \psi(PSL(2,7))$.*

Proof. If G satisfies the the hypothesis of Lemmas 3.1 or 3.3, then the result holds. If G satisfies the hypothesis of Lemma 3.2, then $\psi(G) > 715$, since G is not simple. □

References

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