



www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 6 No. 3 (2017), pp. 45-49.
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AN INFINITE FAMILY OF FINITE 2-GROUPS WITH DEFICIENCY ZERO

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Communicated by Ali Reza Jamali

ABSTRACT. We determine a new infinite sequence of finite 2-groups with deficiency zero. The groups have 2 generators and 2 relations, they have coclass 3 and they are not metacyclic.

1. Introduction

Let G be a finite group given by a finite presentation $\langle X \mid R \rangle$. The *deficiency* of this presentation is the difference $|X| - |R|$. The *deficiency* of G is the maximum of the deficiencies of all finite presentations for G . It is known that the deficiency of a finite group is always non-positive. Interesting is the boundary-case: the groups of deficiency zero. We refer to [4] for a general introduction to the theory of presentations and to [3] for background on groups of deficiency zero.

Each finite group of deficiency zero has trivial Schur multiplier. The converse does not hold in general, see [6]; it is an open question whether it holds for finite p -groups and looking for a deficiency zero presentation for a finite p -group which has trivial Schur Multiplier, is enormously difficult.

In [1] various infinite sequences of finite 2-groups of fixed coclass exhibited which all have deficiency zero. For each of the infinite families there is a single parameterized presentation exhibited that has deficiency zero and defines the groups in the family. There are also various infinite families of finite 2-groups described for which it is conjectured that they have deficiency zero.

Consider the infinite pro-2-group $S = \langle a, u \mid a^2 = u^4, (u^2)^a = u^{-2} \rangle$ of coclass 3. This infinite pro-2-group is denoted with S_4 in [1]. The finite 2-groups associated with S fall into 5 infinite families,

MSC(2010): Primary: 20F05; Secondary: 20D15.

Keywords: finite 2-groups, deficiency zero, Schur multiplier.

Received: 06 December 2016, Accepted: 22 December 2016.

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see [5]. The aim here is to show the existence of deficiency zero presentation for one of the infinite families of the infinite pro-2-group S . More precisely, we show the following.

Theorem 1.1. *For each $n \geq 2$ the group*

$$G_n \cong \langle x, y \mid x^{-2}y^4(x^{-1}y)^{2^n}, x^{-1}y^2xy^2(x^{-1}y)^{2^{n+1}} \rangle$$

is a finite 2-group of order 2^{n+5} and coclass 3 with abelian invariants $(2, 4)$.

More precisely, the groups G_n of Theorem 1.1 is a coclass family associated with the infinite pro-2-group S . It is proved in [1] that all of the groups in the 5 families have deficiency either -1 or 0 . Two of the infinite families consist of groups with non-trivial Schur multiplier and hence the groups in these families have deficiency -1 . The groups in the other three infinite families remained open in [1]; the groups G_n form one of these three infinite families.

2. Proof of Theorem 1.1

The proof of the main theorem of this paper will be done in steps.

Lemma 2.1. *Let $n \in \mathbb{N}$ and let w be a word of even length in $\{x, y\}^\pm$. Then the relations $[(yx)^2, w] = [(xy)^2, w] = 1$ hold in G_n .*

Proof. By the second relation in G_n it follows that

$$(yx)^2 = (y^{-1}x)(x^{-1}y^2xy^2)(y^{-1}x) = (y^{-1}x)^{2^{n+1}+2},$$

therefore $(yx)^2$ is a power of $(y^{-1}x)$ and hence $[(yx)^2, yx] = [(yx)^2, (y^{-1}x)] = 1$. Now $y^{-2} = (y^{-1}x)(yx)^{-1}$, hence $[(yx)^2, y^2] = 1$. On the other hand, by the first relation of G_n the word $(yx)^2$ commutes with $x^{-2}y^4$, therefore $[(yx)^2, x^2] = 1$ holds. Note that $yx^{-1} = (yx)x^{-2}$ and $xy = (yx^{-1})^{-1}y^2$, that is $(yx)^2$ commutes with all words of length two and consequently with any word of even length. As $[(yx)^2, x^2] = [(yx)^2, y^2] = [(yx)^2, yx] = 1$, we obtain that

$$(2.1) \quad (xy)^2 = y^{-1}(yx)^2y = y(yx)^2y^{-1} = x^{-1}(yx)^2x = x(yx)^2x^{-1}.$$

Now let $a, b \in \{x, y\}^\pm$. By (2.1) we have $a^{-1}b^{-1}(xy)^2ba = a^{-1}(yx)^2a = (xy)^2$, that is $(xy)^2$ commutes with every word of length two and therefore with every word of even length. This completes the proof. \square

Lemma 2.2. *Let $n \in \mathbb{N}$. Then the relation $(y^{-1}x)^{2^{n+2}} = 1$ holds in G_n .*

Proof. Let $c = (yx)^2$, $d = (y^{-1}x)^2$, $u = xy^2x^{-1}$, $v = y^{-1}x^2y$ and $t = (xy)^2$. Define $N = \langle c, d, u, v, t \rangle \leq G_n$. We use the modified Todd-Coxeter coset enumeration algorithm in the form as given in [2] to find a presentation for N .

Defining $1.x = 2$ and $2.y = 3$ completes the table of the generator u and we obtain $3.y = u.2$. By defining $3.x = 4$ the table of the generator t is complete and we deduce $4.y = t.1$. Now the table of the generator v completes and we find $4.x = tvt^{-1}.3$. Also by completing the table of d we deduce

$2.x = tv^{-1}d.1$ and by completing the table of c we find $1.y = cd^{-1}vt^{-1}u^{-1}tv^{-1}t^{-1}.4$. Now all the tables are complete and we obtain that

$$N \cong \langle c, d, u, v, t \mid r_1, \dots, r_6 \rangle$$

with

$$\begin{aligned} r_1 &= c^2t^{-1}d^{-1}v(d^{-1}vu^{-1}v^{-1})^2d^{-2^{n-1}} = 1 \\ r_2 &= c^{2^{n-1}}t^{-2^{n-1}-1}d^{-1}vu^2(d^{-1}vu^{-1}v^{-1}u)^{2^{n-1}} = 1 \\ r_3 &= u^2v^{-1}d^{-2^{n-1}} = 1 \\ r_4 &= c^{2^{n-1}+2}t^{-2^{n-1}}v^{-1}(d^{-1}vu^{-1}v^{-1})^2(ud^{-1}vu^{-1}v^{-1})^{2^{n-1}} = 1 \\ r_5 &= cd^{-2^{n-1}} = 1 \\ r_6 &= c^{2^n+1}t^{-2^n}(d^{-1}vu^{-1}v^{-1}u)^{2^n+1} = 1. \end{aligned}$$

Lemma 2.1 implies that c and t are central in N . We have used this to move c and t to the left in all relations of N . Using r_3 to remove the generator v , we can simplify the relations r_1, \dots, r_6 to s_1, s_2, s_4, s_5, s_6 as follows

$$\begin{aligned} s_1 &= c^2t^{-1}d^{-2^{n-1}-1}u^2(d^{-2^{n-1}-1}u^{-1}d^{2^{n-1}})^2 = 1 \\ s_2 &= c^{2^{n-1}}t^{-2^{n-1}-1}d^{-2^{n-1}-1}u^4(d^{-2^{n-1}-1}u^{-1}d^{2^{n-1}}u)^{2^{n-1}} = 1 \\ s_4 &= c^{2^{n-1}+2}t^{-2^{n-1}}u^{-2}d^{2^{n-1}}(d^{-2^{n-1}-1}u^{-1}d^{2^{n-1}})^2(ud^{-2^{n-1}-1}u^{-1}d^{2^{n-1}})^{2^{n-1}} = 1 \\ s_5 &= cd^{-2^{n-1}} = 1 \\ s_6 &= c^{2^n+1}t^{-2^n}(d^{-2^{n-1}-1}u^{-1}d^{2^{n-1}}u)^{2^n+1} = 1. \end{aligned}$$

Using s_5 , by s_1 we have that $(d^{-2^{n-1}-1}u^{-1}d^{2^{n-1}})^2 = c^{-1}tu^{-2}$ and hence

$$(2.2) \quad [u^2, d^{-2^{n-1}-1}u^{-1}d^{2^{n-1}}] = 1.$$

By s_4 this yields that $[u^2, d^{2^{n-1}}] = 1$. Therefore $[u^2, d] = 1$ by s_5 . Now using $[u^2, d] = 1$ the relation s_1 could be written in the form

$$(2.3) \quad udu^{-1} = ct^{-1}d^{-1}.$$

Using (2.3), we obtain that $d^{-2^{n-1}-1}u^{-1}d^{2^{n-1}}u = c^{2^{n-1}-1}t^{-2^{n-1}}$. As c and t are central, we deduce that $[u, d^{-2^{n-1}-1}u^{-1}d^{2^{n-1}}] = 1$. By the latter relation and s_2 it follows that $[u, d^{-2^{n-1}-1}] = 1$ and therefore $[u, d] = 1$ holds, as $d^{2^n+1}(=c)$ is central. Hence N is abelian. Now the relations s_1 and s_6 can be written in the following form

$$\begin{aligned} s'_1 &: ct^{-1} = d^2 \\ s'_6 &: (ct^{-1})^{2^n} = 1. \end{aligned}$$

Replacing ct^{-1} by d^2 in s'_6 gives the relation $d^{2^{n+1}} = 1$. Translating this in the generators of G_n yields that the relation $(y^{-1}x)^{2^{n+2}} = 1$ holds in G_n . □

Lemma 2.3. *Let $n \in \mathbb{N}$ and define*

$$H_n = \langle x, y \mid x^2y^{-4}(y^{-1}x)^{2^n} = 1, x^{-1}y^2xy^2(y^{-1}x)^{2^{n+1}} = 1, (y^{-1}x)^{2^{n+2}} = 1 \rangle.$$

Then the relation $(yx)^4 = (y^{-1}x)^4$ holds in both G_n and H_n .

Proof. Lemma 2.2 implies that the relation $(y^{-1}x)^{2^{n+2}} = 1$ holds in G_n . Therefore

$$(y^{-1}x)^{2^{n+1}} = (x^{-1}y)^{2^{n+1}}$$

holds in both G_n and H_n . Hence by Lemma 2.1, the relation $(yx)^2 = (y^{-1}x)^{2^{n+1}+2}$ holds in G_n and H_n . Now $(yx)^4 = (y^{-1}x)^{2^{n+2}+4} = (y^{-1}x)^4$ holds in both groups. \square

For $n = 1$, it is easy to see that G_1 is of order 64 and nilpotency class 4. This group is the 46-th group of order 64 in the “*SmallGroups*” library of GAP ([7]).

Proof of Theorem 1.1. We recall the relations of G_n and H_n in the relator form in order to treat them as elements of the free group $F(\{x, y\})$. Let $w_1 = x^2y^{-4}(y^{-1}x)^{2^n}$, $w_2 = x^{-1}y^2xy^2(y^{-1}x)^{2^{n+1}}$ and $w_3 = (y^{-1}x)^{2^{n+2}}$ and let $w'_1 = x^{-2}y^4(x^{-1}y)^{2^n}$ and $w'_2 = x^{-1}y^2xy^2(x^{-1}y)^{2^{n+1}}$. Consider the subgroups $T = \langle w_1, w_2, w_3 \rangle$ and $S = \langle w'_1, w'_2 \rangle$ of $F(\{x, y\})$ and denote their normal closures by \bar{T} and \bar{S} , respectively. Then $H_n = F(\{x, y\})/\bar{T}$ and $G_n = F(\{x, y\})/\bar{S}$. Using the results of [1, Sec. 6.1], it is sufficient to show that $\bar{T} = \bar{S}$. By Lemma 2.2 we find that $w_3 \in \bar{S}$. Also $w_2 = w'_2w_3$. Hence $w_2 \in \bar{S}$. On the other hand, $w_1 = x^2w'_1{}^{-1}x^{-2} \in \bar{S}$ as $(y^{-1}x)^4 = (yx)^4$ commutes with y^4 by Lemma 2.1. Therefore $\bar{T} \leq \bar{S}$. The other inclusion $\bar{S} \leq \bar{T}$ is obvious and hence $G_n \cong H_n$ follows. \square

Acknowledgments

The authors wish to thank Prof. Bettina Eick for valuable comments and helpful discussions. The authors are indebted to the referees for providing insightful comments.

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