REGULAR SUBGROUPS, NILPOTENT ALGEBRAS
AND PROJECTIVELY CONGRUENT MATRICES

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Abstract. In this paper we highlight the connection between certain classes of regular subgroups of the affine group $\text{AGL}_n(F)$, $F$ a field, and associative nilpotent $F$-algebras of dimension $n$. We also describe how the classification of projective congruence classes of square matrices is equivalent to the classification of regular subgroups of particular shape.

1. Introduction

Let $F$ be a field. We can identify the affine group $\text{AGL}_n(F)$ with the subgroup of $\text{GL}_{n+1}(F)$ consisting of the matrices having $(1,0,\ldots,0)^T$ as first column. It follows that the group $\text{AGL}_n(F)$ acts on the right on the set $\mathcal{A} = \{(1,v) : v \in F^n\}$ of affine points. A subgroup $R$ of $\text{AGL}_n(F)$ is called regular if it acts regularly on $\mathcal{A}$, namely if, for every $v \in F^n$, there exists a unique element in $R$ having $(1,v)$ as first row. For instance, the translation subgroup $T_n$ of $\text{AGL}_n(F)$ is a regular subgroup.

The problem of classifying, up to conjugation, the regular subgroups of $\text{AGL}_n(F)$ attracted the interest of many authors. For instance, we recall the first systematic works of Caranti, Dalla Volta, Sala [1] and Tamburini [9] or the more recent paper of Catino, Colazzo and Stefanelli [2]. We recall also the work of Hegedűs [4], who constructed (nonabelian) regular subgroups containing no nontrivial translations. Recent generalizations of Hegedűs’ examples have been obtained in [3, 8].


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Following the notation of [7], we write every element \( r \) of a regular subgroup \( R \leq AGL_n(\mathbb{F}) \) as

\[
(1.1) \quad r = \begin{pmatrix}
1 & v \\
0 & \pi_R(r)
\end{pmatrix} = \begin{pmatrix}
1 & v \\
0 & I_n + \delta_R(v)
\end{pmatrix} = \mu_R(v),
\]

considering the functions \( \pi_R : R \to GL_n(\mathbb{F}) \), \( \mu_R : \mathbb{F}^n \to R \), and \( \delta_R : \mathbb{F}^n \to \text{Mat}_n(\mathbb{F}) \). In this paper, as done in [7], we focus our attention on the case where the function \( \delta_R \) is linear. To simplify the notation, we denote by \( \Delta_n(\mathbb{F}) \) the set of the regular subgroups \( R \) of \( AGL_n(\mathbb{F}) \) for which the function \( \delta_R \) is linear.

In Section 2 we illustrate some of the properties of the subgroups \( R \in \Delta_n(\mathbb{F}) \). In particular, the linearity of \( \delta_R \) allows us to highlight the connection between regular subgroups and finite dimensional associative nilpotent algebras. In Section 3 we show how the classification of the regular subgroups \( R \in \Delta_n(\mathbb{F}) \) can be obtained, in principle, working by induction on \( n \). Furthermore, we construct particular regular subgroups \( R_D \in \Delta_n(\mathbb{F}) \) associated to square matrices \( D \in \text{Mat}_{n-1}(\mathbb{F}) \) and we prove that two subgroups \( R_{D_1} \) and \( R_{D_2} \) are conjugate in \( AGL_n(\mathbb{F}) \) if and only if the corresponding matrices \( D_1 \) and \( D_2 \) are projectively congruent.

2. Regular subgroups and nilpotent algebras

A complete classification, up to conjugation, of the regular subgroups of \( AGL_n(\mathbb{F}) \) seems to be a rather difficult problem. For instance, in [7, Example 2.5] it was shown that the group \( AGL_2(\mathbb{R}) \) contains \( 2^{\lfloor |R| \rfloor} \) conjugacy classes of regular subgroups. However, a regular subgroup in \( \Delta_n(\mathbb{F}) \) has interesting properties that should allow a classification (see Section 3 and [6, 7]): for instance, it is unipotent (but the converse is not true), see [7]. We recall that any regular abelian subgroup of \( AGL_n(\mathbb{F}) \) is in \( \Delta_n(\mathbb{F}) \) (also here, for \( n \geq 3 \), the converse is not true).

Proposition 2.1. Let \( R \in \Delta_n(\mathbb{F}) \). Then the following hold:

(a) the set \( \{ v \in \mathbb{F}^n : \mu_R(v) \in \mathbb{Z}(R) \} \) is a subspace of \( \mathbb{F}^n \);
(b) the subspace \( \{ w \in \mathbb{F}^n : w_{\pi_R(r)} = w, \forall r \in R \} \) coincides with \( \text{Ker}(\delta_R) \);
(c) \( \dim \text{Ker}(\delta_R) \geq 1 \).

Proof. (a) First, recall that \( \mu_R(0) = I_{n+1} \) and that \( \mu_R(v) \in \mathbb{Z}(R) \) if and only if \( v \delta_R(w) = w \delta_R(v) \) for all \( w \in \mathbb{F}^n \) (see [7, 9]). The result easily follows from the linearity of \( \delta_R \).

(b) It follows from \( \pi_R(\mu_R(v)) = I_n + \delta_R(v) \).

(c) Since \( \delta_R \) is linear, \( R \) is unipotent and so we may suppose that \( R \) is upper unitriangular. Furthermore, \( \delta_R(v_1 \delta_R(v_2)) = \delta_R(v_1) \delta_R(v_2) \) for all \( v_1, v_2 \in \mathbb{F}^n \), see [7]. Now, let \( m \geq 0 \) be the maximum integer for which there exists \( 0 \neq v \in \mathbb{F}^n \) such that \( \delta_R(v) \) has the last \( m \) rows equal to 0. By way of contradiction, suppose that \( \text{Ker}(\delta_R) = \{0\} \), that is \( m < n \). Let \( 0 \neq w \in \mathbb{F}^n \) be such that \( \delta_R(w) \) has the last \( m \) rows equal to 0 and set \( v = e_{n-m} \delta_R(w) \), where \( \{e_1, \ldots, e_n\} \) is the canonical basis of \( \mathbb{F}^n \).

Since \( R \) is upper unitriangular, we obtain that \( \delta_R(v) = \delta_R(e_{n-m}) \delta_R(w) \) has the last \( m + 1 \) rows equal to 0, in contradiction with the maximality of \( m \).

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Item (c) of previous proposition shows that any regular subgroup \( R \in \Delta_n(\mathbb{F}) \) contains nontrivial translations. In fact, the examples constructed in [4] and in [3, 8] make use of quadratic functions, instead of simpler linear functions.

We now illustrate the connection between regular subgroups and finite dimensional associative nilpotent algebras. Denote by \( \mathcal{N}_n(\mathbb{F}) \) the set of the associative nilpotent \( \mathbb{F} \)-algebras of dimension \( n \) (as \( \mathbb{F} \)-spaces). Following the proof of [7, Theorem 3.1], we can embed a given algebra \( N \in \mathcal{N}_n(\mathbb{F}) \) into \( \text{Mat}_{n+1}(\mathbb{F}) \) via

\[
m \mapsto \begin{pmatrix} 0 & m_B \\ 0 & \delta(m) \end{pmatrix},
\]

where, for any \( m \in N \), \( m_B \) and \( \delta(m) \) denote, respectively, the coordinate row vector of \( m \) and the matrix of the right multiplication by \( m \) with respect to a fixed basis \( B \) of \( N \) over \( \mathbb{F} \). Identifying \( N \) with its image, we have that

\[
\mathcal{L} = \mathbb{F}I_{n+1} + N
\]

is a split local subalgebra of \( \text{Mat}_{n+1}(\mathbb{F}) \), with Jacobson radical \( J(\mathcal{L}) = N \). Clearly, the subset \( R = \{ I_{n+1} + m : m \in N \} \subseteq \mathcal{L} \) consists of invertible matrices and is closed under multiplication, since

\[
\delta(m_1 \delta(m_2)) = \delta(m_1) \delta(m_2)
\]

for all \( m_1, m_2 \in N \). By [7, Lemma 2.1] \( R \) is a regular subgroup, lying in \( \Delta_n(\mathbb{F}) \). This allows us to define a function \( \Phi : \mathcal{N}_n(\mathbb{F}) \to \Delta_n(\mathbb{F}) \) setting \( \Phi(N) = R \) as described before.

Conversely, given a regular subgroup \( R \in \Delta_n(\mathbb{F}) \), we may consider the set

\[
N = R - I_{n+1} = \left\{ \begin{pmatrix} 0 & v \\ 0 & \delta_R(v) \end{pmatrix} : v \in \mathbb{F}^n \right\}.
\]

We have that \( \mathcal{L}_R = \mathbb{F}I_{n+1} + N \) is a split local \( \mathbb{F} \)-algebra of dimension \( n \), by [7, Theorem 3.3]. Notice that \( N = J(\mathcal{L}_R) \in \mathcal{N}_n(\mathbb{F}) \). Hence, we can consider the function \( \Psi : \Delta_n(\mathbb{F}) \to \mathcal{N}_n(\mathbb{F}) \) defined by \( \Psi(R) = J(\mathcal{L}_R) \). When \( \text{char} \mathbb{F} = p > 0 \), following Isaacs’ terminology [5], this shows that any regular subgroup \( R \in \Delta_n(\mathbb{F}) \) is an \( \mathbb{F} \)-algebra group.

Extending the results of [1], where the authors considered the commutative case, we can prove the following.

**Proposition 2.2.** There exists a bijection between the set of conjugacy classes of regular subgroups \( R \) of the affine group \( \text{AGL}_n(\mathbb{F}) \) with linear \( \delta_R \) and the set of isomorphism classes of associative nilpotent \( \mathbb{F} \)-algebras of dimension \( n \).

**Proof.** Consider the maps \( \Phi : \mathcal{N}_n(\mathbb{F}) \to \Delta_n(\mathbb{F}) \) and \( \Psi : \Delta_n(\mathbb{F}) \to \mathcal{N}_n(\mathbb{F}) \) previously defined. By [7, Proposition 3.4] two regular subgroups \( R_1, R_2 \in \Delta_n(\mathbb{F}) \) are conjugate in \( \text{AGL}_n(\mathbb{F}) \) if and only if the corresponding split local algebras \( \mathcal{L}_{R_1}, \mathcal{L}_{R_2} \) are isomorphic. In particular, this holds if and only if the nilpotent algebras \( J(\mathcal{L}_{R_1}) \) and \( J(\mathcal{L}_{R_2}) \) are isomorphic. So, if \( R_1 \) and \( R_2 \) are conjugate, then \( \Psi(R_1) \cong \Psi(R_2) \). Conversely, if \( N_1, N_2 \in \mathcal{N}_n(\mathbb{F}) \) are isomorphic, also the corresponding split local algebras \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) constructed as in (2.1) are isomorphic. Hence, the subgroups \( \Phi(N_1) \) and \( \Phi(N_2) \) are conjugate.

\[\square\]
3. Regular subgroups and projectively congruent matrices

We want to describe a method for constructing regular subgroups of $\text{AGL}_n(\mathbb{F})$ starting from subgroups $R \in \Delta_{n-1}(\mathbb{F})$. First of all, take any matrix $D \in \text{Mat}_{n-1}(\mathbb{F})$ and define

$$R(R, D) = \left\{ \begin{pmatrix} 1 & X & x_n \\ 0 & I_{n-1} + \delta_R(X) & DX^T \\ 0 & 0 & 1 \end{pmatrix} : X \in \mathbb{F}^{n-1}, x_n \in \mathbb{F} \right\}.$$

It can be easily proved that $R(R, D) \in \Delta_n(\mathbb{F})$ if and only if

\begin{equation}
D\delta_R(e_i)^T e_j^T = \delta_R(e_j)D e_j^T \quad \text{for all } i, j = 1, \ldots, n - 1,
\end{equation}

where $\{e_1, \ldots, e_{n-1}\}$ is the canonical basis of $\mathbb{F}^{n-1}$ (by [7, Lemma 2.1] it suffices to study when $R(R, D)$ is closed with respect to multiplication).

Conversely, we want to prove that every regular subgroup $R \in \Delta_n(\mathbb{F})$ can be written as $R(\tilde{R}, D)$ for some $\tilde{R} \in \Delta_{n-1}(\mathbb{F})$ and some matrix $D \in \text{Mat}_{n-1}(\mathbb{F})$ satisfying (3.1). We start with the following result.

**Proposition 3.1.** Let $R \in \Delta_n(\mathbb{F})$ and $m = \dim \ker(\delta_R)$. Then, up to conjugation, $R$ is upper unitriangular with $\delta_R(e_i) = 0$ for all $i \in I = \{n - m + 1, \ldots, n\}$.

**Proof.** First we recall that, by Proposition 2.1, $m \geq 1$ and so $I \neq \emptyset$. Up to conjugation we may suppose that $\delta_R(e_i) = 0$ for all $i \in I$. Now, for all $v \in \mathbb{F}^n$ and for all $i \in I$ we have $\delta_R(e_i)\delta_R(v) = \delta_R(e_i)\delta_R(v) = 0$. Hence, $e_i\delta_R(v) \in \ker(\delta_R) = \langle e_j : j \in I \rangle$. This means that for all $v \in \mathbb{F}^n$ we have $\mu_R(v) = \begin{pmatrix} 1 & X & \tilde{X} \\ 0 & f_1(X) & f_2(X) \\ 0 & 0 & f_3(X) \end{pmatrix}$, where $X \in \mathbb{F}^{n-m}$, $\tilde{X} \in \mathbb{F}^m$, $f_1 : \mathbb{F}^{n-m} \to \text{GL}_{n-m}(\mathbb{F})$, $f_2 : \mathbb{F}^{n-m} \to \text{Mat}_{n-m,m}(\mathbb{F})$ and $f_3 : \mathbb{F}^{n-m} \to \text{GL}_m(\mathbb{F})$. Note that the sets $\{f_1(X) : X \in \mathbb{F}^{n-m}\}$ and $\{f_3(X) : X \in \mathbb{F}^{n-m}\}$ are both unipotent subgroups, so there exist $N \in \text{GL}_{n-m}(\mathbb{F})$ and $M \in \text{GL}_m(\mathbb{F})$ such that both $N^{-1}f_1(X)N$ and $M^{-1}f_2(X)M$ are upper unitriangular for all $X \in \mathbb{F}^{n-m}$. Hence, conjugating by $g = \text{diag}(1, N, M)$, we obtain that $\mu_R(v)$ is unitriangular for all $v \in \mathbb{F}^n$ with $\delta_R(e_i) = 0$ for all $i \in I$ (since the set $\{e_i : i \in I\}$ is fixed by $g$). \hfill \Box

By Proposition 3.1, up to conjugation, $R \in \Delta_n(\mathbb{F})$ can be written as the subgroup

$$R = \left\{ \begin{pmatrix} 1 & X & x_n \\ 0 & f_1(X) & f_2(X)^T \\ 0 & 0 & 1 \end{pmatrix} : X \in \mathbb{F}^{n-1}, x_n \in \mathbb{F} \right\},$$

where $f_1 : \mathbb{F}^{n-1} \to \text{GL}_{n-1}(\mathbb{F})$ and $f_2 : \mathbb{F}^{n-1} \to \mathbb{F}^{n-1}$ are such that $f_1(Xf_1(Y) + Y) = f_1(X)f_1(Y)$ and $f_2(Xf_1(Y) + Y)^T = f_2(X)^T + f_1(X)f_2(Y)^T$ for all $X, Y \in \mathbb{F}^{n-1}$. Since $\delta_R$ is a linear function, $f_1(X) = I_{n-1} + \delta(X)$ and $f_2(X)^T = DX^T$ for some linear function $\delta$ and some matrix $D \in \text{Mat}_{n-1}(\mathbb{F})$ satisfying (3.1). It follows that $\tilde{R} = \left\{ \begin{pmatrix} 1 & X \\ 0 & I_{n-1} + \delta(X) \end{pmatrix} : X \in \mathbb{F}^{n-1} \right\} \in \Delta_{n-1}(\mathbb{F})$ and so $R = R(\tilde{R}, D)$. 

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We now consider the special case of $\mathcal{R}(T_{n-1}, D) \in \Delta_n(\mathbb{F})$, that we simply denote by $\mathcal{R}_D$. Notice that $\delta_{T_{n-1}} = 0$ and so any matrix $D \in \text{Mat}_{n-1}(\mathbb{F})$ satisfies (3.1). So,

$$\mathcal{R}_D = \left\{ \begin{pmatrix} 1 & X & x_n \\ 0 & I_{n-1} & DX^T \\ 0 & 0 & 1 \end{pmatrix} : X \in \mathbb{F}^{n-1}, x_n \in \mathbb{F} \right\}.$$ 

We list here some easy properties of the subgroups $\mathcal{R}_D$. First of all, observe that if $D = 0$ then $\mathcal{R}_D$ coincides with the translation subgroup $T_n$. Furthermore, $\mathcal{R}_D$ is abelian if and only if $D = D^T$. Finally, using the fact the $\mathcal{R}_D$ is unipotent, we set

$$d(\mathcal{R}_D) = \max \{ \deg \min_{\mathbb{F}}(r - I_{n+1}) : r \in \mathcal{R}_D \},$$

$$r(\mathcal{R}_D) = \max \{ \text{rk}(r - I_{n+1}) : r \in \mathcal{R}_D \},$$

$$k(\mathcal{R}_D) = \dim \text{Ker}(\delta_{\mathcal{R}_D})$$

($\min_{\mathbb{F}}(g)$ denotes the minimum polynomial of $g$ over $\mathbb{F}$). Then, we have

$$d(\mathcal{R}_D) = \begin{cases} 2 & \text{if } D^T = -D \text{ and } D \text{ has zero diagonal}, \\ 3 & \text{otherwise}, \end{cases}$$

$$r(\mathcal{R}_D) = \begin{cases} 2 & \text{if } D \neq 0, \\ 1 & \text{if } D = 0, \end{cases}$$

$$k(\mathcal{R}_D) = n - \text{rk}(D).$$

We show that there exists a bijection between conjugacy classes of regular subgroups $\mathcal{R}_D$ of $\text{AGL}_n(\mathbb{F})$ and projective congruent classes of matrices of $\text{Mat}_{n-1}(\mathbb{F})$. Given two matrices $A, B \in \text{Mat}_n(\mathbb{F})$, we say that $A$ and $B$ are projectively congruent if $PAP^T = \lambda B$, for some non-zero element $\lambda \in \mathbb{F}^*$ and some invertible matrix $P \in \text{GL}_n(\mathbb{F})$ (see for instance, [10, 11]). Clearly, when $\mathbb{F}$ is algebraically closed two square matrices are projectively congruent if and only if they are congruent.

**Lemma 3.2.** Given two matrices $A, B \in \text{Mat}_{n-1}(\mathbb{F})$, the subgroups $\mathcal{R}_A$ and $\mathcal{R}_B$ are conjugate in $\text{AGL}_n(\mathbb{F})$ if and only if $A$ and $B$ are projectively congruent.

**Proof.** If $A = 0$ then $\mathcal{R}_A = T_n$ which is normal in $\text{AGL}_n(\mathbb{F})$. Hence $\mathcal{R}_B$ is conjugate to $\mathcal{R}_A$ if and only if $B = 0$. Now, assume $A \neq 0$. By [7, Proposition 3.4] the regular subgroups $\mathcal{R}_A$ and $\mathcal{R}_B$ are conjugate in $\text{AGL}_n(\mathbb{F})$ if and only if there exists an invertible matrix

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & P & M^T \\ 0 & N & \lambda \end{pmatrix} \in \text{GL}_{n+1}(\mathbb{F}),$$

with $P \in \text{Mat}_{n-1}(\mathbb{F})$, $M, N \in \mathbb{F}^{n-1}$, $\lambda \in \mathbb{F}$, such that $\mathcal{R}_A \cdot g = g \cdot \mathcal{R}_B$. This holds if and only if for any $X \in \mathbb{F}^{n-1}$ and any $x_n \in \mathbb{F}$ we have

$$AX^TN = 0, \quad NBY^T = 0 \quad \text{and} \quad \lambda AX^T = PBY^T,$$

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where \( Y = XP + x_n N \). Since \( A \neq 0 \), the condition \( AX^T N = 0 \) for all \( X \in \mathbb{F}^{n-1} \) implies \( N = 0 \). It follows that \( \mathcal{R}_A \) and \( \mathcal{R}_B \) are conjugate in \( \text{AGL}_n(\mathbb{F}) \) if and only if \( \lambda A = PBP^T \) for some \( \lambda \in \mathbb{F}^* \) and some \( P \in \text{GL}_{n-1}(\mathbb{F}) \), that is, if and only if \( A \) and \( B \) are projectively congruent. □

**Example 3.3.** Using Lemma 3.2 we can obtain the classification of the projective congruence classes of matrices in \( \text{Mat}_2(\mathbb{F}) \), for any field \( \mathbb{F} \). By [7, Lemmas 5.3, 5.4 and 7.1], a complete set of representatives of such classes is given, for instance, by the following \( 3 + |\mathbb{F}| + |\mathbb{F}^* : (\mathbb{F}^*)^2| \) matrices:

\[
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 \\
0 & \rho
\end{pmatrix}, \quad \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1 \\
0 & \lambda
\end{pmatrix},
\]

where \( \rho \in \mathbb{F}^*/(\mathbb{F}^*)^2 \) and \( \lambda \in \mathbb{F}^* \).

Since the matrices \( \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \) and \( \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix} \) are projectively congruent, respectively, to \( \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \) and \( \begin{pmatrix}
1 & 1 \\
0 & 0
\end{pmatrix} \), this approach gives an alternative proof for the classification given in [10].

**References**


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