www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650 , ISSN (on-line): 2251-7669
Vol. 2 No. 1 (2013), pp. 37-43.

www.ui.ac.ir

GROUPS WITH ALL SUBGROUPS PERMUTABLE OR SOLUBLE

MARTYN R. DIXON* AND Z. YALCIN KARATAS

Communicated by Patrizia Longobardi

Abstract. In this paper, we consider locally graded groups in which every non-permutable subgroup is soluble of bounded derived length.

1. **Introduction**

A constantly recurring theme in group theory has been the effort to classify groups all of whose proper subgroups have some given property. This type of research was started by R. Dedekind [3], where he classified the finite groups with all subgroups normal. This theme has been continued using several generalizations of normality. One such generalization of course is the notion of subnormality. The literature concerning groups with all subgroups subnormal is long and varied, one highlight being the theorem of Möhres [12] that such a group is soluble. Further details concerning groups with all subgroups subnormal can be found in [10] and [9].

A further generalization of normality is permutability. A subgroup H of a group G is said to be *permutable* if $HK = KH$ for every subgroup K of G and in [8] the structure of infinite groups with all subgroups permutable was obtained. Full details of this work, including generalizations of the notion of permutability, can be found in [18]

In a different direction, Schmidt [17] obtained the structure of finite non-nilpotent groups with all proper subgroups nilpotent and he showed in particular that such groups are soluble. It is not possible to extend this theorem to infinite groups since, as is well-known, there are 2-generator infinite simple groups with all proper subgroups abelian (see, for example, the book of A. Yu Ol'shanskii [13]). On

MSC(2010): Primary: 20E07; Secondary: 20F15.

Keywords: permutable, soluble, locally graded.

Received: 29 October 2012, Accepted: 22 November 2012.

*Corresponding author.

the other hand, one consequence of the amazing paper [1] shows that an infinite locally graded group with all proper subgroups nilpotent-by-Chernikov is necessarily soluble. As usual a group is locally graded if every nontrivial finitely generated subgroup has a nontrivial finite image.

Recently many authors have been interested in groups in which only certain subgroups have some given property and a common theme has been to consider groups whose normal (or subnormal or permutable) subgroups are abelian (or nilpotent or soluble). Among the first results of this type are those of Romalis and Sesekin [15, 19, 16] who discussed meta-Hamiltonian groups–those groups with all non-normal subgroups abelian. Bruno and Phillips [2] continued this theme by considering groups whose non-normal subgroups are locally nilpotent. Because of the existence of Tarski monsters they restricted attention to groups G such that every finitely generated non-nilpotent subgroup of G has a finite non-nilpotent image. Groups with all non-subnormal subgroups nilpotent have been the subject of papers of H. Smith [20, 21]. To round out this short relevant history of the subject we mention the paper [6] where the authors prove, among other things, that a locally graded group with all non-permutable subgroups abelian is soluble of derived length at most 4.

The purpose of our paper is to discuss locally graded groups in which all non-permutable subgroups are soluble. Our research has been stimulated by a recent paper of K. Ersoy, A. Tortora and M. Tota [5] who have obtained some nice results concerning locally graded groups in which all non-subnormal subgroups are soluble of bounded derived length and we are indebted to these authors for giving us a preprint of their work. Work in this particular area is trickier for a couple of reasons. Unlike with nilpotent groups, a finite group in which all proper subgroups are soluble need not be soluble. However, J. Thompson [23] has exhibited the finite minimal simple groups–those non-abelian simple groups in which every proper subgroup is soluble. A further problem is that, as yet, the structure of a locally graded group in which every proper subgroup is soluble is not fully understood, although in [7] it is shown that such groups are hyperabelian.

We shall prove several results that complement the results of [5] and partially extend those results in [6] and [2]. Our main result is the following theorem.

Theorem 1.1. *Let G be a locally graded group in which every subgroup is permutable or soluble of derived length at most d.*

(i) *If G is soluble then G has derived length at most $d+3$;*
(ii) *If G is not soluble, then G'' is finite and perfect. Also all proper subgroups of G'' are soluble of derived length at most d.*

It seems to be unknown in the soluble case whether this bound is the best possible, even in the case when $d = 1$. There are some consequences of this result that are also of interest and these will become evident during the course of the proofs. Much of our notation is standard and can be found in [14]. We note that $\mathfrak{N}_c$ denotes the class of nilpotent groups of nilpotency class at most c and $\mathfrak{S}_d$ denotes the class of soluble groups of derived length at most d.

2. **Proof of the Theorems**

We begin the proof of Theorem 1.1 with the following lemma.

Lemma 2.1. *Let G be an (infinite torsion-free abelian)-by-finite group with all subgroups permutable or soluble of derived length at most d. Then G is not perfect.*

Proof. Suppose that G is perfect. Let T be an abelian normal subgroup of G such that G/T is finite. By our assumption, $G \neq T$. Then there is a finitely generated subgroup F of G such that $G = TF$ and since G is perfect, $F \notin \mathfrak{S}_d$. Hence F is permutable in G. On the other hand, we have $T \cap F \lhd G$ and hence

$$\frac{G}{T \cap F} = \frac{T}{T \cap F} \rtimes \frac{F}{T \cap F}.$$

Certainly $F/T \cap F$ is permutable in $G/T \cap F$. Also $F/T \cap F \cong G/T$ is finite and perfect. Hence, by [22, Corollary C1], we have $F/T \cap F \lhd G/T \cap F$. Consequently, $F \lhd G$. Now, however, G/F is abelian and perfect, so $G = F$. Therefore G is finitely generated, as is T since $|G : T|$ is finite. A standard argument, which we now give, allows us to complete the proof. Let p be a prime not dividing $|G : T|$. Then, G/T^p is a finite group. Since $|T/T^p|$ and $|G/T^p : T/T^p|$ are relatively prime, the Schur-Zassenhaus Theorem implies that G/T^p splits over T/T^p and we have

$$\frac{G}{T^p} = \frac{T}{T^p} \rtimes \frac{E}{T^p},$$

where E/T^p is a p'-group. Now E/T^p is perfect since $E/T^p \cong G/T$ and hence E/T^p is permutable in G/T^p. Hence by [22, Corollary C1], $E/T^p \lhd G/T^p$ and we have $E \lhd G$. However, G/E is abelian and perfect, so $G = E$ which gives us a contradiction. The result follows.

□

We can now proceed directly to the proof of Theorem 1.1.

Proof. If $G^{(3)} \in \mathfrak{S}_d$, then clearly G is soluble of derived length at most $d + 3$ and (i) holds. Hence we may assume that $G^{(3)} \notin \mathfrak{S}_d$. Then every subgroup containing $G^{(3)}$ is permutable, as is every subgroup of $G/G^{(3)}$. Hence $G/G^{(3)}$ is metabelian by [18, 2.4.22 Theorem] so that $G'' = G^{(3)}$ is perfect and we may suppose that G is not soluble.

Let $T \lhd G''$. If $T \notin \mathfrak{S}_d$, then every subgroup of G''/T is permutable, and hence G''/T is metabelian by [18, 2.4.22 Theorem]. Then G''/T is metabelian and perfect which implies that $G'' = T$ and we conclude that every proper normal subgroup of G'' is soluble of derived length at most d.

Let S be the product of all proper normal subgroups of G''. If $N_1, N_2, \ldots, N_k$ are proper normal subgroups of G'', then $N_1N_2 \ldots N_k$ is soluble, so, as G'' is perfect, we have $G'' \neq N_1N_2 \ldots N_k$. Hence $N_1N_2 \ldots N_k$ is a proper normal subgroup of G'', so $N_1N_2 \ldots N_k \in \mathfrak{S}_d$. Hence $S \in \mathbf{L}\mathfrak{S}_d = \mathfrak{S}_d$. Therefore, $S \neq G''$ and G''/S is simple.

We note that by a theorem of Longobardi, Maj and Smith [11], G''/S is also locally graded. A simple group contains no proper nontrivial permutable subgroups, by [22, Corollary C2], so the proper subgroups of G''/S are all soluble of derived length at most d. Thus if G''/S is not finitely generated then G''/S is soluble of derived length at most d, a contradiction since G''/S is perfect. Hence G''/S is finitely generated and therefore finite. Thus G''/S is a minimal simple group.

If all proper subgroups of G'' are soluble of derived length at most d, then by [4, Lemma 2.1], either G'' is an infinite soluble group or G'' is finite. Since G'' is perfect this implies that G'' is finite in this case. Therefore suppose that G'' contains a proper subgroup $X \notin \mathfrak{S}_d$. Then, X is permutable in G'', and XS/S is permutable in G''/S. Since G''/S is simple, [22, Corollary C2], implies that $XS = S$ or $G'' = XS$. We have $G'' = XS$ as $X \notin \mathfrak{S}_d$.

Suppose now that S is infinite and that i is the first index such that $S^{(i)}/S^{(i+1)}$ is infinite. Thus $G''/S^{(i)}$ is finite. Let $T/S^{(i+1)}$ be the torsion subgroup of $S^{(i)}/S^{(i+1)}$. Suppose that $T \neq S^{(i)}$. Then $G''/T \cong (G''/S^{(i+1)})/(T/S^{(i+1)})$ is (abelian torsion-free)-by-finite. This is not possible by Lemma 2.1 since G''/T is perfect. Hence $S^{(i)}/S^{(i+1)}$ is periodic and $G''/S^{(i+1)}$ is a locally finite group. Hence, there is a finite subgroup $F/S^{(i+1)}$ such that

$$\frac{G''}{S^{(i+1)}} = \frac{S^{(i)}}{S^{(i+1)}} \cdot \frac{F}{S^{(i+1)}}.$$

Now, $F/S^{(i+1)} \notin \mathfrak{S}_d$, otherwise $G''/S^{(i+1)}$ is soluble which is not possible as G'' is perfect. Hence $F \notin \mathfrak{S}_d$, so F is permutable in G''.

Suppose that $E/S^{(i+1)}$ is a proper subgroup of $F/S^{(i+1)}$ such that $E/S^{(i+1)} \notin \mathfrak{S}_d$. Then $E \notin \mathfrak{S}_d$ so E is permutable in G'' and $G'' = ES$. Now, $ES^{(i)}/S^{(i)}$ is permutable in $G''/S^{(i)}$, which is finite and perfect. By [18, 5.2.6 Corollary] we have $ES^{(i)}/S^{(i)} \triangleleft G''/S^{(i)}$, so $ES^{(i)} \triangleleft G''$. Then $G''/ES^{(i)} = ES/ES^{(i)}$ is perfect and soluble, and $G'' = ES^{(i)}$ also. So we have

$$\frac{G''}{S^{(i+1)}} = \frac{S^{(i)}}{S^{(i+1)}} \cdot \frac{E}{S^{(i+1)}}.$$

We deduce that there is a finite permutable supplement to $S^{(i)}/S^{(i+1)}$ in which every proper subgroup is soluble of derived length at most d. Let $E/S^{(i+1)}$ be such a supplement and note that it is not soluble. Then $E/S^{(i+1)}$ must be a perfect, permutable subgroup of $G''/S^{(i+1)}$, so $E/S^{(i+1)} \triangleleft G''/S^{(i+1)}$ again by [22, Corollary C1]. Hence, $E \triangleleft G''$. Then, $G''/E = ES^{(i)}/E$ is both perfect and soluble, so $G'' = E$. Then, $G''/S^{(i+1)} = E/S^{(i+1)}$ is a finite group, which is again a contradiction. Hence S is finite and so is G''.

Also, we know that $G'' = XS$ where X is permutable in G'' and $X \notin \mathfrak{S}_d$. Since G'' is finite, $X \triangleleft G''$ by [22, Corollary C1]. Therefore, G''/X is perfect and soluble, so $G'' = X$. Hence we conclude that all proper subgroups of G'' are soluble of derived length at most d.

□

The following result is rather straightforward.

Lemma 2.2. *Let G be a (finite non-abelian simple)-by-metabelian group. Then G is metabelian-by-finite.*

Proof. Let S be a normal subgroup of G such that S is finite simple and G/S is metabelian. Then we can find an embedding

$$\frac{G}{C_G(S)} \hookrightarrow Aut(S).$$

Thus, $G/C_G(S)$ is finite. Also, $C_G(S)S/S \cong C_G(S)/C_G(S) \cap S \cong C_G(S)$ is a subgroup of G/S, so $C_G(S)$ is metabelian and we are done.

□

Next we obtain a result analogous to a result in [5], and proved in the same way.

Lemma 2.3. *Let G be a locally graded group in which every subgroup is permutable or soluble of derived length at most d. If G is not soluble, then G is (soluble of derived length d)-by-(finite almost minimal simple).*

Proof. By Theorem 1.1 we know that G'' is finite perfect and all proper subgroups of G'' are soluble of derived length at most d. Let S be the unique maximal normal subgroup of G''. Then G/S is (finite simple)-by-metabelian, so G/S is metabelian-by-finite by Lemma 2.2. Hence G is soluble-by-finite. Let T be the soluble radical of G. If $T \notin \mathfrak{S}_d$, then, as usual, $G'' \leq T$ and G'' is soluble, a contradiction. Hence, G is (soluble of derived length d)-by-finite.

Next we consider G/T. This has trivial soluble radical. If $\bar{M} = M/T$ is a minimal normal subgroup of $\bar{G} = G/T$ then $\bar{M}$ is a direct product of isomorphic finite non-abelian simple groups. If $\bar{N} = N/T$ is one of these simple groups then $\bar{N}$ is a perfect, perumutable subgroup of $\bar{G}$ so, by [22, Corollary C1], $N \triangleleft G$. Hence $\bar{M}$ is a finite simple group. Since a simple group cannot contain nontrivial proper permutable subgroups, all proper subgroups of $\bar{M}$ are soluble of derived length at most d, so $\bar{M}$ is a minimal simple group. Then $C_{\bar{G}}(\bar{M}) \cap \bar{M} = 1$ and since G/M is metabelian so is $C_{\bar{G}}(\bar{M})$. Since $\bar{G}$ has trivial soluble radical it follows that $C_{\bar{G}}(\bar{M})$ is trivial. Hence $\bar{G}$ embeds as a subgroup of $\mathrm{Aut}\,\bar{M}$. This completes the proof.

□

Finally we have the following theorem, which is the true analogue of [6, Lemma 2.2]

Theorem 2.4. *Let G be a locally graded group with all subgroups permutable or nilpotent of class at most c. Then G is soluble of derived length at most $4 + [\log_2 c]$.*

Proof. It is well-known that a nilpotent group of class c has derived length at most $d = 1 + [\log_2 c]$. If $X = G^{(3)} \in \mathfrak{S}_d$ then G is soluble of derived length at most $4 + [\log_2 c]$ as required. Consequently we may suppose that $X \notin \mathfrak{S}_d$ so that certainly $X \notin \mathfrak{N}_\mathfrak{c}$. As in the proof of Theorem 1.1, $X = G''$ and G'' is perfect. Let S be the product of the proper normal subgroups of G''. If $S \neq G''$ then we can show, as in the proof of Theorem 1.1, that $S \in \mathfrak{N}_c$ and that all proper subgroups of G''/S are nilpotent of class at most c. Since G''/S is locally graded, if it is finitely generated then it must be finite. By Schmidt's Theorem [17], mentioned earlier, G''/S is then soluble, which is a contradiction. On the other hand, if G''/S is not finitely generated then $G''/S \in \mathbf{L}\mathfrak{N}_c = \mathfrak{N}_c$, again a contradiction.

Hence $G'' = S$ in which case every finitely generated subgroup of G'' is a product of finitely many normal nilpotent subgroups each of class c. Since G'' is perfect this implies that G'' is not finitely generated and in this case we obtain the final contradiction that $G'' \in \mathbf{L}\mathfrak{N}_c = \mathfrak{N}_c$. □

Acknowledgments

The authors would like to extend special thanks to K. Ersoy, A. Tortora and M. Tota for giving us a preprint of their work. The first author would like to thank the University of Alabama and the Ischia Group Theory 2012 conference sponsors for financial support.

References

[1] A. O. Asar, Locally nilpotent p-groups whose proper subgroups are hypercentral or nilpotent-by-Chernikov, *J. London Math. Soc. (2)*, **61** (2000) 412–422.

[2] B. Bruno and R. E. Phillips, Groups with restricted nonnormal subgroups, *Math. Z.*, **176** no. 2 (1981) 199–221.

[3] R. Dedekind, Über Gruppen deren sämtliche Teiler Normalteiler sind, *Math. Ann.*, **48** (1897) 548–561.

[4] M. R. Dixon and M. J. Evans, Groups with the minimum condition on insoluble subgroups, *Arch. Math. (Basel)*, **72** (1999) 241–251.

[5] K. Ersoy, A. Tortora, and M. Tota, On Groups with all subgroups subnormal or soluble of bounded derived length, *manuscript.*

[6] M. De Falco, F. De Giovanni, C. Musella, and R. Schmidt, Groups in which every non-abelian subgroup is permutable, *Rend. Circ. Mat. Palermo* (2), **52** no. 1 (2003) 70–76.

[7] S. Franciosi, F. de Giovanni, and M. L. Newell, Groups with polycyclic non-normal subgroups, *Algebra Colloq.*, **7** no. 1 (2000) 33–42.

[8] K. Iwasawa, On the structure of infinite M-groups, *Jap. J. Math.*, **18** (1943) 709–728.

[9] J. C. Lennox and D. J. S. Robinson, The Theory of Infinite Soluble Groups, *Oxford Mathematical Monographs*, Oxford University Press, Oxford, 2004.

[10] J. C. Lennox and S. E. Stonehewer, Subnormal Subgroups of Groups, *Oxford University Press*, Oxford, 1987.

[11] P. Longobardi, M. Maj, and H. Smith, A note on locally graded groups, *Rend. Sem. Mat. Univ. Padova*, **94** (1995) 275–277.

[12] W. Möhres, Auflösbarkeit von Gruppen, deren Untergruppen alle subnormal sind, *Arch. Math. (Basel)*, **54** no. 3 (1990) 232–235.

[13] A. Yu Ol'shanskii, Geometry of Defining Relations in Groups, *Mathematics and its Applications*, **70**, Kluwer Academic Publishers, Dordrecht, 1991.

[14] D. J. S. Robinson, Finiteness Conditions and Generalized Soluble Groups, **1 and 2**, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Springer-Verlag, Berlin, Heidelberg, New York, 1972, Band 62 and 63.

[15] G. M. Romalis and N. F. Sesekin, Metahamiltonian groups, *Ural. Gos. Univ. Mat. Zap.* **5** no. 3 (1966) 101–106.

[16] ———, The metahamiltonian groups. III, *Ural. Gos. Univ. Mat. Zap.*, **7** no. 3 (1969/1970) 195–199.

[17] O. J. Schmidt, Groups all of whose subgroups are nilpotent, *Mat. Sb.*, **31** (1924) 366–372 (Russian).

[18] R. Schmidt, Subgroup Lattices of Groups, *De Gruyter expositions in mathematics* **14**, de Gruyter, Berlin, New York, 1994.

[19] N. F. Sesekin and G. M. Romalis, Metahamiltonian groups. II, *Ural. Gos. Univ. Mat. Zap.*, **6** no. 3 (1968) 50–52.

[20] H. Smith, Groups with all non-nilpotent subgroups subnormal, *Topics in infinite groups*, Quad. Mat., Dept. Math., Seconda Univ. Napoli, Caserta, **8** (2001) 309–326.

[21] ———, Torsion-free groups with all non-nilpotent subgroups subnormal, *Topics in infinite groups*, Quad. Mat., Dept. Math., Seconda Univ. Napoli, Caserta, **8** (2001) 297–308.

[22] S. E. Stonehewer, Permutable subgroups of infinite groups, *Math. Z.*, **125** (1972) 1–16.

[23] J. G. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable, *Bull. Amer. Math. Soc.*, **74** (1968) 383–437.

Martyn R. Dixon

Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350, U.S.A

Email: `mdixon@gp.as.ua.edu`

Z. Yalcin Karatas

Department of Mathematics, University of Georgia, Athens, GA 30602, U.S.A.

Email: `yzkaratas@crimson.ua.edu`