www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650 , ISSN (on-line): 2251-7669
Vol. 2 No. 3 (2013), pp. 1-8.

www.ui.ac.ir

ON THE ORDER OF THE SCHUR MULTIPLIER OF A PAIR OF FINITE p-GROUPS II

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Communicated by Hamid Mousavi

Abstract. Let G be a finite p-group and N be a normal subgroup of G with $|N| = p^n$ and $|G/N| = p^m$. A result of Ellis (1998) shows that the order of the Schur multiplier of such a pair (G, N) of finite p-groups is bounded by $p^{\frac{1}{2}n(2m+n-1)}$ and hence it is equal to $p^{\frac{1}{2}n(2m+n-1)-t}$ for some non-negative integer t. Recently, the authors have characterized the structure of (G, N) when N has a complement in G and $t \leq 3$. This paper is devoted to classification of pairs (G, N) when N has a normal complement in G and $t = 4, 5$.

1. Introduction

By a pair of groups (G, N) we mean a group G with a normal subgroup N. In 1998, Ellis [2] defined the Schur multiplier of a pair (G, N) to be the abelian group $M(G, N)$ appearing in a natural exact sequence

$$\begin{aligned} H_3(G) &\to H_3(G/N) \to M(G,N) \to M(G) \to M(G/N) \\ &\to N/[N,G] \to (G)^{ab} \to (G/N)^{ab} \to 0 \end{aligned}$$

in which $H_3(G)$ is the third homology of G with integer coefficients. He [2] also noted that for any pair (G, N) of groups,

$$M(G,N) \cong \ker(N \wedge G \to G),$$

where $N \wedge G$ is the exterior product of N and G. In particular, if $N = G$, then $M(G, G)$ is the usual Schur multiplier of G.

MSC(2010): Primary: 20E34; Secondary: 20D15.

Keywords: Pair of groups, Schur multiplier, Finite p-groups.

Received: 13 April 2012, Accepted: 17 November 2012.

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In 1956, Green [4] showed that if G is a group of order p^n, then its Schur multiplier is of order at most $p^{\frac{n(n-1)}{2}}$ and hence equals to $p^{\frac{n(n-1)}{2}-t}$ for some non-negative integer t. Berkovich [1], Zhou [11], Ellis [3] and Niroomand [7, 8] determined the structure of G for $t = 0, 1, 2, 3, 4, 5$ by different methods.

In 1998, Ellis [2] gave an upper bound for the order of the Schur multiplier of a pair of finite p-groups. He proved that if G is a finite p-group with a normal subgroup N of order p^n and its quotient G/N of order p^m, then the Schur multiplier of (G, N) is bounded by $p^{\frac{1}{2}n(2m+n-1)}$ and hence equals to $p^{\frac{1}{2}n(2m+n-1)-t}$ for some non-negative integer t.

Let (G, N) be a pair of groups and K be the complement of N in G. In 2004, Salemkar, Moghaddam and Saeedi [10] characterized the structure of such a pair (G, N) when $t = 0, 1$ under some conditions. Recently, the authors [5] determined the structure of the pair (G, N), for $t = 0, 1$ without any condition and also gave the structure of (G, N) for $t = 2, 3$ when K is normal. In this paper, we are going to determine the structure of (G, N) for $t = 4, 5$ when K is a normal subgroup of G.

In this paper, D and Q denote the dihedral and the quaternion group of order 8, D_{16} denotes the dihedral group of order 16 and, E_1 and E_2 denote the extra special p-groups of order p^3 of odd exponent p and p^2, respectively. Also E_4 denotes the unique central product of a cyclic group of order p^2 and a non-abelian group of order p^3, and $\mathbf{Z}_n^{(m)}$ denotes the direct product of m copies of $\mathbf{Z}_n$.

The following result is essential to prove the main theorems.

Theorem 1.1. [2]*Let (G, N) be a pair of groups and K be the complement of N in G. Then*

$$M(G) \cong M(G, N) \times M(K).$$

In 1907, Schur [6] gave an structure for the Schur multiplier of a direct product of finite groups. He showed that

$$M(G_1 \times G_2) = M(G_1) \times M(G_2) \times (G_1^{ab} \otimes G_2^{ab}).$$

As a consequence of this fact we have the following important result.

Corollary 1.2. *Let (G, N) be a pair of groups and K be the complement of N in G. Then*

$$|M(G, N)| = |M(N)||N^{ab} \otimes K^{ab}|.$$

The following theorems give the structure of a finite p-group in terms of the order of its Schur multiplier.

Theorem 1.3. [3] *Let G be a group of prime-power order p^n with $|M(G)| = p^{\frac{1}{2}n(n-1)-t}$. Then*
i) $t = 0$ if and only if G is elementary abelian;
ii) $t = 1$ if and only if $G \cong \mathbf{Z}_{p^2}$ or $G \cong E_1$;
iii) $t = 2$ if and only if $G \cong \mathbf{Z}_p \times \mathbf{Z}_{p^2}$, $G \cong D$ or $G \cong \mathbf{Z}_p \times E_1$;
iv) $t = 3$ if and only if $G \cong \mathbf{Z}_{p^3}$, $G \cong \mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_{p^2}$, $G \cong Q$, $G \cong E_2$, $G \cong D \times \mathbf{Z}_2$ or $G \cong E_1 \times \mathbf{Z}_p \times \mathbf{Z}_p$.

Theorem 1.4. [9] *Let G be an abelian group of order p^n with $|M(G)| = p^{\frac{1}{2}n(n-1)-4}$. Then G is isomorphic to $\mathbf{Z}_{p^2} \times \mathbf{Z}_{p^2}$ or $\mathbf{Z}_{p^2} \times {\mathbf{Z}_p}^{(3)}$.*

The following result can be easily obtained by using a method similar to the proof of Theorem 1.4.

Theorem 1.5. *Let G be an abelian group of order p^n with $|M(G)| = p^{\frac{1}{2}n(n-1)-5}$. Then G is isomorphic to $\mathbf{Z}_{p^3} \times \mathbf{Z}_p$ or $\mathbf{Z}_{p^2} \times {\mathbf{Z}_p}^{(4)}$.*

Theorem 1.6. [7] *Let G be a non-abelian group of order p^n with $|M(G)| = p^{\frac{1}{2}n(n-1)-4}$. Then G is isomorphic to one of the following groups.*
For $p = 2$,
1) $D \times \mathbf{Z}_p^{(2)}$;
2) $Q \times \mathbf{Z}_2$;
3) $\langle a, b | a^4 = b^4 = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^2b^2\rangle$;
4) $\langle a, b, c | a^2 = b^2 = c^2 = 1, abc = bca = cab\rangle$;

For $p \neq 2$
5) E_4;
6) $E_1 \times \mathbf{Z}_p^{(3)}$;
7) $\mathbf{Z}_p^{(4)} \rtimes_\theta \mathbf{Z}_p$;
8) $E_2 \times \mathbf{Z}_p$;
9) $\langle a, b | a^{p^2} = b^p = 1, [a, b, a] = [a, b, b] = 1\rangle$;
10) $\langle a, b | a^9 = b^3 = 1, [a, b, a] = 1, [a, b, b] = a^6, [a, b, b, b] = 1\rangle$;
11) $\langle a, b | a^p = b^p = 1, [a, b, a] = [a, b, b, a] = [a, b, b, b] = 1\rangle (p \neq 3)$;

Theorem 1.7. [8] *Let G be a non-abelian group of order p^n with $|M(G)| = p^{\frac{1}{2}n(n-1)-5}$. Then G is isomorphic to one of the following groups.*
1) $D \times \mathbf{Z}_2^{(3)}$;
2) $E_1 \times \mathbf{Z}_p^{(4)}$;
3) $E_2 \times \mathbf{Z}_p^{(2)}$;
4) $E_4 \times \mathbf{Z}_p$;
5) *extra special p-group of order p^5;*
6) $\langle a, b | a^{p^2} = b^{p^2} = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^p\rangle$;
7) $\langle a, b | a^{p^2} = b^p = 1, [a, b, a] = [a, b, b] = a^p, [a, b, b, b,] = 1\rangle$;
8) $\langle a, b | a^{p^2} = b^p = 1, [a, b, a] = [a, b, b, b,] = 1, [a, b, b] = a^{np}\rangle$
where n is a fixed quadratic non-residue of p and $p \neq 3$;
9) $\langle a, b | a^{p^2} = 1, b^3 = a^3, , [a, b, a] = [a, b, b, b,] = 1, [a, b, b] = a^6\rangle$;
10) $\langle a, b | a^p = 1, b^p = [a, b, b], [a, b, a] = [a, b, b, b,] = [a, b, b, a] = 1\rangle$;
11) D_{16};
12) $\langle a, b | a^4 = b^4 = 1, a^{-1}ba = b^{-1}\rangle$;

13) $Q \times \mathbf{Z}_2^{(2)}$*;*
14) $(D \times \mathbf{Z}_2) \rtimes \mathbf{Z}_2$*;*
15) $(Q \times \mathbf{Z}_2) \rtimes \mathbf{Z}_2$*;*
16) $\mathbf{Z}_2 \times \langle a, b, c | a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$;

2. Main Results

In this section, let (G, N) be a pair of groups such that $G \cong N \times K$ with $|N| = p^n$ and $|K| = p^m$. As mentioned before, Ellis [2] showed that $|M(G, N)| = p^{\frac{1}{2}n(2m+n-1)-t}$ for some non-negative integer t. Recently, all these pairs of finite p-groups are listed in [5] by the authors, when $t = 0, 1, 2, 3$. The aim of this paper is to characterize the structure of such pairs of finite p-groups, when $t = 4, 5$.

Theorem 2.1. *By the above assumption,* $t = 4$ *if and only if* G *is isomorphic to one of the following groups.*
1) $G \cong N \times K$ *where* $N \cong \mathbf{Z}_p$ *and* K *is any group with* $d(K) = m - 4$*;*
2) $G \cong N \times K$ *where* $N \cong \mathbf{Z}_p \times \mathbf{Z}_p$ *and* K *is any group with* $d(K) = m - 2$*;*
3) $G \cong N \times K$ *where* $N \cong {\mathbf{Z}_p}^{(4)}$ *and* K *is any group with* $d(K) = m - 1$*;*
4) $G = N \cong D \times \mathbf{Z}_2^{(2)}$;
5) $G = N \cong Q \times \mathbf{Z}_2$;
6) $G = N \cong \langle a, b | a^4 = b^4 = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^2 b^2 \rangle$;
7) $G = N \cong \langle a, b, c | a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$;
8) $G = N \cong E_4$*;*
9) $G = N \cong E_1 \times \mathbf{Z}_p^{(3)}$*;*
10) $G = N \cong \mathbf{Z}_p^{(4)} \rtimes_\theta \mathbf{Z}_p$*;*
11) $G = N \cong E_2 \times \mathbf{Z}_p$*;*
12) $G = N \cong \langle a, b | a^{p^2} = b^p = 1, [a, b, a] = [a, b, b] = 1 \rangle$*;*
13) $G = N \cong \langle a, b | a^9 = b^3 = 1, [a, b, a] = 1, [a, b, b] = a^6, [a, b, b, b] = 1 \rangle$;
14) $G = N \cong \langle a, b | a^p = b^p = 1, [a, b, a] = [a, b, b, a] = [a, b, b, b] = 1 \rangle$
$(p \neq 3)$;
15) $G = N \cong \mathbf{Z}_{p^2} \times \mathbf{Z}_{p^2}$;
16) $G = N \cong \mathbf{Z}_{p^2} \times {\mathbf{Z}_p}^{(3)}$;
17) $G \cong N \times K$ *where* $K = \mathbf{Z}_p$ *and* $N \cong {\mathbf{Z}_p}^{(2)} \times \mathbf{Z}_{p^2}$;
18) $G \cong N \times K$ *where* $K = \mathbf{Z}_p$ *and* $N \cong Q$;
19) $G \cong N \times K$ *where* $K = \mathbf{Z}_p$ *and* $N \cong E_2$;
20) $G \cong N \times K$ *where* $K = \mathbf{Z}_p$ *and* $N \cong D \times \mathbf{Z}_2$;
21) $G \cong N \times K$ *where* $K = \mathbf{Z}_p$ *and* $N \cong E_1 \times {\mathbf{Z}_p}^{(2)}$*;*
22) $G \cong N \times K$ *where* $K = {\mathbf{Z}_p}^{(2)}$ *and* $N \cong \mathbf{Z}_{p^2} \times \mathbf{Z}_p$*;*
23) $G \cong N \times K$ *where* $K = {\mathbf{Z}_p}^{(2)}$ *and* $N \cong D$*;*

24) $G \cong N \times K$ where $K = \mathbf{Z}_p{}^{(2)}$ and $N \cong E_1 \times \mathbf{Z}_p$;
25) $G \cong N \times K$ where $K = \mathbf{Z}_{p^2} \times \mathbf{Z}_p$ and $N \cong \mathbf{Z}_{p^2}$;
26) $G \cong N \times K$ where $K = \mathbf{Z}_p{}^{(3)}$ and $N \cong E_1$;
27) $G \cong N \times K$ where $K = \mathbf{Z}_p{}^{(3)}$ and $N \cong \mathbf{Z}_{p^2}$.

Proof. The necessity of theorem follows from the fact that $G = N \times K$ and Corollary 1.2. For sufficiency, first suppose that N is an elementary abelian p-group. Then $nm - 4 = nd(K)$ and using Corollary 1.2 we have $|N \otimes K^{ab}| = p^{nm-4} = p^{nd(K)}$. Hence $n(m - d(K)) = 4$ which implies that $n = 1, 2$ or 4. Therefore $N \cong \mathbf{Z}_p$ and K is any group with $d(K) = m - 4$ or $N \cong \mathbf{Z}_p{}^{(2)}$ and K is any group with $d(K) = m - 2$, or $N \cong \mathbf{Z}_p{}^{(4)}$ and K is any group with $d(K) = m - 1$.

Now suppose that N is not an elementary abelian p-group . Then using Corollary 1.2 we have $|N^{ab} \otimes K^{ab}| > p^{nm-4}$ and so $md(N) > nm - 4$ which implies that $m(n - d(N)) < 4$. Therefore $m = 0, 1, 2, 3$.

If $m = 0$, then $K = 1$ and N is one of the groups which are listed in Theorems 1.4 and 1.6.

If $m = 1$, then $K = \mathbf{Z}_p$ and $d(N) = n - 1, n - 2$ or $n - 3$. It follows that $|N^{ab} \otimes K| = p^{n-1}, p^{n-2}$ or p^{n-3} and so Corollary 1.2 implies that $|M(N)| = p^{\frac{n^2-n}{2}-3}, p^{\frac{n^2-n}{2}-2}$, or $p^{\frac{n^2-n}{2}-1}$, respectively. In the first case N is $\mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_{p^2}, Q, E_2, , D \times \mathbf{Z}_2$ or $E_1 \times \mathbf{Z}_p \times \mathbf{Z}_p$ and the other cases are impossible by Theorem 1.3.

If $m = 2$, then $d(N) = n - 1$ and $K = \mathbf{Z}_p \times \mathbf{Z}_p$ or $K = \mathbf{Z}_{p^2}$. In the first case $|N^{ab} \otimes K| = p^{2(n-1)}$ and so $|M(N)| = p^{\frac{n^2-n}{2}-2}$. Therefore N is $\mathbf{Z}_p \times \mathbf{Z}_{p^2}, D$ or $\mathbf{Z}_p \times E_1$. In the second case $N^{ab} \cong \mathbf{Z}_p{}^{(n-1)}$ or $N^{ab} \cong \mathbf{Z}_{p^2} \times \mathbf{Z}_p{}^{(n-2)}$. If N^{ab} is an elementary abelian p-group, then $|N^{ab} \otimes K| = p^{(n-1)}$ and so $|M(N)| = p^{\frac{n^2+n-6}{2}}$ which is impossible. If $N^{ab} \cong \mathbf{Z}_{p^2} \times \mathbf{Z}_p{}^{(n-2)}$, then $|M(N)| = p^{\frac{n^2+n-8}{2}}$ which is impossible too.

If $m = 3$, then $d(N) = n - 1$ and K is an abelian p-group of order p^3 or an extra special p-group of order p^3. In the first case we have three possibilities for K. The first possibility is $K \cong \mathbf{Z}_p^{(3)}$, and similar to the previous part, one can see that $|M(N)| = p^{\frac{n^2-n}{2}-1}$ and so $N \cong E_1$ or $\mathbf{Z}_{p^2}$. The second possibility is $K \cong \mathbf{Z}_{p^3}$. This implies that $n = 1$ which is a contradiction. The third possibility is $K \cong \mathbf{Z}_{p^2} \times \mathbf{Z}_p$ which implies that $n = 2$ and $N \cong \mathbf{Z}_{p^2}$.
In the second case, if K is an extra special p-group of order p^3, then $N^{ab} \cong \mathbf{Z}_p^{(n-1)}$ or $N^{ab} \cong \mathbf{Z}_{p^2} \times \mathbf{Z}_p^{(n-2)}$. This implies that $|N^{ab} \otimes K| = |N^{ab} \otimes \mathbf{Z}_p^{(2)}| = p^{2n-2}$ and so $n = 1$ which is a contradiction. Hence the proof is complete. □

Theorem 2.2. *By the previous assumption, $t = 5$ if and only if G is isomorphic to one of the following groups.*
1) $G \cong N \times K$ where $N \cong \mathbf{Z}_p$ and K is any group with $d(K) = m - 5$;
2) $G \cong N \times K$ where $N \cong \mathbf{Z}_p{}^{(5)}$ and K is any group with $d(K) = m - 1$;
3) $G = N \cong D \times \mathbf{Z}_2^{(3)}$;
4) $G = N \cong E_1 \times \mathbf{Z}_p^{(4)}$;
5) $G = N \cong E_2 \times \mathbf{Z}_p^{(2)}$;

6) $G = N \cong E_4 \times \mathbf{Z}_p$;

7) $G = N \cong$ *an extra special p-group of order* p^5;

8) $G = N \cong \langle a, b | a^{p^2} = b^{p^2} = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^p \rangle$;

9) $G = N \cong \langle a, b | a^{p^2} = b^p = 1, [a, b, a] = [a, b, b] = a^p, [a, b, b, b,] = 1 \rangle$;

10) $G = N \cong \langle a, b | a^{p^2} = b^p = 1, [a, b, a] = [a, b, b, b,] = 1, [a, b, b] = a^{np} \rangle$,
where n *is a fixed quadratic non-residue of* p *and* $p \neq 3$;

11) $G = N \cong \langle a, b | b^3 = a^3, a^{p^2} = [a, b, a] = [a, b, b, b,] = 1, [a, b, b] = a^6 \rangle$;

12) $G = N \cong \langle a, b | a^p = 1, b^p = [a, b, b], [a, b, a] = [a, b, b, b,] = [a, b, b, a] = 1 \rangle$;

13) $G = N \cong D_{16}$;

14) $G = N \cong \langle a, b | a^4 = b^4 = 1, a^{-1}ba = b^{-1} \rangle$;

15) $G = N \cong Q \times \mathbf{Z}_2^{(2)}$;

16) $G = N \cong (D \times \mathbf{Z}_2) \rtimes \mathbf{Z}_2$;

17) $G = N \cong (Q \times \mathbf{Z}_2) \rtimes \mathbf{Z}_2$;

18) $G = N \cong \mathbf{Z}_2 \times < a, b, c | a^2 = b^2 = c^2 = 1, abc = bca = cab >$;

19) $G = N \cong \mathbf{Z}_{p^3} \times \mathbf{Z}_p$;

20) $G = N \cong \mathbf{Z}_{p^2} \times \mathbf{Z}_p{}^{(4)}$;

21) $G \cong N \times K$ *where* $K = \mathbf{Z}_p$ *and* $N \cong D \times \mathbf{Z}_2{}^{(2)}$;

22) $G \cong N \times K$ *where* $K = \mathbf{Z}_p$ *and* $N \cong Q \times \mathbf{Z}_2$;

23) $G \cong N \times K$ *where* $K = \mathbf{Z}_p$ *and* $N \cong < a, b, c | a^2 = b^2 = c^2 = 1, abc = bca = cab >$;

24) $G \cong N \times K$ *where* $K = \mathbf{Z}_p$ *and* $N \cong E_4$;

25) $G \cong N \times K$ *where* $K = \mathbf{Z}_p$ *and* $N \cong E_1 \times \mathbf{Z}_p{}^{(3)}$;

26) $G \cong N \times K$ *where* $K = \mathbf{Z}_p$ *and* $N \cong \mathbf{Z}_p^{(4)} \rtimes \mathbf{Z}_p$;

27) $G \cong N \times K$ *where* $K = \mathbf{Z}_p$ *and* $N \cong E_2 \times \mathbf{Z}_p$;

28) $G \cong N \times K$ *where* $K = \mathbf{Z}_p$ *and* $N \cong E_2 \times \mathbf{Z}_p$;

29) $G \cong N \times K$ *where* $K = \mathbf{Z}_p{}^{(2)}$ *and* $N \cong E_1 \times \mathbf{Z}_p{}^{(2)}$;

30) $G \cong N \times K$ *where* $K = \mathbf{Z}_p{}^{(2)}$ *and* $N \cong \mathbf{Z}_p{}^{(2)} \times \mathbf{Z}_{p^2}$;

31) $G \cong N \times K$ *where* $K = \mathbf{Z}_p{}^{(2)}$ *and* $N \cong Q$;

32) $G \cong N \times K$ *where* $K = \mathbf{Z}_p{}^{(2)}$ *and* $N \cong E_2$;

33) $G \cong N \times K$ *where* $K = \mathbf{Z}_p{}^{(2)}$ *and* $N \cong D \times \mathbf{Z}_2$;

34) $G \cong N \times K$ *where* $K = \mathbf{Z}_{p^2}$ *and* $N \cong E_1$;

35) $G \cong N \times K$ *where* $K = \mathbf{Z}_{p^2}$ *and* $N \cong \mathbf{Z}_p \times \mathbf{Z}_{p^2}$;

36) $G \cong N \times K$ *where* $K = \mathbf{Z}_{p^3}$ *and* $N \cong \mathbf{Z}_{p^2}$;

37) $G \cong N \times K$ *where* $K = \mathbf{Z}_p{}^{(3)}$ *and* $N \cong D$;

38) $G \cong N \times K$ *where* $K = \mathbf{Z}_p{}^{(3)}$ *and* $N \cong E_1 \times \mathbf{Z}_p$;

39) $G \cong N \times K$ *where* $K = \mathbf{Z}_p{}^{(3)}$ *and* $N \cong \mathbf{Z}_p \times \mathbf{Z}_{p^2}$;

40) $G \cong N \times K$ *where* K *is an extra special* P*-group and* $N \cong \mathbf{Z}_{p^2}$;

41) $G \cong N \times K$ *where* $K = \mathbf{Z}_p{}^{(4)}$ *and* $N \cong E_1$;

42) $G \cong N \times K$ where $K = \mathbf{Z}_p{}^{(2)} \times \mathbf{Z}_p$ and $N \cong \mathbf{Z}_{p^2}$.

Proof. The proof of this theorem is similar to the proof of the previous theorem so we left the details to the reader. Necessity is straightforward. For sufficiency, first suppose that N is an elementary abelian p-group . Then $n(m - d(K)) = 5$, so $n = 1$ or 5. If $n = 1$, then $N \cong \mathbf{Z}_p$ and K is any group with $d(K) = m - 5$ and $n = 5$ which implies that $N \cong \mathbf{Z}_p^{(5)}$ and K is any group with $d(K) = m - 1$.

Suppose that N is not an elementary abelian p-group. Then we have $|N^{ab} \otimes K^{ab}| > p^{nm-5}$ by Corollary 1.2. It follows that $md(N) > nm - 5$ and thus $m(n - d(N)) < 5$ which implies that $m = 0, 1, 2, 3$ or 4. If $m = 0$, then $K = 1$ and N is one of the groups that are listed in Theorems 1.5 and 1.7.

If $m = 1$, then $K = \mathbf{Z}_p$ and $d(N) = n - i$, for $1 \leq i \leq 4$. Therefore by Corollary 1.2 $|M(N)| = p^{\frac{n(n-1)}{2}-(5-i)}$, for $1 \leq i \leq 4$, respectively. It follows that $N \cong \mathbf{Z}_{p^2} \times \mathbf{Z}_p{}^{(3)}$, $D \times \mathbf{Z}_2^{(2)}$, $Q \times \mathbf{Z}_2$, E_4, $E_1 \times \mathbf{Z}_p^{(3)}$, $E_2 \times \mathbf{Z}_p$, $N \cong \mathbf{Z}_{p^3}$ or $\langle a, b, c | a^2 = b^2 = c^2 = 1, abc = bca = cab\rangle$ by Theorems 1.6, 1.4 and 1.3.

If $m = 2$, then $K = \mathbf{Z}_{p^2}$ or $K = \mathbf{Z}_p \times \mathbf{Z}_p$ and $d(N) = n - 1$ or $d(N) = n - 2$. First suppose that $K = \mathbf{Z}_p \times \mathbf{Z}_p$. If $d(N) = n - 1$, then $|M(N)| = p^{\frac{n^2-n}{2}-3}$. It follows that $N \cong \mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_{p^2}$, $N \cong E_1 \times \mathbf{Z}_p \times \mathbf{Z}_p$, $N \cong Q$, $N \cong E_2$ or $N \cong D \times \mathbf{Z}_2$. If $d(N) = n - 2$, then $|N \otimes K| = p^{2(n-2)}$. This implies that $|M(N)| = p^{\frac{n^2+n-2}{2}}$ and so $n < 1$ which is impossible.
Now suppose that $K = \mathbf{Z}_{p^2}$. If $d(N) = n - 1$ and $N^{ab} \cong \mathbf{Z}_p{}^{(n-1)}$, then $|M(N)| = p^{\frac{n^2+n-8}{2}}$ which implies that $n = 3$ and $|M(N)| = p^2$. Therefore $N \cong E_1$. If $d(N) = n - 1$ and $N^{ab} \cong \mathbf{Z}_p^2 \times \mathbf{Z}_p{}^{(n-2)}$, then $n = 3$ or $n = 4$. For $n = 3$ there is not any structure for N and $n = 4$ implies that $N \cong \mathbf{Z}_{p^2} \times \mathbf{Z}_p$. If $d(N) = n - 2$, then $N^{ab} \cong \mathbf{Z}_p^{(n-2)}$ or $N^{ab} \cong \mathbf{Z}_p^3 \times \mathbf{Z}_p^{(n-3)}$ or $N^{ab} \cong \mathbf{Z}_p^2 \times \mathbf{Z}_p^2 \times \mathbf{Z}_p^{(n-4)}$. Therefore similar to the previous case one can see that $n < 5$ which is impossible.

If $m = 3$, then $d(N) = n - 1$ and K is an abelian p-group of order p^3 or is an extra special p-group of order p^3. In the first case we have three possibilities for K. The first possibility is $K \cong \mathbf{Z}_{p^3}$. If $N^{ab} \cong \mathbf{Z}_p^{(n-1)}$, then $n = 1$ which is impossible and if $N^{ab} \cong \mathbf{Z}_{p^2} \times \mathbf{Z}_p^{(n-2)}$, then $N \cong \mathbf{Z}_{p^2}$. The second possibility is $K \cong \mathbf{Z}_{p^2} \times \mathbf{Z}_p$. In this case, there is no structure for N. The third possibility is $K \cong \mathbf{Z}_p^{(3)}$. Thus $|M(N)| = p^{\frac{n^2-n}{2}-2}$ and so $N \cong D$, $N \cong E_1 \times \mathbf{Z}_p$ or $N \cong \mathbf{Z}_{p^2} \times \mathbf{Z}_p$ by Theorem 1.3.
Now suppose that K is an extra special p-group of order p^3. Then $N^{ab} \cong \mathbf{Z}_p^{(n-1)}$ or $N^{ab} \cong \mathbf{Z}_{p^2} \times \mathbf{Z}_p^{(n-2)}$. If N^{ab} is an elementary abelian, then there is no structure for N. Otherwise $N = \mathbf{Z}_{p^2}$.

If $m = 4$, then $d(N) = n-1$. So $N^{ab} \cong \mathbf{Z}_p^{(n-1)}$ or $N^{ab} \cong \mathbf{Z}_p^2 \times \mathbf{Z}_p^{(n-2)}$. In the first case $|N^{ab} \otimes K^{ab}| = |\mathbf{Z}_p^{(n-1)} \otimes K^{ab}| \leq p^{d(K)(n-1)}$. Now suppose that $d(K) < 4$. Then we have $|N^{ab} \otimes K^{ab}| \leq p^{3(n-1)}$. Therefore $|M(N)| \geq p^{-3(n-1)+(8n+n^2-n-10)/2}$ by Corollary 1.2. So $n < 2$ which is impossible. If $d(K) = 4$, then we have $K \cong \mathbf{Z}_p^{(4)}$. Hence $|N^{ab} \otimes K^{ab}| = p^{4(n-1)}$ which implies that $|M(N)| = p^{(n^2-n)/2-1}$. So $N \cong E_1$ by Theorem 1.3.
In the second case, suppose that K is not abelian. Then $|N^{ab} \otimes K^{ab}| = |\mathbf{Z}_{p^2} \times \mathbf{Z}_p^{(n-2)} \otimes K^{ab}| = |\mathbf{Z}_{p^2} \otimes K^{ab}||\mathbf{Z}_p^{(n-2)} \otimes K^{ab}| \leq p^3 p^{d(K)(n-2)} \leq p^{3(n-1)}$ which implies that $n < 2$ and it is impossible.
If K is abelian, then $K \cong \mathbf{Z}_{p^2} \times \mathbf{Z}_p^{(2)}$. Thus $|N^{ab} \otimes K| = p^{3n-2}$. It follows that $N \cong \mathbf{Z}_{p^2}$. This completes the proof. □

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