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AN INFINITE FAMILY OF FINITE 2-GROUPS WITH DEFICIENCY ZERO

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Abstract. We determine a new infinite sequence of finite 2-groups with deficiency zero. The groups have 2 generators and 2 relations, they have coclass 3 and they are not metacyclic.

1. Introduction

Let G be a finite group given by a finite presentation $\langle X \mid R\rangle$. The *deficiency* of this presentation is the difference $|X|-|R|$. The *deficiency* of G is the maximum of the deficiencies of all finite presentations for G. It is known that the deficiency of a finite group is always non-positive. Interesting is the boundary-case: the groups of deficiency zero. We refer to [4] for a general introduction to the theory of presentations and to [3] for background on groups of deficiency zero.

Each finite group of deficiency zero has trivial Schur multiplicator. The converse does not hold in general, see [6]; it is an open question whether it holds for finite p-groups and looking for a deficiency zero presentation for a finite p-group which has trivial Schur Multiplicator, is enormously difficult.

In [1] various infinite sequences of finite 2-groups of fixed coclass exhibited which all have deficiency zero. For each of the infinite families there is a single parameterized presentation exhibited that has deficiency zero and defines the groups in the family. There are also various infinite families of finite 2-groups described for which it is conjectured that they have deficiency zero.

Consider the infinite pro-2-group $S = \langle a, u \mid a^2 = u^4, (u^2)^a = u^{-2}\rangle$ of coclass 3. This infinite pro-2-group is denoted with S_4 in [1]. The finite 2-groups associated with S fall into 5 infinite families,

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see [5]. The aim here is to show the existence of deficiency zero presentation for one of the infinite families of the infinite pro-2-group S. More precisely, we show the following.

Theorem 1.1. *For each $n \geq 2$ the group*

$$G_n \cong \langle x, y \mid x^{-2}y^4(x^{-1}y)^{2^n}, x^{-1}y^2xy^2(x^{-1}y)^{2^{n+1}} \rangle$$

is a finite 2-group of order 2^{n+5} and coclass 3 with abelian invariants $(2,4)$.

More precisely, the groups G_n of Theorem 1.1 is a coclass family associated with the infinite pro-2-group S. It is proved in [1] that all of the groups in the 5 families have deficiency either -1 or 0. Two of the infinite families consist of groups with non-trivial Schur multiplicator and hence the groups in these families have deficiency -1. The groups in the other three infinite families remained open in [1]; the groups G_n form one of these three infinite families.

2. Proof of Theorem 1.1

The proof of the main theorem of this paper will be done in steps.

Lemma 2.1. *Let $n \in \mathbb{N}$ and let w be a word of even length in $\{x, y\}^{\pm}$. Then the relations $[(yx)^2, w] = [(xy)^2, w] = 1$ hold in G_n.*

Proof. By the second relation in G_n it follows that

$$(yx)^2 = (y^{-1}x)(x^{-1}y^2xy^2)(y^{-1}x) = (y^{-1}x)^{2^{n+1}+2},$$

therefore $(yx)^2$ is a power of $(y^{-1}x)$ and hence $[(yx)^2, yx] = [(yx)^2, (y^{-1}x)] = 1$. Now $y^{-2} = (y^{-1}x)(yx)^{-1}$, hence $[(yx)^2, y^2] = 1$. On the other hand, by the first relation of G_n the word $(yx)^2$ commutes with $x^{-2}y^4$, therefore $[(yx)^2, x^2] = 1$ holds. Note that $yx^{-1} = (yx)x^{-2}$ and $xy = (yx^{-1})^{-1}y^2$, that is $(yx)^2$ commutes with all words of length two and consequently with any word of even length. As $[(yx)^2, x^2] = [(yx)^2, y^2] = [(yx)^2, yx] = 1$, we obtain that

$$(xy)^2 = y^{-1}(yx)^2y = y(yx)^2y^{-1} = x^{-1}(yx)^2x = x(yx)^2x^{-1}. \tag{2.1}$$

Now let $a, b \in \{x, y\}^{\pm}$. By (2.1) we have $a^{-1}b^{-1}(xy)^2ba = a^{-1}(yx)^2a = (xy)^2$, that is $(xy)^2$ commutes with every word of length two and therefore with every word of even length. This completes the proof. □

Lemma 2.2. *Let $n \in \mathbb{N}$. Then the relation $(y^{-1}x)^{2^{n+2}} = 1$ holds in G_n.*

Proof. Let $c = (yx)^2$, $d = (y^{-1}x)^2$, $u = xy^2x^{-1}$, $v = y^{-1}x^2y$ and $t = (xy)^2$. Define $N = \langle c, d, u, v, t \rangle \leq G_n$. We use the modified Todd-Coxeter coset enumeration algorithm in the form as given in [2] to find a presentation for N.

Defining $1.x = 2$ and $2.y = 3$ completes the table of the generator u and we obtain $3.y = u.2$. By defining $3.x = 4$ the table of the generator t is complete and we deduce $4.y = t.1$. Now the table of the generator v completes and we find $4.x = tvt^{-1}.3$. Also by completing the table of d we deduce

$2.x = tv^{-1}d.1$ and by completing the table of c we find $1.y = cd^{-1}vt^{-1}u^{-1}tv^{-1}t^{-1}.4$. Now all the tables are complete and we obtain that

$$N \cong \langle c, d, u, v, t \mid r_1, \dots, r_6 \rangle$$

with

$$\begin{aligned}
r_1 &= c^2t^{-1}d^{-1}v(d^{-1}vu^{-1}v^{-1})^2d^{-2^{n-1}} = 1\\
r_2 &= c^{2^{n-1}}t^{-2^{n-1}-1}d^{-1}vu^2(d^{-1}vu^{-1}v^{-1}u)^{2^{n-1}} = 1\\
r_3 &= u^2v^{-1}d^{-2^{n-1}} = 1\\
r_4 &= c^{2^{n-1}+2}t^{-2^{n-1}}v^{-1}(d^{-1}vu^{-1}v^{-1})^2(ud^{-1}vu^{-1}v^{-1})^{2^{n-1}} = 1\\
r_5 &= cd^{-2^n-1} = 1\\
r_6 &= c^{2^n+1}t^{-2^n}(d^{-1}vu^{-1}v^{-1}u)^{2^n+1} = 1.
\end{aligned}$$

Lemma 2.1 implies that c and t are central in N. We have used this to move c and t to the left in all relations of N. Using r_3 to remove the generator v, we can simplify the relations $r_1, \dots, r_6$ to s_1, s_2, s_4, s_5, s_6 as follows

$$\begin{aligned}
s_1 &= c^2t^{-1}d^{-2^n-1}u^2(d^{-2^{n-1}-1}u^{-1}d^{2^{n-1}})^2 = 1\\
s_2 &= c^{2^{n-1}}t^{-2^{n-1}-1}d^{-2^{n-1}-1}u^4(d^{-2^{n-1}-1}u^{-1}d^{2^{n-1}}u)^{2^{n-1}} = 1\\
s_4 &= c^{2^{n-1}+2}t^{-2^{n-1}}u^{-2}d^{2^{n-1}}(d^{-2^{n-1}-1}u^{-1}d^{2^{n-1}})^2(ud^{-2^{n-1}-1}u^{-1}d^{2^{n-1}})^{2^{n-1}} = 1\\
s_5 &= cd^{-2^n-1} = 1\\
s_6 &= c^{2^n+1}t^{-2^n}(d^{-2^{n-1}-1}u^{-1}d^{2^{n-1}}u)^{2^n+1} = 1.
\end{aligned}$$

Using s_5, by s_1 we have that $(d^{-2^{n-1}-1}u^{-1}d^{2^{n-1}})^2 = c^{-1}tu^{-2}$ and hence

$$[u^2, d^{-2^{n-1}-1}u^{-1}d^{2^{n-1}}] = 1. \tag{2.2}$$

By s_4 this yields that $[u^2, d^{2^{n-1}}] = 1$. Therefore $[u^2, d] = 1$ by s_5. Now using $[u^2, d] = 1$ the relation s_1 could be written in the form

$$udu^{-1} = ct^{-1}d^{-1}. \tag{2.3}$$

Using (2.3), we obtain that $d^{-2^{n-1}-1}u^{-1}d^{2^{n-1}}u = c^{2^{n-1}-1}t^{-2^{n-1}}$. As c and t are central, we deduce that $[u, d^{-2^{n-1}-1}u^{-1}d^{2^{n-1}}] = 1$. By the latter relation and s_2 it follows that $[u, d^{-2^{n-1}-1}] = 1$ and therefore $[u, d] = 1$ holds, as $d^{2^n+1}(= c)$ is central. Hence N is abelian. Now the relations s_1 and s_6 can be written in the following form

$$\begin{aligned}
s_1' &: ct^{-1} = d^2\\
s_6' &: (ct^{-1})^{2^n} = 1.
\end{aligned}$$

Replacing ct^{-1} by d^2 in s_6' gives the relation $d^{2^{n+1}} = 1$. Translating this in the generators of G_n yields that the relation $(y^{-1}x)^{2^{n+2}} = 1$ holds in G_n. □

Lemma 2.3. *Let $n \in \mathbb{N}$ and define*

$$H_n = \langle x, y \mid x^2y^{-4}(y^{-1}x)^{2^n} = 1, x^{-1}y^2xy^2(y^{-1}x)^{2^{n+1}} = 1, (y^{-1}x)^{2^{n+2}} = 1\rangle.$$

Then the relation $(yx)^4 = (y^{-1}x)^4$ holds in both G_n and H_n.

Proof. Lemma 2.2 implies that the relation $(y^{-1}x)^{2^{n+2}} = 1$ holds in G_n. Therefore

$$(y^{-1}x)^{2^{n+1}} = (x^{-1}y)^{2^{n+1}}$$

holds in both G_n and H_n. Hence by Lemma 2.1, the relation $(yx)^2 = (y^{-1}x)^{2^{n+1}+2}$ holds in G_n and H_n. Now $(yx)^4 = (y^{-1}x)^{2^{n+2}+4} = (y^{-1}x)^4$ holds in both groups. □

For $n = 1$, it is easy to see that G_1 is of order 64 and nilpotency class 4. This group is the 46-th group of order 64 in the "*SmallGroups*" library of GAP ([7]).

Proof of Theorem 1.1. We recall the relations of G_n and H_n in the relator form in order to treat them as elements of the free group $F(\{x, y\})$. Let $w_1 = x^2y^{-4}(y^{-1}x)^{2^n}$, $w_2 = x^{-1}y^2xy^2(y^{-1}x)^{2^{n+1}}$ and $w_3 = (y^{-1}x)^{2^{n+2}}$ and let $w'_1 = x^{-2}y^4(x^{-1}y)^{2^n}$ and $w'_2 = x^{-1}y^2xy^2(x^{-1}y)^{2^{n+1}}$. Consider the subgroups $T = \langle w_1, w_2, w_3\rangle$ and $S = \langle w'_1, w'_2\rangle$ of $F(\{x, y\})$ and denote their normal closures by $\overline{T}$ and $\overline{S}$, respectively. Then $H_n = F(\{x, y\})/\overline{T}$ and $G_n = F(\{x, y\})/\overline{S}$. Using the results of [1, Sec. 6.1], it is sufficient to show that $\overline{T} = \overline{S}$. By Lemma 2.2 we find that $w_3 \in \overline{S}$. Also $w_2 = w'_2w_3$. Hence $w_2 \in \overline{S}$. On the other hand, $w_1 = x^2{w'_1}^{-1}x^{-2} \in \overline{S}$ as $(y^{-1}x)^4 = (yx)^4$ commutes with y^4 by Lemma 2.1. Therefore $\overline{T} \leq \overline{S}$. The other inclusion $\overline{S} \leq \overline{T}$ is obvious and hence $G_n \cong H_n$ follows. □

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References

[1] H. Abdolzadeh and B. Eick, On efficient presentations for infinite sequences of 2-groups with fixed coclass, *Algebra Colloq.*, **20** (2013) 561–572.

[2] M. J. Beetham and C. M. Campbell, A note on the Todd-Coxeter coset enumeration algorithm, *P. Edinburgh Math. Soc.*, **20** (1976) 73–79.

[3] G. Havas, M. F. Newman and E. A. O'Brien, Groups of deficiency zero, *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, **25** (1994) 53–67.

[4] D. L. Johnson, *Topics in the theory of group presentations*, *London Mathematical Society Lecture Note Series*, **42**, Cambridge University Press, Cambridge, 1980.

[5] M. F. Newman, E.A. O'Brien, Classifying 2-groups by coclass, *Trans. Amer. Math. Soc.*, **351** (1999) 131–169.

[6] R. G. Swan, Representations of polycyclic groups, *Proc. Amer. Math. Soc.*, **18** (1967) 573 – 574.

[7] The GAP Group, GAP — Groups, Algorithms and Programming, Version 4.4 (available from www.gap-system.org), 2005.

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