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FINITE 2-GROUPS OF CLASS 2 WITH A SPECIFIC AUTOMORPHISM GROUP

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ABSTRACT. In this paper we determine all finite 2-groups of class 2 in which every automorphism of order 2 leaving the Frattini subgroup elementwise fixed is inner.

1. Introduction

One of the most widely known, although non-trivial, properties of finite p-groups is that, with the exception of the groups of order at most p, they always have a non-inner automorphism α of p-power order. This fact was proved by Gaschütz in 1966 [7]. His original result and a number of subsequent variations and improvements show that, obvious exceptions apart, α may be taken such that it acts trivially on some prescribed subgroups or quotients of the group. Even before Gaschütz' result the question had been raised of whether such an α must exist which has order p. Indeed, in 1964 Hans Liebeck proved that if p is an odd prime and G is a finite p-group of class 2 then G has a non-inner automorphism of order p acting trivially on the Frattini subgroup $\Phi(G)$. The corresponding result for 2-groups is generally false, as Liebeck himself showed it by giving an example of a group of class 2 and order 128 in which every automorphism of order 2 fixing $\Phi(G)$ elementwise is inner. Another example of a group with the same property but of order 64 is exhibited in [1]. In [2] it is shown that every 2-group of class 2 has a non-inner automorphism of order 2 fixing $\Omega_1(Z(G))$ or $\Phi(G)$ elementwise (actually the same proof shows that such an automorphism fixes $Z(G)$ or $\Phi(G)$ elementwise). For the results on the existence of noninner automorphisms of order p for finite p-groups we refer the reader to [3, 5] and their bibliographies.

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In this paper we determine all finite 2-groups of class 2 in which every automorphism of order 2 leaving the Frattini subgroup elementwise fixed is inner.

Theorem 1.1. *Let G be a finite 2-group of class 2. Then G has no non-inner automorphism of order 2 leaving the Frattini subgroup $\Phi(G)$ elementwise fixed if and only if*

$$G = \langle a, b \mid a^{2^n} = b^{2^r} = 1, \ \ a^{2^{n-r}} = [a, b]\rangle,$$

where $2 < 2r \leq n$.

2. Proof of Theorem 1.1

For a group G, $Z(G)$, $\Phi(G)$ and $d(G)$ denote the center, the Frattini subgroup and the minimum number of generators of G, respectively. All further unexplained notations are standard as in Gorenstein [8].

Let G be a finite 2-group of class 2 in which every automorphism of order 2 leaving the Frattini subgroup elementwise fixed is inner. Then it follows from [9, Theorem 1.1], that

$$G^\star \leq C_G(G^\star) = \Phi(G), \tag{2.1}$$

where $G^\star = \{x \in G \mid x^2 \in Z(G)\}$. Therefore $Z(G) \leq \Phi(G)$. Moreover, by [1, Lemma 2.2], we have

$$d(Z_2(G)/Z(G)) = d(G)d(Z(G)). \tag{2.2}$$

Thus $d(Z(G)) = 1$. Now suppose that $d(G^\star) > 2$. Then $G^\star$ has an elementary abelian subgroup $\langle u\rangle \times \langle v\rangle$ of order 4 such that $\langle u\rangle \times \langle v\rangle \cap Z(G) = 1$. Set $M = C_G(u)$, $N = C_G(v)$. If $M = N$, then for $x \in G - M$, we have

$$[x, uv] = [x, u][x, v] = z^2 = 1$$

where z is the unique central element of order 2. Hence, x commutes with uv, and so $C_G(uv) \neq M$. Thus one can replace v by uv and assume that $M \neq N$. Let $x \in N \setminus M$ and $y \in M \setminus N$. By [4, Lemma 2.1], the map α given by $x \mapsto xv$ and $y \mapsto yu$ can be extended to an automorphism of order 2 that fixes $\Phi(G)$ elementwise. Clearly α is noninner, a contradiction. Hence we must have $d(G^\star) = 2$. Thus (2.2), implies that $d(G) = 2$.

Next, let $G/Z(G) = \langle aZ(G)\rangle \times \langle bZ(G)\rangle$. Since G is of nilpotency class 2, we have $o(aZ(G)) = o(bZ(G)) = 2^r = o([a, b])$, for some positive integer r. If $r = 1$, then $\Phi(G) \leq Z(G)$ and therefore $C_G(\Phi(G)) \nleq \Phi(G)$, that contradicts (2.1). Hence $r > 1$.

Let $a^ib^j[a, b]^k \in C_G(\Phi(G))$. Then $[a^ib^j[a, b]^k, a^2] = 1 = [a^ib^j[a, b]^k, b^2]$, which implies 2^{r-1} divides i, j. Thus we have

$$C_G(\Phi(G)) = \langle a^{2^{r-1}}, b^{2^{r-1}}, [a, b]\rangle = Z(\Phi(G)) = G^\star \tag{2.3}$$

If $\exp(Z(G)) = \exp(G^\star)$, then $G^\star = Z(G)\times U$, for some $U \leq G^\star$. Since $d(G^\star/Z(G)) = 2$, it follows that $d(U) = 2$ and $d(G^\star) = 3$, a contradiction. Thus $\exp(Z(G)) < \exp(G^\star)$. Without loss of generality we may assume that $\exp(G^\star) = o(a^{2^{r-1}})$ and since $a^{2^r} \in Z(G)$, we get $Z(G) = \langle a^{2^r}\rangle$. Because $[a, b] \in Z(G)$ and $o([a, b]) = 2^r$, it follows that $o(a) \geq 2^{2r}$.

Now, we show that one may assume $b^{2^r} = 1$. Since $b^{2^r} \in Z(G)$, we have $b^{2^r} = a^{i2^r}$, for some integer i. If i is even, then ba^{-i}, has order 2^r and therefore it suffices to replace b by ba^{-i}. If i is an odd integer, then ba^{-i} is of order 2^{r+1}. Replacing b by ba^{-i}, we get $G = \langle a, b\rangle$, $b^{2^{r+1}} = 1$. Because $b^{2^r} \in Z(G)$, it follows that $b^{2^r} = a^{i2^r}$, for some integer i. If i is odd, then $a^{2^{r+1}} = 1$. Hence $|Z(G)| = 2$. This implies that $r = 1$, a contradiction. Thus we may suppose that i is even and hence $(ba^{-i})^{2^r} = 1$. Replacing b by ba^{-i}, we get that $b^{2^r} = 1$ and therefore

$$G = \langle a, b \mid a^{2^n} = b^{2^r} = 1,\ \ a^{2^{n-r}} = [a, b]\rangle, \tag{2.4}$$

for some positive integers r, n such that $2 < 2r \leq n$. In the literature a group with the latter presentation is denoted by $Q(n, r)$ (See for instance [10]).

After that, let $G = Q(n, r) = \langle a, b \mid a^{2^n} = b^{2^r} = 1,\ \ a^{2^{n-r}} = [a, b]\rangle$ for some positive integers r, n such that $2 < 2r \leq n$. We show that every automorphism of G of order 2 leaving the Frattini subgroup elementwise fixed is inner. To this end, we use the following result to verify whether a given automorphism of G is inner.

Remark 2.1. *Suppose that G is a finite 2-generator 2-group of class 2, such that G' is cyclic. Then $\alpha \in Aut(G)$ is inner if and only if $[G, \alpha] \leq G'$ (See part (ii) of Lemma 1 in [6]).*

We also use the following proposition, which is of some independent interest.

Proposition 2.2. *Let G be a finite 2-group such that $C_G(\Phi(G)) = Z(\Phi(G))$. If $d(Z(\Phi(G))) = 2$, then every automorphism of G of order 2 leaving $\Phi(G)$ elementwise fixed is central.*

Proof. Set $A = Z(\Phi(G))$ and let α be an automorphism of G fixing $\Phi(G)$ elementwise. For each $n \in \Phi(G)$ and $x \in G$, we have

$$n = ((n^{x^{-1}})^x)^\alpha = n^{x^{-1}x^\alpha}.$$

Therefore, $x^{-1}x^\alpha \in C_G(\Phi(G)) = A$. If, in addition, the order of α is two, then $x^{-1}x^\alpha \in \Omega_1(A)$. We claim that α is a central automorphism. Indeed, assume for a contradiction that there exist $x \in G$ and $u \in \Omega_1(A) \setminus Z(G)$ such that $x^\alpha = xu$. Thus $\Omega_1(A) = \langle u\rangle\Omega_1(Z(G))$, as $d(A) = 2$. Since $x^2 = (x^2)^\alpha = (xu)^2 = x^2[u, x]$, we get $[u, x] = 1$. Let $y \in G \setminus C_G(u) = C_G(\Omega_1(A))$. Then $y^\alpha = yv$ for some $v \in \Omega_1(A)$. We have $[v, y] = 1$, since $y^2 = (y^2)^\alpha$. If $v \notin Z(G)$, then $y \in C_G(v) = C_G(\Omega_1(A)) = C_G(u)$, a contradiction. Thus $v \in Z(G)$. As $[x, y]^\alpha = [x, y]$, we get $[u, y] = 1$ which again contradicts the choice of y. Therefore $u \in Z(G)$ and α is a central automorphism. □

Now, Let α be an automorphism of $G = Q(n, r)$, $2 < 2r \leq n$, of order 2 leaving $\Phi(G)$ elementwise fixed. Since $Z(\Phi(G)) = \langle a^{2^{r-1}}, b^{2^{r-1}}\rangle$, it follows from Proposition 2.2 that $[G, \alpha] \leq \Omega_1(Z(G)) \leq G'$. Therefore α is inner, by Remark 2.1. This completes the proof of Theorem 1.1.

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