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CERTAIN FINITE ABELIAN GROUPS WITH THE RÉDEI k-PROPERTY

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Abstract. Three infinite families of finite abelian groups will be described such that each member of these families has the Rédei k-property for many non-trivial values of k.

1. **Introduction**

Let G be a finite abelian group written multiplicatively. The identity element of G will be denoted by e. For the subsets $A_1, \ldots, A_k$ of G we define the product $A_1 \cdots A_k$ to be the subset

$$\{a_1 \cdots a_k \ : \ a_1 \in A_1, \ldots, a_k \in A_k\}$$

of G. The product $A_1 \cdots A_k$ is direct if

$$a_1 \cdots a_k = a'_1 \cdots a'_k \text{ and } a_1, a'_1 \in A_1, \ldots, a_k, a'_k \in A_k$$

imply $a_1 = a'_1, \ldots, a_k = a'_k$. If G is the direct product of its subsets $A_1, \ldots, A_k$, then we say that the equation $G = A_1 \cdots A_k$ is a factorization of G. It is clear that $G = A_1 \cdots A_n$ is a factorization of G if and only if $G = A_1 \cdots A_k$ and $|G| = |A_1| \cdots |A_k|$ hold. A subset A of G is called normalized if $e \in A$. A factorization is called normalized if each of its factors is a normalized subset. A subset A of G is defined to periodic if there is a $g \in G \setminus \{e\}$ for which $gA = A$. A factorization is defined to be periodic if at least one its factors is periodic. A finite abelian group possesses the Hajós k-property if it admits only periodic factorizations into k factors. For a subset A of G the notation $\langle A \rangle$ stands for the smallest subgroup of G that contains A, that is, $\langle A \rangle$ denotes the span of A in G. A normalized subset A of G is called a full-rank subset if $\langle A \rangle = G$. In other words the normalized subset A of G is a full-rank

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subset of G if it spans the whole G. The normalized subset A of G is not a full-rank subset of G if A is contained in a proper subgroup of G. A normalized factorization is called a full-rank factorization if each of its factors is a full-rank subset. A finite abelian group possesses the Rédei k-property if it does not admit any full-rank normalized factorization into k factors. The concept of Hajós k-property was introduced in [2] and the concept of Rédei k-property was introduced in [1]. In [1] it was shown that the Hajós k-property implies the Rédei k-property.

In this short note we will show that certain large families of finite abelian groups have the Rédei k-property for certain non-trivial values of k.

Let n be the number of the not necessarily distinct prime divisors of $|G|$. In other words $|G|$ is equal to the product of n not necessarily distinct primes. Let $k > n$ and consider a normalized factorization $G = A_1 \cdots A_k$ of G. From $|G| = |A_1| \cdots |A_k|$ it follows that $|A_i| = 1$ must hold for some i, $1 \leq i \leq k$. Therefore $A_i = \{e\}$ and so $\langle A_i \rangle \neq G$. Thus G has the Rédei k-property for each $k > n$. According to a famous theorem of L. Rédei [3] if $G = A_1 \cdots A_k$ is a normalized factorization of G such that $|A_i|$ is a prime for each i, $1 \leq i \leq k$, then at least one of the factors is a subgroup of G. As a consequence each finite abelian group G has the Rédei k-property for $k = n$.

It is clear that for $k = 1$ no finite abelian group can possess the Rédei k-property. The non-trivial values of k for the Rédei k-property are those for which $2 \leq k \leq n - 1$. We will describe three infinite families of finite abelian groups such that if G is a member of one of these families, then G has the Rédei k-property for each k with $2k > n$. The families of these finite abelian groups are the following.

(i) G is a finite cyclic group.

(ii) G is a finite abelian group such that the p-component of G is the direct product of at most three cyclic groups and $p \leq 13$ for each prime divisor p of $|G|$.

(iii) G is a finite abelian group for which $p \leq 7$ for each prime divisor p of $|G|$.

2. **The results**

First we prove a result on factorizations containing only two factors.

Theorem 2.1. *Let G be a finite abelian group whose type is one of* (i), (ii), (iii). *Let $G = AB$ be a normalized factorization of G such that $|A|$ is a prime. Then either $\langle A \rangle \neq G$ or $\langle B \rangle \neq G$. In other words the factorization $G = AB$ is not a full-rank factorization.*

Proof. Let n be the number of the not necessarily distinct prime divisors of $|G|$. Let us denote the prime $|A|$ by p. The factorization $G = AB$ implies that $|G| = |A||B|$. In the special case $n = 1$ it follows that $|G| = |A| = p$ and $|B| = 1$. Therefore $G = A$ and $B = \{e\}$. Plainly $\langle B \rangle \neq G$ and so for the special case $n = 1$ the theorem is proved. For the remaining part of the proof we assume that $n \geq 2$ and we start an induction on n.

Let us consider a normalized factorization $G = AB$ such that $|A| = p$. If $\langle A \rangle \neq G$ or $\langle B \rangle \neq G$, then there is nothing to prove and so we assume that $\langle A \rangle = G$ and $\langle B \rangle = G$. Choose an element $a \in A \setminus \{e\}$. By Lemma 3 of [6], in the factorization $G = AB$ the factor A can be replaced by

$C = \{e, a, a^2, \ldots, a^{p-1}\}$ to get the normalized factorization $G = CB$. From the factorization $G = CB$, by the proof of Lemma 1 of [5], one can read off that $a^p B = B$. This means that B is a periodic subset of G unless $a^p = e$.

If $a^p \neq e$, then B is periodic. There is a subgroup H of G and there is a normalized subset D of G such that the product DH is direct and $DH = B$. We get the normalized factorization $G = AB = ADH$ of G. From this factorization by considering the factor group G/H we get the normalized factorization

$$G/H = [(AH)/H][(DH)/H]$$

of the factor group G/H, where

$$(AH)/H = \{aH \ : \ a \in A\},\ (DH)/H = \{dH \ : \ d \in D\}.$$

Clearly G/H is a finite cyclic group and its order is the product of less than n primes.

By the inductive hypotheses it follows that either $(AH)/H$ or $(DH)/H$ does not span the whole factor group G/H. We claim that on the other hand from the assumptions $\langle A \rangle = G$ and $\langle B \rangle = G$ it follows that both of $(AH)/H$ and $(DH)/H$ span the whole factor group G/H.

To verify the claim we argue in the following way. Since $G = \langle B \rangle = \langle DH \rangle$ for each $g \in G$ there are elements $d_1, \ldots, d_r \in D$ and $h_1, \ldots, h_r \in H$ further there are integers $\alpha(1), \ldots, \alpha(r)$ such that

$$g = (d_1 h_1)^{\alpha(1)} \cdots (d_r h_r)^{\alpha(r)}.$$

From this it follows that

$$\begin{aligned} gH &= [(d_1 h_1)^{\alpha(1)} \cdots (d_r h_r)^{\alpha(r)}]H \\ &= [(d_1 h_1)^{\alpha(1)}]H \cdots [(d_r h_r)^{\alpha(r)}]H \\ &= (d_1 h_1 H)^{\alpha(1)} \cdots (d_r h_r H)^{\alpha(r)} \\ &= (d_1 H)^{\alpha(1)} \cdots (d_r H)^{\alpha(r)}. \end{aligned}$$

This means that each element gH of G/H can be represented in the form $(d_1 H)^{\alpha(1)} \cdots (d_r H)^{\alpha(r)}$ and so $(DH)/H$ spans the whole G/H. The fact that $(AH)/H$ spans the whole G/H can be proved in a similar way.

We may summarize the above argument by saying that the assumption $a^p \neq e$ leads to a contradiction and therefore $a^p = e$ must hold for each $a \in A \setminus \{e\}$. It follows that $|a| = p$ for each $a \in A \setminus \{e\}$. Thus $\langle A \rangle$ is an elementary p-group. As $\langle A \rangle = G$ we get that G is an elementary p-group.

Let us consider first the case when G is a type (i) group, that is, consider the case when G is a finite cyclic group. As G is an elementary p-group, it follows that G is a cyclic group of order p. This means that $|G|$ has only one prime divisor and so $n = 1$. This contradicts to our assumption that $n \geq 2$.

Let us consider next the case when G is a type (ii) group. As G is an elementary p-group and using the fact the p-component of G is a direct product of at most three cyclic groups we get that $|G| \leq p^3$. If $|G| = p$, then we get a contradiction as before. If $|G| = p^2$, then in the normalized factorization $G = AB$ clearly $|A| = |B| = p$ hold. Consequently by Rédei's theorem either A or B is a subgroup of G. Thus either $\langle A \rangle \neq G$ or $\langle B \rangle \neq G$. If $|G| = p^3$, then in the normalized factorization $G = AB$ clearly

$|A| = p$, $|B| = p^2$ must hold. Using the assumption that $p \leq 13$, by Theorem 1 of [7], it follows that $\langle B \rangle \neq G$.

Finally, let us turn to the case when G is a type (iii) group. Using the facts that G is an elementary p-group and $p \leq 7$, by Theorem 2 of [8], it follows that $\langle B \rangle \neq G$. □

After these preparations we are ready to prove the main result of this note.

Theorem 2.2. *Let G be a finite abelian group belonging to one of the families* (i), (ii), (iii) *and let $G = A_1 \cdots A_k$ be a normalized factorization of G. If $|G|$ is the product of n not necessarily distinct primes and $2k > n$, then $\langle A_i \rangle \neq G$ for some i, $1 \leq i \leq k$.*

Proof. Assume on the contrary that $\langle A_i \rangle = G$ for each i, $1 \leq i \leq k$. Let $|A_i|$ be the product of n_i not necessarily distinct primes. We claim that $n_i \leq 1$ for some i, $1 \leq i \leq k$. In order to prove the claim assume on the contrary that $n_i \geq 2$ for each i, $1 \leq i \leq k$.

The factorization $G = A_1 \cdots A_k$ implies $|G| = |A_1| \cdots |A_k|$ and so it follows that $n = n_1 + \cdots + n_k$. Using $n_i \geq 2$ we get $n \geq 2 + \cdots + 2 = 2k$ which in turn contradicts the assumption $2k > n$. Thus there is an n_i with $n_i \leq 1$.

We may assume that $n_1 \leq 1$ since this is only a matter of rearranging the factors $A_1, \ldots, A_k$. Set $A = A_1$ and set $B = A_2 \cdots A_k$. If $n_1 = 0$, then $A = \{e\}$. In this case clearly $\langle A_1 \rangle \neq G$. If $n_1 = 1$, then from the normalized factorization $G = AB$, by Theorem 2.1, it follows that either $\langle A \rangle \neq G$ or $\langle B \rangle \neq G$.

By the indirect assumption $\langle A_1 \rangle = G$ and so $\langle A \rangle = G$. Thus we are left with the case $\langle B \rangle \neq G$ to consider. Again by the indirect assumption $\langle A_2 \rangle = G$ must hold. Since the factors $A_2, \ldots, A_k$ are normalized subsets of G it follows that $A_2 \subseteq A_2 \cdots A_k = B$. We get that $G = \langle A_2 \rangle \subseteq \langle B \rangle \subseteq G$. Thus $\langle B \rangle = G$. Therefore the possibility $\langle B \rangle \neq G$ is ruled out as well. This final contradiction completes the proof. □

3. An open problem

Many years ago in a conversation K. Corrádi proposed the following problem.

Problem 3.1. *Let p be a prime and let s be an integer $3 \leq s \leq p-1$. Further let G be an elementary p-group of order p^s. Suppose that $G = AB$ is a normalized factorization of G, where $|A| = p$ and suppose that $\langle A \rangle = G$. Does it follow that $\langle B \rangle \neq G$?*

The problem is motivated by Problem 5 of [4]. If the answer for Problem 3.1 is in the affirmative, then Theorem 2.1 and consequently Theorem 2.2 hold for each finite abelian group not only for the special classes of groups we specified.

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