CHARACTERIZATION OF THE SYMMETRIC GROUP BY ITS NON-COMMUTING GRAPH

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Abstract. The non-commuting graph \( \nabla(G) \) of a non-abelian group \( G \) is defined as follows: its vertex set is \( G - Z(G) \) and two distinct vertices \( x \) and \( y \) are joined by an edge if and only if the commutator of \( x \) and \( y \) is not the identity. In this paper we prove that if \( G \) is a finite group with \( \nabla(G) \cong \nabla(S_n) \), then \( G \cong S_n \), where \( S_n \) is the symmetric group of degree \( n \), where \( n \) is a natural number.

1. Introduction

Let \( G \) be a group. The non-commuting graph \( \nabla(G) \) of \( G \) is defined as follows: the set of vertices of \( \nabla(G) \) is \( G - Z(G) \), where \( Z(G) \) is the center of \( G \) and two vertices are connected whenever they do not commute. Also we define the prime graph \( \Gamma(G) \) of \( G \) as follows: the vertices of \( \Gamma(G) \) are the prime divisors of the order of \( G \) and two distinct vertices \( p \) and \( q \) are joined by an edge and we write \( p \sim q \), if there is an element in \( G \) of order \( pq \). We denote by \( \pi_e(G) \) the set of orders of elements of \( G \). The connected components of \( \Gamma(G) \) are denoted by \( \pi_i, i = 1, 2, \ldots, t(G) \), where \( t(G) \) is the number of components. We can express the order of \( G \) as a product of some positive integer \( m_i, i = 1, 2, \ldots, t(G) \) with \( \pi(m_i) = \pi_i \). The integers \( m_i \)'s are called the order components of \( G \). In 2006, A. Abdollahi, S. Akbari and H. R. Maimani put forward a conjecture in [1] as follows.

AAM’s Conjecture: If \( M \) is a finite non abelian simple group and \( G \) is a group such that \( \nabla(G) \cong \nabla(M) \), then \( G \cong M \).

Ron Solomon and Andrew Woldar proved the above conjecture in [6]. In this paper we will prove that if \( G \) is a finite group with \( \nabla(G) \cong \nabla(S_n) \), then \( G \cong S_n \), where \( S_n \) is the symmetric group of degree \( n \).

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Lemma 2.2. Let \( G \) be a finite group such that \( \nabla(G) \cong \nabla(S_n) \), \( n \geq 3 \). Then \( |G| = |S_n| \).

2. Preliminaries

The following result was proved in part(1) of Theorem 3.16 of [1].

**Lemma 2.1.** Let \( G \) be a finite group such that \( \nabla(G) \cong \nabla(S_n) \), \( n \geq 3 \). Then \( |G| = |S_n| \).

**Lemma 2.2.** Let \( G \) and \( H \) be two non-abelian groups. If \( \nabla(G) \cong \nabla(H) \), then

\[
\nabla(C_G(A)) \cong \nabla(C_H(\varphi(A)))
\]

for all \( \emptyset \neq A \subseteq G - Z(G) \), where \( \varphi \) is the isomorphism from \( \nabla(G) \) to \( \nabla(H) \) and \( C_G(A) \) is non-abelian.

**Proof.** It is sufficient to show that \( \varphi \mid_{V(C_G(A))} \) is the restriction of \( \varphi \) to \( V(C_G(A)) \) and

\[
\begin{align*}
V(C_G(A)) &= C_G(A) - Z(C_G(A)), \\
V(C_H(\varphi(A))) &= C_H(\varphi(A)) - Z(C_H(\varphi(A))).
\end{align*}
\]

Assume \( d \) is an element of \( V(C_H(\varphi(A))) \), then \( d \in H - Z(H) \) and so there exists an element \( c \) of \( G - Z(G) \) such that \( \varphi(c) = d \). From

\[
d = \varphi(c) \in C_H(\varphi(A)),
\]

it follows that \( [\varphi(c), \varphi(g)] = 1 \) for all \( g \in A \) and since \( \varphi \) is an isomorphism from \( \nabla(G) \) to \( \nabla(H) \), \( [c, g] = 1 \) for all \( g \in A \). Therefore \( c \in C_G(A) \). But \( d \not\in Z(C_H(\varphi(A))) \), so for an element \( x \in C_H(\varphi(A)) \) we have \( [x, d] \neq 1 \). Hence \( x \) is an element of \( H \) that does not commute with \( d \in H \). This implies that \( x \in H - Z(H) \). Thus there exists \( x' \in G - Z(G) \), such that \( \varphi(x') = x \). It is easy to see that \( [x', c] \neq 1 \) and therefore \( c \not\in Z(G(A)) \). Hence

\[
c \in C_G(A) - Z(C_G(A)) = V(C_G(A))
\]

and \( \varphi(c) = d \). \( \square \)

The following result was proved by E. Artin in [2] and [3] and together with the classification of finite simple groups can be stated as follows:

**Lemma 2.3.** Let \( G \) and \( M \) be finite simple groups, \( |G| = |M| \), then the following holds:

1. If \( |G| = |A_8| = |L_3(4)| \), then \( G \cong A_8 \) or \( G \cong L_3(4) \);
2. If \( |G| = |B_n(q)| = |C_n(q)| \), where \( n \geq 3 \), and \( q \) is odd, then \( G \cong B_n(q) \) or \( G \cong C_n(q) \);
3. If \( M \) is not in the above cases, then \( G \cong M \).

As an immediate consequence of Lemma 2.3, we get the following corollary.

**Corollary 2.4.** Let \( G \) be a finite simple group with \( |G| = |A_n| \), where \( n \) is a natural number, \( n \geq 5 \), \( n \neq 8 \), then \( G \cong A_n \).
Lemma 2.5. Let $G$ and $H$ be two finite groups with $\nabla(G) \cong \nabla(H)$ and $|G| = |H|$. Then $p_1p_2\cdots p_t \in \pi_e(G)$ if and only if $p_1p_2\cdots p_t \in \pi_e(H)$, where $p_i$'s are distinct prime numbers for $i = 1, 2, \ldots, t$. In particular, $\Gamma(G) = \Gamma(H)$.

Proof. If $\varphi$ is an isomorphism from $\nabla(G)$ to $\nabla(H)$ and $|G| = |H|$, then we can easily see that

$$|Z(C_G(x))| = |Z(C_H(\varphi(x)))|$$

for all $x \in G$. If $p_1p_2\cdots p_t \in \pi_e(G)$, then there exists an element $z \in G$ such that $o(z) = p_1p_2\cdots p_t$. Thus

$$p_1p_2\cdots p_t = |\langle z \rangle||Z(C_G(z))|$$

and so

$$p_1p_2\cdots p_t |Z(C_H(\varphi(z)))|.$$ 

Hence $H$ has an abelian subgroup of order $p_1p_2\cdots p_t$, which is a cyclic group. Therefore $p_1p_2\cdots p_t \in \pi_e(H)$. By a similar argument we see that if $p_1p_2\cdots p_t \in \pi_e(H)$, then $p_1p_2\cdots p_t \in \pi_e(G)$. □

Lemma 2.6. Let $G$ be a finite group with $\nabla(G) \cong \nabla(S_n)$, where $3 \leq n \leq 8$ or $11 \leq n \leq 14$, then $G \cong S_n$.

Proof. Since $\nabla(G) \cong \nabla(S_n)$, by Lemma 2.1, $|G| = |S_n|$. Also by Lemma 2.5 $\Gamma(G) = \Gamma(S_n)$, where $\Gamma$ denotes the prime graph. Thus the order components of $G$ and $S_n$ are the same. In [7] it is proved that $S_p$ and $S_{p+1}$ are characterizable by their order components, where $p \geq 3$ is a prime number. Hence $S_n$, where $3 \leq n \leq 8$ or $11 \leq n \leq 14$ is characterizable by their order components and so $G \cong S_n$, where $3 \leq n \leq 8$ or $11 \leq n \leq 14$. □

Lemma 2.7. Let $G$ be a finite group with $\nabla(G) \cong \nabla(S_n)$, $n = 9, 10, 15, 16$, then $G \cong S_n$.

Proof. We give the proof in the case $n = 9$, the proof in other cases is similar. Set

$$T = \{\alpha \in S_9|(i)\alpha = i, i = 4, 5, \ldots, 9\}.$$ 

Obviously

$$T \leq S_9,$$

$$T \cong S_3$$

and $C_{S_9}(T - \{1\}) \cong S_6$. By Lemma 2.2 we have

$$\nabla(C_{S_9}(T - \{1\})) \cong \nabla(C_G(\varphi(T - \{1\}))),$$

where $\varphi$ is an isomorphism from $\nabla(S_9)$ to $\nabla(G)$. Thus by Lemma 2.6 $C_G(\varphi(T - \{1\})) \cong S_6$.

Let $N$ be a minimal normal subgroup of $G$. If

$$N \cap C_G(\varphi(T - \{1\})) = 1,$$
then since 
\[ |NC_G(\varphi(T - \{1\}))| \cdot |G| = 9! \]
and 
\[ |C_G(\varphi(T - \{1\}))| = 6!, \]
we have \(|N|9 \cdot 8 \cdot 7\). We know that \(N\) is a union of conjugacy classes of \(G\) and the size of conjugacy class of \(G\) containing \(x\) is equal to the size of conjugacy class of \(S_9\) containing \(\varphi^{-1}(x)\) for all \(x \in G - \{1\}\).

We can see that all conjugacy class sizes in \(S_9\) less than \(9 \cdot 8 \cdot 7\) are 1, \(\frac{9 \cdot 8 \cdot 7}{2}\), \(\frac{9 \cdot 8 \cdot 7}{3}\) and \(\frac{9 \cdot 8 \cdot 7}{8}\). Let \(y\) be an arbitrary element in \(N - \{1\}\). Thus the size of conjugacy class of \(y\) in \(G\) and so the size of conjugacy class of \(\varphi^{-1}(y)\) in \(S_9\) is equal to \(\frac{9 \cdot 8 \cdot 7}{2}\), \(\frac{9 \cdot 8 \cdot 7}{3}\) or \(\frac{9 \cdot 8 \cdot 7}{8}\).

Therefore we have one of the possibilities: \(\varphi^{-1}(y)\) is a 2-cycle, \(\varphi^{-1}(y)\) is a 3-cycle or \(\varphi^{-1}(y)\) is a permutation of type \(2^2\).

In any case there exists a subgroup of \(S_9\), say \(K\) isomorphic to \(S_3\) such that 
\[ \varphi^{-1}(y) \in C_{S_9}(K - \{1\}) \]
and 
\[ C_{S_9}(K - \{1\}) \cong S_6. \]
Hence 
\[ y \in N \cap C_G(\varphi(K - \{1\})). \]

By Lemma 2.6 \(C_G(\varphi(K - \{1\})) \cong S_6\) and since 
\[ N \cap C_G(\varphi(K - \{1\})) \neq 1, \]
\(A_6\) is embedded in \(N\).

If 
\[ N \cap C_G(\varphi(T - \{1\})) \neq 1, \]
then since \(C_G(\varphi(T - \{1\})) \cong S_6\) and 
\[ N \cap C_G(\varphi(T - \{1\})) \subseteq C_G(\varphi(T - \{1\})), \]
we conclude that \(A_6\) is embedded in \(N\) in this case too.

Thus \(2^3 \cdot 3^2 \cdot 5 \cdot |N|\). We know that \(N\) is a direct product of isomorphic simple groups. But \(5 \mid |N|\) and \(5^2 \nmid |N|\), hence \(N\) is a simple group.

Moreover \(5 \sim 7\) in \(\Gamma(S_9)\) and since \(\Gamma(G) = \Gamma(S_9)\) by Lemma 2.5 ,5 \(\not\sim 7\) in \(\Gamma(G)\) too. By Frattini’s argument \(N_G(N_5)N = G\), where \(N_5\) is a Sylow 5-subgroup of \(N\) and since \(7 \mid |G|\), \(7 \mid |N_G(N_5)|\) or \(7 \mid |N|\).

If \(7 \mid |N_G(N_5)|\), then there exists an element \(z\) of order 7 in \(N_G(N_5)\) and so \(\langle z \rangle N_5\) is a subgroup of \(N_G(N_5)\) of order 5.7. Hence \(\langle z \rangle N_5\) is a cyclic group. It means that \(5 \sim 7\) in \(\Gamma(G)\), which is a contradiction. Thus \(7 \mid |N|\).

Now we assert that \(C_G(N) = 1\). Otherwise there is a minimal normal subgroup \(T\) of \(G\) such that
$T \leq C_G(N)$. By the same argument as above we see that $2^3 \cdot 3^2 \cdot 5 \cdot 7 \mid |T|$. Therefore $2^3 \cdot 3^2 \cdot 5 \cdot 7 \mid |C_G(N)|$. Hence $5 \mid |C_G(N)|$ and so there is an element $a \in C_G(N)$ such that $o(a) = 5$ and since $7 \mid |N|$, there is an element of order $7$, say $b$ in $N$. $o(ab) = 5 \cdot 7$, because $ab = ba$. But $5 \not\sim 7$ in $\Gamma(G)$ and this is a contradiction. Thus $C_G(N) = 1$.

It implies that

$$G \cong \frac{G}{1} = \frac{G}{C_G(N)} \hookrightarrow Aut(N).$$

Therefore

$$9! = |G||Aut(N)|.$$ 

So we proved that $N$ is a simple group with

$$2^3 \cdot 3^2 \cdot 5 \cdot 7 \mid |N|,$$

$$9! = 2^7 \cdot 3^4 \cdot 5 \cdot 7 \mid |Aut(N)|$$

and $|N|2^7 \cdot 3^4 \cdot 5 \cdot 7$. By table 1 of [5], we conclude that $N \cong A_9$. But

$$G \hookrightarrow Aut(N),$$

$$|G| = |S_9|$$

and

$$Aut(N) \cong Aut(A_9) \cong S_9.$$

Hence $G \cong S_9$. 

\[ \square \]

**Lemma 2.8.** Let $T$ be a finite group and $T \cong S_1 \times S_2 \times \cdots \times S_t$, where $S_i$s are isomorphic simple groups, $1 \leq i \leq t$. Let $T$ contain a copy of the alternating group $A_{n-3}$, $n \geq 16$ and $|T||n!$. Then $T$ is a simple group.

**Proof.** Without loss of generality we may assume that

$$T = S_1 \times S_2 \times \cdots \times S_t.$$ 

Suppose that $\pi_1 : S_1 \times S_2 \times \cdots \times S_t \rightarrow S_1 \times 1 \times \cdots \times 1$ is defined by

$$\pi_1(s_1, s_2, \ldots, s_t) = (s_1, 1, \ldots, 1)$$

and $K$ is a subgroup of $T$ isomorphic to $A_{n-3}$. Set

$$\overline{S_1} = S_1 \times 1 \times \cdots \times 1$$

and

$$\overline{S_2} \times \cdots \times \overline{S_t} = 1 \times S_2 \times \cdots \times S_t.$$ 

Now we consider the following three cases.

Case 1) $K \cap \overline{S_1} = K \cap \overline{S_2} \times \cdots \times \overline{S_t} = 1$. 


In this case \( \phi : K \to \pi_1(K) \) defined by \( \phi(k) = \pi_1(k) \) for all \( k \in K \) is an isomorphism from \( K \) onto \( \pi_1(K) \). This means that \( K \cong \pi_1(K) \). Thus we have

\[
\mathbb{A}_{n-3} \cong K \cong \pi_1(K) \leq \mathbb{S}_1 \cong S_1
\]

and so

\[
\frac{(n-3)!}{2} = |\mathbb{A}_{n-3}||S_1|.
\]

But \( S_i \)s are isomorphic simple groups, \( 1 \leq i \leq t \) and thus

\[
\frac{(n-3)!}{2} = |\mathbb{A}_{n-3}||S_i|,
\]

for \( 1 \leq i \leq t \).

Therefore

\[
\left[ \frac{(n-3)!}{2} \right]^t |T|
\]

and since \( |T|n! \), we obtain \( \left[ \frac{(n-3)!}{2} \right]^t |n!\). But

\[
\left[ \frac{(n-3)!}{2} \right]^2 \mid n!
\]

for \( n \geq 16 \) and so \( t = 1 \) and \( T \) is a simple group.

Case 2) \( K \cap \mathbb{S}_1 \neq 1 \)

Since \( \mathbb{S}_1 \leq T \), we have

\[
1 \neq K \cap \mathbb{S}_1 \leq K \cong \mathbb{A}_{n-3},
\]

which implies that \( K \cap \mathbb{S}_1 = K \) and so

\[
\mathbb{A}_{n-3} \cong K \leq \mathbb{S}_1 \cong S_1.
\]

Now similar argument as in Case (1) shows that \( T \) is a simple group.

Case 3) \( K \cap \mathbf{S}_2 \times \cdots \times \mathbf{S}_t \neq 1 \)

Since

\[
\mathbf{S}_2 \times \cdots \times \mathbf{S}_t \leq T,
\]

we have

\[
1 \neq K \cap \mathbf{S}_2 \times \cdots \times \mathbf{S}_t \leq K \cong \mathbb{A}_{n-3},
\]

which implies that

\[
K \cap \mathbf{S}_2 \times \cdots \times \mathbf{S}_t = K
\]
and so

\[ A_{n-3} \cong K \leq S_2 \times \cdots \times S_t \cong S_2 \times \cdots \times S_t. \]

Thus \( A_{n-3} \) is embedded in \( S_2 \times \cdots \times S_t \).

By repeating above argument for

\[ T_i = S_i \times \cdots \times S_t, \quad 2 \leq i \leq t, \]

we conclude that \( T \) is a simple group. \( \square \)

**Lemma 2.9.** Let \( a, b \) be two natural numbers. Then:

1) \( a^b b! \leq (ab)! \) and \( a^0 0! = (a 0)! \)

2) If \( a \geq 4 \), then \( a^{b-1} b! \leq (a(b-1))! \)

3) \( 3^{b-1} b! \leq (3b - 3)! \)

4) If \( b \geq 3 \), then \( 2^{b-1} b! \leq (2b - 2)! \)

5) If \( b \geq 5 \), then \( 2^{b-2} b! \leq 2(2b - 4)! \)

6) If \( b \geq 4 \), then \( 2^{b-2} b! \leq 2(3b - 6)! \)

**Proof.**

1) We prove Lemma 2.9 part 1 by induction on \( b \). If \( b = 1 \), then clearly (1) holds. Suppose that \( a^k k! \leq (ak)! \). We prove that \( a^{k+1} (k+1)! \leq (ak + a)! \).

By induction hypothesis

\[ a^{k+1} (k+1)! \leq (ak)! a(k+1). \]

But clearly

\[ (ak)! a(k+1) \leq (ak + a)! \]

and so

\[ a^{k+1} (k+1)! \leq (ak + a)! \]

and this completes the proof of (1).

2) We prove part 2 by induction on \( b \). If \( b = 1 \), then clearly (2) holds. Suppose that

\[ a^{k-1} k! \leq (a(k-1))! \]

for \( k \geq 1 \) and \( a \geq 4 \). We prove that

\[ a^k (k+1)! \leq (ak)! \]

By induction hypothesis,

\[ a^k (k+1)! \leq (a(k-1))! a(k+1). \]
But since \( ak \geq 4 \), we have \( ak - 1 \geq 3 \) and so
\[
(ak)(ak - 1) \cdots (ak - a + 1) \geq ak + a.
\]
Thus \((ak)! \geq (ak - a)!a(k + 1)\). Hence
\[
a^k(k + 1)! \leq (ak)!
\]
and this completes the proof of (2).

3) We prove this part by induction on \( b \) too. If \( b = 1 \), then (3) clearly holds. Suppose that
\[
3^{k-1}k! \leq (3k - 3)!. 
\]
We prove that
\[
3^k(k + 1)! \leq (3k)!. 
\]
By induction hypothesis we obtain
\[
3^k(k + 1)! \leq (3k - 3)!3(k + 1).
\]
It is easy to know that
\[
k + 1 \leq k(3k - 1)(3k - 2)
\]
for \( k \geq 1 \). Thus
\[
(3k - 3)!3(k + 1) \leq (3k)!
\]
and so
\[
3^k(k + 1)! \leq (3k)!
\]
and this completes the proof of (3).

4) We prove part (4) by induction on \( b \). If \( b = 3 \), then clearly (4) holds. Suppose that
\[
2^{k-1}k! \leq (2k - 2)!
\]
for \( k \geq 3 \). We prove that
\[
2^k(k + 1)! \leq (2k)!.
\]
By induction hypothesis we obtain \( 2^k(k + 1)! \leq (2k - 2)!2(k + 1) \). It is easy to see that \( k + 1 \leq k(2k - 1) \) for \( k \geq 3 \). Thus
\[
(2k - 2)!2(k + 1) \leq (2k)!
\]
and so
\[
2^k(k + 1)! \leq (2k)!
\]
and this completes the proof of (4).

5) We prove this part by induction on \( b \). If \( b = 5 \), then (5) clearly holds. Suppose that
\[
2^{k-2}k! \leq 2(2k - 4)!
\]
for $k \geq 5$. We prove that

$$2^{k-1}(k + 1)! \leq 2(2k - 2)!.$$  

By induction hypothesis

$$2^{k-1}(k + 1)! \leq 2(2k - 4)!2(k + 1).$$

But since

$$k^2 - 3k + 1 \geq 0$$

for $k \geq 5$, we have

$$k + 1 \leq (k - 1)(2k - 3)$$

and so

$$2(2k - 4)!2(k + 1) \leq 2(2k - 2)!.$$  

Hence

$$2^{k-1}(k + 1)! \leq 2(2k - 2)!$$

and this completes the proof of (5).

6) We prove (6) by induction on $b$ too. If $b = 4$, then (6) clearly holds. Suppose that

$$3^{k-2}k! \leq 2(3k - 6)!$$

for $k \geq 4$. We prove that

$$3^{k-1}(k + 1)! \leq 2(3k - 3)!.$$  

By induction hypothesis

$$3^{k-1}(k + 1)! \leq 2(3k - 6)!3(k + 1).$$

It is easy to see that

$$3(k + 1) \leq (3k - 3)(3k - 4)(3k - 5)$$

for $k \geq 4$ and so

$$2(3k - 6)!3(k + 1) \leq 2(3k - 3)!$$

for $k \geq 4$. Hence

$$3^{k-1}(k + 1)! \leq 2(3k - 3)!$$

and this completes the proof of (6).  

□

**Lemma 2.10.** Let $a \geq 0$, $b \geq 0$ be two integers. Then $a!b! \leq (a + b)!$. 


Proof. If \(a \geq 1, b \geq 1\), then since
\[
 a + b > b, \\
 a + b - 1 > b - 1, \ldots, \\
 a + 1 > 1,
\]
we have
\[
(a + b)(a + b - 1) \cdots (a + 1) > b!
\]
and so
\[
(a + b)! \\
= (a + b)(a + b - 1) \cdots (a + 1) a! > b! a!.
\]
If \(a = 0\) or \(b = 0\), then clearly \(a! b! = (a + b)!\). \(\square\)

Lemma 2.11. Let \(a_1, a_2, \ldots, a_m\) be integers with \(a_i \geq 0, 1 \leq i \leq m\). Then \(a_1! a_2! \cdots a_m! \leq (a_1 + \cdots + a_m)!\).

Proof. We prove Lemma by induction on \(m\). If \(m = 1\), then clearly Lemma holds. Assume that
\[
a_1! a_2! \cdots a_k! \\
\leq (a_1 + a_2 + \cdots + a_k)!
\]
We prove that
\[
a_1! a_2! \cdots a_k! a_{k+1}! \\
\leq (a_1 + a_2 + \cdots + a_k + a_{k+1})!.
\]
By induction hypothesis
\[
a_1! a_2! \cdots a_k! a_{k+1}! \\
\leq (a_1 + a_2 + \cdots + a_k)! a_{k+1}!.
\]
But by Lemma 2.10 we have
\[
(a_1 + \cdots + a_k)! a_{k+1}! \\
\leq (a_1 + a_2 + \cdots + a_k + a_{k+1})!.
\]
Thus
\[
a_1! a_2! \cdots a_{k+1}! \\
\leq (a_1 + a_2 + \cdots + a_{k+1})!.
\]
\(\square\)

Lemma 2.12. Let \(l, m, n\) be three natural numbers with \(n \geq 13\). Then the following holds.
1) If there exists a $m$-cycle, $m \geq 4$ in a cycle type of $x \in S_n$, then $|C_{S_n}(x)| \leq m(n - m)!$

2) If there exists two $l$-cycles in a cycle type of $x \in S_n$, where $l = 2$ or $l = 3$, then $|C_{S_n}(x)| \leq l^2! (n - 2l)!$

3) If there exist a 2-cycle and a 3-cycle in a cycle type of $x \in S_n$, then $|C_{S_n}(x)| \leq 2.3.(n - 5)!$

Proof. 1) Assume that $x \in S_n$ is a permutation of type

$$1^{\alpha_1} \cdot 2^{\alpha_2} \cdot m^{\alpha_m} \cdot n^{\alpha_n},$$

where $\alpha_i \geq 0$, $1 \leq i \leq n$. By assumption $\alpha_m \geq 1$. Thus

$$|C_{S_n}(x)| = 1^{\alpha_1} \cdot 2^{\alpha_2} \cdot m^{\alpha_m} \cdot n^{\alpha_n},$$

where $\alpha_m \geq 1$. By Lemma 2.9 part 1 and 2 we conclude that

$$|C_{S_n}(x)| \leq \alpha_1! \cdot 2^{\alpha_2} \cdot m^{(\alpha_m - 1)} \cdot n^{\alpha_n}!$$

and so by Lemma 2.11, we have

$$|C_{S_n}(x)| \leq m(\alpha_1 + 2\alpha_2 + \cdots + m(\alpha_m - 1) + \cdots + n\alpha_n)! = m(n - m)!$$

and this completes the proof of (1).

2) Assume that $x \in S_n$ is a permutation of type

$$1^{\alpha_1} \cdot 2^{\alpha_2} \cdot n^{\alpha_n}$$

where $\alpha_i \geq 0$, $1 \leq i \leq n$. By assumption $\alpha_l \geq 2$, where $l = 2$ or $l = 3$. First suppose that $l = 2$.

We have

$$|C_{S_n}(x)| = 1^{\alpha_1} \cdot 2^{\alpha_2}\cdot n^{\alpha_n}!.$$

If $\alpha_2 \geq 5$, then by Lemma 2.9 part 5 and 1 we conclude that

$$|C_{S_n}(x)| \leq \alpha_1! \cdot 2^{\alpha_2} \cdot (2\alpha_2 - 4)! \cdots (n\alpha_n)!$$

and so by Lemma 2.11 we have

$$|C_{S_n}(x)| \leq 2^3(\alpha_1 + 2\alpha_2 - 4 + \cdots + n\alpha_n)! = 2^3(n - 4)! = 2^22!(n - 4)!.$$
If $\alpha_2 = 2$, then

$$|C_{S_n}(x)| = 1^{\alpha_1} \cdot \alpha_1! \cdot 2^2 \cdot 2! \cdot \cdots \cdot n^{\alpha_n} \cdot \alpha_n!.$$ 

By part 1 of Lemma 2.9 and Lemma 2.11 we conclude that

$$|C_{S_n}(x)| \leq 2^2 \cdot 2! \cdot \alpha_1! \cdot (3\alpha_3)! \cdot \cdots \cdot (n\alpha_n)! \leq 2^2 \cdot 2!(n-4)!.$$ 

If $\alpha_2 = 3$ or $\alpha_2 = 4$, then similar argument as case $\alpha_2 = 2$ shows us that

$$|C_{S_n}(x)| \leq 2^3 \cdot 3!(n-6)!$$ 

or

$$|C_{S_n}(x)| \leq 2^4 \cdot 4!(n-8)!$$

respectively and since

$$2^3 \cdot 3!(n-6)! \leq 2^2 \cdot 2!(n-4)!$$

and

$$2^4 \cdot 4!(n-8)! \leq 2^2 \cdot 2!(n-4)!$$

for $n \geq 13$, we have

$$|C_{S_n}(x)| \leq 2^2 \cdot 2!(n-4)!$$

in this case too.

Now suppose that $l = 3$. If $\alpha_3 \geq 4$, then by Lemma 2.9 part 6 and 1 we have

$$|C_{S_n}(x)| \leq \alpha_1!(2\alpha_2)! \cdot 3^2 \cdot 2(3\alpha_3 - 6)! \cdot \cdots \cdot (n\alpha_n)!$$

and so by Lemma 2.11 we have

$$|C_{S_n}(x)| \leq 3^2 \cdot 2!(\alpha_1 + 2\alpha_2 + 3\alpha_3 - 6 + \cdots + n\alpha_n)! = 3^2 \cdot 2!(n-6)!.$$
If $\alpha_3 = 2$, then
\[
|C_{S_n}(x)| = 1^{\alpha_1}\alpha_1!2^{\alpha_2}\alpha_2!3^{2}\alpha_2!\cdots n^{\alpha_n}\alpha_n!.
\]

By Lemma 2.9 part 1 and Lemma 2.11 we conclude that
\[
|C_{S_n}(x)| \leq 3^2\alpha_1!(2\alpha_2)!4\alpha_2!\cdots (n\alpha_n)! \\
\leq 3^2!(n-6)!.
\]

If $\alpha_3 = 3$, then similar argument as case $\alpha_3 = 2$ shows us that
\[
|C_{S_n}(x)| \leq 3^3.3!(n-9)!
\]
and since
\[
3^3.3!(n-9)! \leq 3^22!(n-6)!
\]
for $n \geq 13$, we have
\[
|C_{S_n}(x)| \leq 3^22!(n-6)!
\]
in this case too and so the proof of (2) is complete.

3) Again assume that $x \in S_n$ is a permutation of type
\[
1^{\alpha_1}\cdot 2^{\alpha_2}\cdots n^{\alpha_n},
\]
where $\alpha_i \geq 0$, $1 \leq i \leq n$. By assumption $\alpha_2 \geq 1$ and $\alpha_3 \geq 1$. We have
\[
|C_{S_n}(x)| = 1^{\alpha_1}\alpha_1!2^{\alpha_2}\alpha_2!3^{\alpha_3}\alpha_3!\cdots n^{\alpha_n}\alpha_n!.
\]
If $\alpha_2 \geq 3$, then by Lemma 2.9 part 4,3 and 1 we have
\[
|C_{S_n}(x)| \leq \alpha_1!2(2\alpha_2-2)!3(3\alpha_3-3)!\cdots (n\alpha_n)!
\]
and so by Lemma 2.11
\[
|C_{S_n}(x)| \leq 2.3.(\alpha_1 + 2\alpha_2 - 2 + 3\alpha_3 - 3 + \cdots + n\alpha_n)! \\
= 2.3.(n-5)!. 
\]
If $\alpha_2 = 1$, then we have
\[
|C_{S_n}(x)| = 1^{\alpha_1}\alpha_1!2.3^{\alpha_3}\alpha_3!\cdots n^{\alpha_n}\alpha_n!.
\]
By Lemma 2.9 part 1 and 3 we have

\[ |C_{S_n}(x)| \leq \alpha_1! \cdot 2 \cdot 3 \cdot (3\alpha_3 - 3)! \cdots (n\alpha_n)! \]

and so by Lemma 2.11

\[ |C_{S_n}(x)| \leq 2 \cdot 3 \cdot (\alpha_1 + 3\alpha_3 - 3 + \cdots + n\alpha_n)! \]

\[ = 2 \cdot 3 \cdot (n - 5)!. \]

If \( \alpha_2 = 2 \), then similar argument as case \( \alpha_2 = 1 \) shows us that

\[ |C_{S_n}(x)| \leq \alpha_1! 2^2 \cdot 2! \cdot 3 \cdot (3\alpha_3 - 3)! \cdots (n\alpha_n)! \]

\[ \leq 2^2 \cdot 2! \cdot 3(\alpha_1 + 3\alpha_3 - 3 + \cdots + n\alpha_n)! \]

\[ = 2^2 \cdot 2! \cdot 3(n - 7)! \]

and since

\[ 2^2 \cdot 2! \cdot 3(n - 7)! \leq 2.3(n - 5)! \]

for \( n \geq 13 \), we have

\[ |C_{S_n}(x)| \leq 2.3(n - 5)! \]

in this case too and the proof of (3) is complete. \( \Box \)

**Lemma 2.13.** Let \( l, k \) be two natural numbers with \( l > 1 \) and \( 1 < l + k < n - 1 \), where \( n \geq 13 \) is a natural number. Then \( l(n - l)! > (l + k)(n - l - k)! \).

**Proof.** We prove Lemma 2.12 by induction on \( k \). If \( k = 1 \), then since \( n - l > 2 \), \( l > 1 \), we have \( l(n - l) > l + 1 \) and so

\[ l(n - l)! > (l + 1)(n - l - 1)!. \]

Thus the lemma holds whenever \( k = 1 \). Suppose that if

\[ 1 < l + k < n - 1, \]

\[ l > 1, \]

then

\[ l(n - l)! > (l + k)(n - l - k)!. \]
We prove the lemma for \( k + 1 \).

Suppose that

\[
1 < l + k + 1 < n - 1, \\
l > 1.
\]

Since

\[
(n - l - k) > 2, \\
l + k > 1,
\]

we have

\[
(l + k)(n - l - k) \\
> 2(l + k) > l + k + 1
\]

and so

\[
(l + k)(n - l - k)! \\
> (l + k + 1)(n - l - k - 1)!.
\]

Thus by induction hypothesis we conclude that

\[
l(n - l)! \\
> (l + k + 1)(n - l - k - 1)!.
\]

Hence the lemma is proved. \( \square \)

**Lemma 2.14.** Let \( l, m, n \) be three natural numbers with \( l > 1 \), \( n \geq 13 \), \( m \neq n \) and \( l \leq m \). Then \( l(n - l)! \geq m(n - m)! \)

*Proof.* If \( l = m \), then clearly Lemma holds. If \( l < m \) and \( 1 < m < n - 1 \), then since \( l > 1 \), Lemma 2.14 concluded from Lemma 2.13. But if \( l < m \) and \( m = n - 1 \), then we have

\[
m(n - m)! \\
= (n - 1)!
\]

\[
= n - 1.
\]

We have \( (n - 1) < (n - 2)2 \) for \( n \geq 13 \) and since \( 1 < n - 2 < n - 1 \) by above argument for all \( 1 < l \leq n - 2 \) we have

\[
l(n - l)! \geq (n - 2)2!.
\]

Hence \( l(n - l)! > n - 1 \), also if \( l = n - 1 \), clearly

\[
l(n - l)! \geq n - 1.
\]

So the proof is complete. \( \square \)
Lemma 2.15. If \( x \in S_n \) and \( |x^{S_n}| \leq n(n-1)(n-2) \), where \( x^{S_n} \) is the conjugacy class of \( S_n \), \( n \geq 13 \) containing \( x \). Then \( x = 1 \), \( x \) is a 2-cycle or \( x \) is a 3-cycle and \( |x^{S_n}| = (n-3)! \).

Proof. Suppose that \( |x^{S_n}| \leq n(n-1)(n-2) \). Then

\[
|C_{S_n}(x)| \geq \frac{n!}{n(n-1)(n-2)} = (n-3)!.
\]

If there exists a \( m \)-cycle, \( m \geq 4 \) in a cycle type of \( x \), then by Lemma 2.12 part 1

\[
|C_{S_n}(x)| \leq m(m-1)!\]

and by Lemma 2.14 we conclude that if \( m \neq n \), then

\[ m(n-m)! \leq 4(n-4)! \]

But if \( m = n \), then \( m(n-m)! = n \). It is easy to know that

\[ n < 4(n-4)! \]

for \( n \geq 13 \). Therefore if there exists a \( m \)-cycle, \( m \geq 4 \) in a cycle type of \( x \), then

\[
|C_{S_n}(x)| \leq 4(n-4)!.
\]

But we have \( |C_{S_n}(x)| \geq (n-3)! \) and so

\[ (n-3)! \leq 4(n-4)! \]

which is a contradiction, because \( n \geq 13 \). Thus there is no \( m \)-cycle, \( m \geq 4 \) in a cycle type of \( x \). If there exist two 2-cycles or two 3-cycles in a cycle type of \( x \), then by Lemma 2.12 part 2 we conclude that

\[
|C_{S_n}(x)| \leq 2^22!(n-4)!
\]

or

\[
|C_{S_n}(x)| \leq 3^22!(n-6)!
\]

respectively and so

\[ (n-3)! \leq 2^22!(n-4)! \]

or

\[ (n-3)! \leq 3^22!(n-6)! \]

which is a contradiction, because \( n \geq 13 \). Also if there exists a 3-cycle and a 2-cycle in a cycle type of \( x \), then by Lemma 2.12 part 3 we conclude that

\[ |C_{S_n}(x)| \leq 2.3.(n-5)! \]
and so
\[(n - 3)! \leq 2.3(n - 5)!,\]
which is a contradiction with \(n \geq 13\). Thus \(x = 1\) or \(x\) is a 2-cycle or \(x\) is a 3-cycle. Hence \(|xS_n| = 1\) or \(|xS_n| = \frac{n(n-1)}{2}\) or \(xS_n = \frac{n(n-1)(n-2)}{3}\).

**Lemma 2.16.** Let \(x\) be an element of \(S_n\), \(n \geq 13\). If \(|C_{S_n}(x)| = 3(n - 3)!\), then \(x\) is a 3-cycle.

**Proof.** If \(|C_{S_n}(x)| = 3(n - 3)!\), then
\[|C_{S_n}(x)| \geq (n - 3)!\]
and so by Lemma 2.15 we conclude that \(x = 1\) or \(x\) is a 2-cycle or \(x\) is a 3-cycle. But if \(x = 1\) or \(x\) is a 2-cycle, then clearly
\[|C_{S_n}(x)| \neq 3(n - 3)!\]
\((n \neq 3(n - 3)!\) and \(2(n - 2)! \neq 3(n - 3)!\) and so \(x\) is a 3-cycle. \(\square\)

3. Main result

In this section we will prove our main result.

**Theorem 3.1.** Let \(G\) be a finite group with \(\nabla(G) \cong \nabla(S_n)\), where \(S_n\) is the symmetric group of degree \(n\) and \(n \geq 3\), then \(G \cong S_n\).

**Proof.** By Lemma 2.1, we have \(|G| = |S_n|\). Since \(\nabla(G) \cong \nabla(S_n)\),
\[|G - Z(G)| = |S_n - Z(S_n)| = |S_n| - 1\]
and so \(|Z(G)| = 1\).

By Lemmas 2.6 and 2.7 we may assume that \(n \geq 16\). Without loss of generality we can assume that \(\varphi : S_n \to G\) and \(\varphi(1) = 1\), where \(\varphi\) is an isomorphism from \(\nabla(S_n)\) to \(\nabla(G)\).

Now we prove the theorem by induction on \(n\), where \(n \geq 16\). If \(n = 16\), then theorem holds by Lemma 2.7. Suppose the theorem is true for all \(m < n\) and assume that \(n \geq 16\). We will prove that the result is valid for \(S_n\).

Set
\[A = \{\alpha \in S_n|(i)\alpha = i, i = 4, 5, \ldots, n\}.\]

Clearly
\[A \leq S_n,\]
\[A \cong S_3.\]
By Lemma 2.2 we have
\[ \nabla(C_{S_n}(A)) \cong \nabla(C_G(\varphi(A))) \]
and since \( C_{S_n}(A) \cong S_{n-3} \), we have
\[ \nabla(S_{n-3}) \cong \nabla(C_G(\varphi(A))). \]
Thus by induction hypothesis \( C_G(\varphi(A)) \cong S_{n-3} \). Therefore \( G \) has a subgroup isomorphic to \( S_{n-3} \).

Let \( H = C_G(\varphi(A)) \). Now we assume that \( N \) is an arbitrary minimal normal subgroup of \( G \). We will prove that \( N \) is a simple group and that \( H \rightarrow N \cap P \)

for all subgroups \( P \) of \( G \) isomorphic to \( S_{n-3} \). In particular \( N \) contains all even permutations of \( P \), for all \( P \leq G, P \cong S_{n-3} \).

Let \( P \) be an arbitrary subgroup of \( G \) isomorphic to \( S_{n-3} \). We have \( N \cap P \leq P \). We assert that \( N \cap P \neq 1 \). If \( N \cap P = 1 \), then we have
\[ |NP| = |N||P||G| = n!. \]
Thus
\[ |N|.(n-3)!\cdot n!, \]
since \( |P| = (n-3)! \). This implies that \( |N|n(n-1)(n-2) \). Moreover \( N \) is a union of conjugacy classes of \( G \) and the size of conjugacy class of \( G \) containing \( x \) is equal to the size of conjugacy class of \( S_n \) containing \( \varphi^{-1}(x) \) for all \( x \in G - \{1\} \).

By Lemma 2.15 we see that all conjugacy class sizes less than \( n(n-1)(n-2) \) in \( S_n \), \( n \geq 16 \) are 1, \( \frac{n(n-1)}{2} \) and \( \frac{n(n-1)(n-2)}{3} \).

Let \( y \) be an arbitrary element of \( N - \{1\} \). Thus the size of the conjugacy class of \( G \) containing \( y \) and so the size of conjugacy class of \( S_n \) containing \( \varphi^{-1}(y) \) is equal to \( \frac{n(n-1)}{2} \) or \( \frac{n(n-1)(n-2)}{3} \). Also by Lemma 2.15 \( \varphi^{-1}(y) \) is a 2-cycle or \( \varphi^{-1}(y) \) is a 3-cycle.

In any case there exists a subgroup of \( S_n \), say \( E \) isomorphic to \( S_3 \) such that \( \varphi^{-1}(y) \in C_{S_n}(E) \) and
\[ C_{S_n}(E) \cong S_{n-3}. \]
So \( y \in C_G(\varphi(E)) \), also we know that \( y \in N - \{1\} \). Therefore
\[ y \in N \cap C_G(\varphi(E)) \]
and
\[ N \cap C_G(\varphi(E)) \neq 1. \]

By Lemma 2.2
\[ \nabla(S_{n-3}) \cong \nabla(C_{S_n}(E)) \]
\[ \cong \nabla(C_G(\varphi(E))) \]
and so by induction hypothesis
\[ C_G(\varphi(E)) \cong S_{n-3}. \]

Since
\[ 1 \neq N \cap C_G(\varphi(E)) \]
\[ \leq C_G(\varphi(E)) \cong S_{n-3}, \]
we conclude that
\[ A_{n-3} \hookrightarrow N \cap C_G(\varphi(E)). \]

Set \( R = N \cap C_G(\varphi(E)) \). Therefore
\[ \frac{(n-3)!}{2} ||R||. \]

Since \( P \cap N = 1 \),
\[ P \cap R \]
\[ \subseteq P \cap N = 1 \]
and so \( P \cap R = 1 \). Thus \( |PR| = |P||R| \). On the other hand \(|P| = (n-3)!| \) and
\[ \frac{(n-3)!}{2} ||R||. \]

So
\[ \frac{(n-3)!}{2}^2 ||P||^2 = |PR|. \]

But since \( PR \subseteq G \), we have
\[ |PR| \leq |G| = n!. \]

So
\[ \frac{(n-3)!}{2} \leq n!, \]
which is a contradiction, since we assumed that \( n \geq 16 \). Hence \( P \cap N \neq 1 \) for all subgroup \( P \) of \( G \) isomorphic to \( S_{n-3} \). In particular \( N \cap H \neq 1 \). Also
\[ 1 \neq N \cap P \leq P \cong S_{n-3} \]
implies that
\[ A_{n-3} \hookrightarrow N \cap P \]
for all \( P \leq G, P \cong S_{n-3} \).
Since \( N \) is a minimal normal subgroup of \( G \), \( N \) is a direct product of isomorphic simple groups, say
\[ N \cong S_1 \times \cdots \times S_t, \]
where \( S_i \)'s are isomorphic simple groups, \( 1 \leq i \leq t \). Also since
\[ A_{n-3} \hookrightarrow N \cap H, \]
\( A_{n-3} \hookrightarrow N \). Thus by Lemma 2.8 \( N \) is a simple group.

Next set
\[ B = \{ \beta \in S_n | (i) \beta = i, i = 1, 2, \ldots, n - 3 \}. \]
Clearly
\[ B \leq S_n, \]
\[ B \cong S_3 \]
and
\[ C_{S_n}(B) \cong S_{n-3}. \]
It is easy to see that
\[ C_{S_n}(A) \cap C_{S_n}(B) \cong S_{n-6}. \]
By Lemma 2.2 we have
\[
\nabla(S_{n-6}) \\
\cong \nabla(C_{S_n}(A) \cap C_{S_n}(B)) \\
= \nabla(C_{S_n}(A \cup B)) \\
\cong \nabla(C_G(\varphi(A \cup B))) \\
= \nabla(C_G(\varphi(A) \cup \varphi(B))) \\
= \nabla(C_G(\varphi(A) \cap C_G(\varphi(B))))
\]
and so by induction hypothesis
\[ C_G(\varphi(A)) \cap C_G(\varphi(B)) \cong S_{n-6}. \]
Similarly \( C_G(\varphi(B)) \cong S_{n-3} \).
By above argument
\[ A_{n-3} \hookrightarrow N \cap C_G(\varphi(A)) \]
and

\[ A_{n-3} \hookrightarrow N \cap C_G(\varphi(B)). \]

Let

\[ L \leq N \cap C_G(\varphi(A)), \]
\[ K \leq N \cap C_G(\varphi(B)) \]

and \( L \cong K \cong A_{n-3} \). We have

\[ L \cap K \leq N \cap C_G(\varphi(A)) \cap C_G(\varphi(B)) \leq C_G(\varphi(A)) \cap C_G(\varphi(B)) \cong S_{n-6}. \]

Now we will prove the following claim.

Claim: \( L \cap K \neq C_G(\varphi(A)) \cap C_G(\varphi(B)) \)

Suppose by way of contradiction, that

\[ L \cap K = C_G(\varphi(A)) \cap C_G(\varphi(B)). \]

Assume that \( a = (1\ 2\ 3) \in S_n \). Clearly \( a \in C_{S_n}(B) \). Since

\[ |C_{S_n}(B) \cap C_{S_n}(a)| = |C_{C_{S_n}(B)}(a)| = 3(n - 6)!, \]

we conclude that

\[ |C_G(\varphi(B)) \cap C_G(\varphi(a))| = |C_{C_G(\varphi(B))}(\varphi(a))| = 3(n - 6)! \]

But

\[ C_G(\varphi(B)) \cong S_{n-3} \]

and by Lemma 2.16 if

\[ y \in S_{n-3}, \]
\[ |C_{S_{n-3}}(y)| = 3(n - 6)!, \]
\[ n \geq 16, \]
then \( y \) is a 3-cycle. Thus \( \varphi(a) \) is a 3-cycle in \( C_G(\varphi(B)) \cong S_{n-3} \). Therefore \( \varphi(a) \) is an even permutation in \( C_G(\varphi(B)) \cong S_{n-3} \) and so
\[
\varphi(a) \in K \cong A_{n-3}
\]
(Note that \( K \leq C_G(\varphi(B)) \)). Also we have
\[
C_{S_n}(A) \cap C_{S_n}(B) \subseteq C_{S_n}(a),
\]
so
\[
L \cap K = C_G(\varphi(A)) \cap C_G(\varphi(B)) \leq C_G(\varphi(a)),
\]
which implies that
\[
C_G(\varphi(A)) \cap C_G(\varphi(B)) \leq C_G(\varphi(a)) \cap K = C_K(\varphi(a)).
\]
On the other hand \( \langle \varphi(a) \rangle \leq C_K(\varphi(a)) \). Since
\[
C_{S_n}(C_{S_n}(a)) \cap C_{S_n}(A) \cap C_{S_n}(B) = 1,
\]
we have
\[
\langle \varphi(a) \rangle \cap C_G(\varphi(A)) \cap C_G(\varphi(B)) \subseteq C_G(\varphi(a)) \cap C_G(\varphi(A)) \cap C_G(\varphi(B)) = 1
\]
and so
\[
\langle \varphi(a) \rangle \cap C_G(\varphi(A)) \cap C_G(\varphi(B)) = 1.
\]
Therefore
\[
|\langle \varphi(a) \rangle C_G(\varphi(A)) \cap C_G(\varphi(B))| = |\langle \varphi(a) \rangle| |C_G(\varphi(A)) \cap C_G(\varphi(B))| = 3(n - 6)!.
\]
Moreover since \( a \) commutes with all elements of \( C_{S_n}(A) \cap C_{S_n}(B) \), \( \varphi(a) \) commutes with all elements of
\[
C_G(\varphi(A)) \cap C_G(\varphi(B)).
\]
So
\[
\langle \varphi(a) \rangle C_G(\varphi(A)) \cap C_G(\varphi(B)) \leq G.
\]
But we have

\[ \langle \varphi(a) \rangle \leq C_K(\varphi(a)), \]
\[ C_G(\varphi(A)) \cap C_G(\varphi(B)) \leq C_K(\varphi(a)) \]

and thus

\[ \langle \varphi(a) \rangle C_G(\varphi(A)) \cap C_G(\varphi(B)) \leq C_K(\varphi(a)). \]

Hence

\[ 3(n-6)! \frac{\left| C_K(\varphi(a)) \right|}{2}, \]

where \( K \cong A_{n-3} \). But this is impossible, because \( \varphi(a) \) is a 3-cycle in \( K \) and so

\[ |C_K(\varphi(a))| = |C_{A_{n-3}}(\varphi(a))| \]
\[ = 3 \cdot \frac{(n-6)!}{2}. \]

Hence

\[ L \cap K \neq C_G(\varphi(A)) \cap C_G(\varphi(B)) \]

and the claim is proved. For the order of \( N \) we will prove the followings:

1. \( |N| > \frac{n!}{4} \)

We know that \( L, K \leq N \) and \( |L| = |K| = \frac{(n-3)!}{2} \). Also

\[ L \cap K \leq C_G(\varphi(A)) \cap C_G(\varphi(B)) \]

and so

\[ |L \cap K| \leq \frac{|C_G(\varphi(A)) \cap C_G(\varphi(B))|}{2}. \]

From \( L, K \leq N \), we deduce that \( LK \leq N \). Thus

\[
\frac{|N|}{|LK|} = \frac{|L||K|}{|L \cap k|} \geq \frac{|L||K|}{|C_G(\varphi(A)) \cap C_G(\varphi(B))|} \geq \frac{\frac{(n-3)!}{2} \cdot \frac{(n-3)!}{2}}{\frac{(n-6)!}{2}} \frac{(n-6)!}{2} = \frac{n!}{4}.
\]

On the other hand

\[
\frac{(n-3)! \cdot \frac{(n-3)!}{2} \cdot \frac{(n-6)!}{2}}{2} > \frac{n!}{4}.
\]
for $n \geq 16$. Thus $|N| > \frac{n!}{4}$.

2. $|N| \neq \frac{n!}{3}$

We know that

$$C_G(\varphi(A)) \cong C_G(\varphi(B)) \cong S_{n-3}$$

and

$$N \cap C_G(\varphi(A)) \neq 1$$

and

$$N \cap C_G(\varphi(B)) \neq 1.$$  

If $C_G(\varphi(A)) \leq N$ and $C_G(\varphi(B)) \leq N$, then

$$C_G(\varphi(A))C_G(\varphi(B)) \subseteq N.$$  

Thus

$$|N| \geq |C_G(\varphi(A))C_G(\varphi(B))| = \frac{|C_G(\varphi(A))||C_G(\varphi(B))|}{|C_G(\varphi(A)) \cap C_G(\varphi(B))|} = \frac{(n-3)!(n-3)!}{(n-6)!}.$$  

But

$$\frac{(n-3)!(n-3)!}{(n-6)!} > \frac{n!}{2}$$

for $n \geq 16$, which implies that $|N| = |G|$ and since $N$ is an arbitrary minimal normal subgroup of $G$, we conclude that $G$ is a simple group. By assumption $\nabla(G) \cong \nabla(S_n)$ and [6] we have $G \cong S_n$, so $S_n$ must be a simple group too, which is a contradiction.

Hence

$$N \cap C_G(\varphi(A)) \neq C_G(\varphi(A))$$

or

$$N \cap C_G(\varphi(B)) \neq C_G(\varphi(B)).$$

Suppose that

$$N \cap C_G(\varphi(A)) \neq C_G(\varphi(A)).$$
We know that
\[ 1 \neq N \cap C_G(\varphi(A)) \leq C_G(\varphi(A)) \cong S_{n-3}. \]

Therefore
\[ |N \cap C_G(\varphi(A))| = |A_{n-3}| = \frac{(n-3)!}{2} \]

and so we have
\[ |N\mathbf{C}_G(\varphi(A))| = \frac{|N||C_G(\varphi(A))|}{|N \cap C_G(\varphi(A))|} = \frac{|N|(n-3)!}{(n-3)!} = 2|N|. \]

Moreover \( N \trianglelefteq G \) implies that \( N\mathbf{C}_G(\varphi(A)) \leq G \). Thus
\[ |N\mathbf{C}_G(\varphi(A))| = 2|N||G| = n!. \]

Now if \(|N| = \frac{n!}{4}\), then we have \( 2n!|n!| = n! \), a contradiction. This shows that \(|N| \neq \frac{n!}{4}\).

3. \(|N| = \frac{n!}{2}\)

From \(|N| > \frac{n!}{4}\) and \(|N||G| = n!\), we conclude that \(|N|\) is equal to one of \( \frac{n!}{4}, \frac{n!}{2} \) or \( n! \). By \( 2 |N| \neq \frac{n!}{4} \).

If \(|N| = |G| = n!\), then \( G \) is a simple group, since \( N \) is an arbitrary minimal normal subgroup of \( G \). By assumption \( \nabla(G) \cong \nabla(S_n) \). Now since \( G \) is a simple group, by [6] \( G \cong S_n \). So \( S_n \) must be a simple group too, a contradiction. Hence \(|N| = \frac{n!}{2}\).

From \(|N| = \frac{n!}{2}\), simplicity of \( N \) and by corollary 2.4, \( N \cong A_n \). We assert that \( C_G(N) = 1 \). Otherwise there is a minimal normal subgroup of \( G \), say \( M \) such that \( M \leq C_G(N) \). We proved that all minimal normal subgroups of \( G \) are isomorphic to \( A_n \). Thus \( M \cong A_n \) and since
\[ N \cap C_G(N) = Z(N) = 1, \]
\[ M \cap N = 1. \]

On the other hand \( MN \leq G \) and so
\[ |MN| = |M||N||G|. \]

It follows that \( (\frac{n!}{2})^2|G| = n!\), a contradiction. Hence \( C_G(N) = 1 \) and so
\[ G \cong \frac{G}{C_G(N)} \hookrightarrow Aut(N) \]
and since for $n \geq 16$,

$$\text{Aut}(N) \cong \text{Aut}(\mathbb{A}_n) \cong S_n,$$

we conclude that $G$ is embedded into $S_n$. But $|G| = |S_n|$ and so $G \cong S_n$.

\[\square\]

REFERENCES


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