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## REGULAR SUBGROUPS, NILPOTENT ALGEBRAS AND PROJECTIVELY CONGRUENT MATRICES

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**ABSTRACT.** In this paper we highlight the connection between certain classes of regular subgroups of the affine group  $\text{AGL}_n(\mathbb{F})$ ,  $\mathbb{F}$  a field, and associative nilpotent  $\mathbb{F}$ -algebras of dimension  $n$ . We also describe how the classification of projective congruence classes of square matrices is equivalent to the classification of regular subgroups of particular shape.

### 1. Introduction

Let  $\mathbb{F}$  be a field. We can identify the affine group  $\text{AGL}_n(\mathbb{F})$  with the subgroup of  $\text{GL}_{n+1}(\mathbb{F})$  consisting of the matrices having  $(1, 0, \dots, 0)^T$  as first column. It follows that the group  $\text{AGL}_n(\mathbb{F})$  acts on the right on the set  $\mathcal{A} = \{(1, v) : v \in \mathbb{F}^n\}$  of affine points. A subgroup  $R$  of  $\text{AGL}_n(\mathbb{F})$  is called regular if it acts regularly on  $\mathcal{A}$ , namely if, for every  $v \in \mathbb{F}^n$ , there exists a unique element in  $R$  having  $(1, v)$  as first row. For instance, the translation subgroup  $\mathcal{T}_n$  of  $\text{AGL}_n(\mathbb{F})$  is a regular subgroup.

The problem of classifying, up to conjugation, the regular subgroups of  $\text{AGL}_n(\mathbb{F})$  attracted the interest of many authors. For instance, we recall the first systematic works of Caranti, Dalla Volta, Sala [1] and Tamburini [9] or the more recent paper of Catino, Colazzo and Stefanelli [2]. We recall also the work of Hegedűs [4], who constructed (nonabelian) regular subgroups containing no nontrivial translations. Recent generalizations of Hegedűs' examples have been obtained in [3, 8].

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Following the notation of [7], we write every element  $r$  of a regular subgroup  $R \leq \text{AGL}_n(\mathbb{F})$  as

$$(1.1) \quad r = \begin{pmatrix} 1 & v \\ 0 & \pi_R(r) \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & I_n + \delta_R(v) \end{pmatrix} = \mu_R(v),$$

considering the functions  $\pi_R : R \rightarrow \text{GL}_n(\mathbb{F})$ ,  $\mu_R : \mathbb{F}^n \rightarrow R$ , and  $\delta_R : \mathbb{F}^n \rightarrow \text{Mat}_n(\mathbb{F})$ . In this paper, as done in [7], we focus our attention on the case where the function  $\delta_R$  is linear. To simplify the notation, we denote by  $\Delta_n(\mathbb{F})$  the set of the regular subgroups  $R$  of  $\text{AGL}_n(\mathbb{F})$  for which the function  $\delta_R$  is linear.

In Section 2 we illustrate some of the properties of the subgroups  $R \in \Delta_n(\mathbb{F})$ . In particular, the linearity of  $\delta_R$  allows us to highlight the connection between regular subgroups and finite dimensional associative nilpotent algebras. In Section 3 we show how the classification of the regular subgroups  $R \in \Delta_n(\mathbb{F})$  can be obtained, in principle, working by induction on  $n$ . Furthermore, we construct particular regular subgroups  $\mathcal{R}_D \in \Delta_n(\mathbb{F})$  associated to square matrices  $D \in \text{Mat}_{n-1}(\mathbb{F})$  and we prove that two subgroups  $\mathcal{R}_{D_1}$  and  $\mathcal{R}_{D_2}$  are conjugate in  $\text{AGL}_n(\mathbb{F})$  if and only if the corresponding matrices  $D_1$  and  $D_2$  are projectively congruent.

## 2. Regular subgroups and nilpotent algebras

A complete classification, up to conjugation, of the regular subgroups of  $\text{AGL}_n(\mathbb{F})$  seems to be a rather difficult problem. For instance, in [7, Example 2.5] it was shown that the group  $\text{AGL}_2(\mathbb{R})$  contains  $2^{\lfloor \mathbb{R} \rfloor}$  conjugacy classes of regular subgroups. However, a regular subgroup in  $\Delta_n(\mathbb{F})$  has interesting properties that should allow a classification (see Section 3 and [6, 7]): for instance, it is unipotent (but the converse is not true), see [7]. We recall that any regular abelian subgroup of  $\text{AGL}_n(\mathbb{F})$  is in  $\Delta_n(\mathbb{F})$  (also here, for  $n \geq 3$ , the converse is not true).

**Proposition 2.1.** *Let  $R \in \Delta_n(\mathbb{F})$ . Then the following hold:*

- (a) *the set  $\{v \in \mathbb{F}^n : \mu_R(v) \in \mathbf{Z}(R)\}$  is a subspace of  $\mathbb{F}^n$ ;*
- (b) *the subspace  $\{w \in \mathbb{F}^n : w\pi_R(r) = w, \forall r \in R\}$  coincides with  $\text{Ker}(\delta_R)$ ;*
- (c)  *$\dim \text{Ker}(\delta_R) \geq 1$ .*

*Proof.* (a) First, recall that  $\mu_R(0) = I_{n+1}$  and that  $\mu_R(v) \in \mathbf{Z}(R)$  if and only if  $v\delta_R(w) = w\delta_R(v)$  for all  $w \in \mathbb{F}^n$  (see [7, 9]). The result easily follows from the linearity of  $\delta_R$ .

(b) It follows from  $\pi_R(\mu_R(v)) = I_n + \delta_R(v)$ .

(c) Since  $\delta_R$  is linear,  $R$  is unipotent and so we may suppose that  $R$  is upper unitriangular. Furthermore,  $\delta_R(v_1\delta_R(v_2)) = \delta_R(v_1)\delta_R(v_2)$  for all  $v_1, v_2 \in \mathbb{F}^n$ , see [7]. Now, let  $m \geq 0$  be the maximum integer for which there exists  $0 \neq v \in \mathbb{F}^n$  such that  $\delta_R(v)$  has the last  $m$  rows equal to 0. By way of contradiction, suppose that  $\text{Ker}(\delta_R) = \{0\}$ , that is  $m < n$ . Let  $0 \neq w \in \mathbb{F}^n$  be such that  $\delta_R(w)$  has the last  $m$  rows equal to 0 and set  $v = e_{n-m}\delta_R(w)$ , where  $\{e_1, \dots, e_n\}$  is the canonical basis of  $\mathbb{F}^n$ . Since  $R$  is upper unitriangular, we obtain that  $\delta_R(v) = \delta_R(e_{n-m})\delta_R(w)$  has the last  $m+1$  rows equal to 0, in contradiction with the maximality of  $m$ .  $\square$

Item (c) of previous proposition shows that any regular subgroup  $R \in \Delta_n(\mathbb{F})$  contains nontrivial translations. In fact, the examples constructed in [4] and in [3, 8] make use of quadratic functions, instead of simpler linear functions.

We now illustrate the connection between regular subgroups and finite dimensional associative nilpotent algebras. Denote by  $\mathcal{N}_n(\mathbb{F})$  the set of the associative nilpotent  $\mathbb{F}$ -algebras of dimension  $n$  (as  $\mathbb{F}$ -spaces). Following the proof of [7, Theorem 3.1], we can embed a given algebra  $N \in \mathcal{N}_n(\mathbb{F})$  into  $\text{Mat}_{n+1}(\mathbb{F})$  via

$$m \mapsto \begin{pmatrix} 0 & m_{\mathcal{B}} \\ 0 & \delta(m) \end{pmatrix},$$

where, for any  $m \in N$ ,  $m_{\mathcal{B}}$  and  $\delta(m)$  denote, respectively, the coordinate row vector of  $m$  and the matrix of the right multiplication by  $m$  with respect to a fixed basis  $\mathcal{B}$  of  $N$  over  $\mathbb{F}$ . Identifying  $N$  with its image, we have that

$$(2.1) \quad \mathcal{L} = \mathbb{F}I_{n+1} + N$$

is a split local subalgebra of  $\text{Mat}_{n+1}(\mathbb{F})$ , with Jacobson radical  $\mathbf{J}(\mathcal{L}) = N$ . Clearly, the subset  $R = \{I_{n+1} + m : m \in N\} \subseteq \mathcal{L}$  consists of invertible matrices and is closed under multiplication, since  $\delta(m_1\delta(m_2)) = \delta(m_1)\delta(m_2)$  for all  $m_1, m_2 \in N$ . By [7, Lemma 2.1]  $R$  is a regular subgroup, lying in  $\Delta_n(\mathbb{F})$ . This allows us to define a function  $\Phi : \mathcal{N}_n(\mathbb{F}) \rightarrow \Delta_n(\mathbb{F})$  setting  $\Phi(N) = R$  as described before.

Conversely, given a regular subgroup  $R \in \Delta_n(\mathbb{F})$ , we may consider the set

$$N = R - I_{n+1} = \left\{ \begin{pmatrix} 0 & v \\ 0 & \delta_R(v) \end{pmatrix} : v \in \mathbb{F}^n \right\}.$$

We have that  $\mathcal{L}_R = \mathbb{F}I_{n+1} + N$  is a split local  $\mathbb{F}$ -algebra of dimension  $n$ , by [7, Theorem 3.3]. Notice that  $N = \mathbf{J}(\mathcal{L}_R) \in \mathcal{N}_n(\mathbb{F})$ . Hence, we can consider the function  $\Psi : \Delta_n(\mathbb{F}) \rightarrow \mathcal{N}_n(\mathbb{F})$  defined by  $\Psi(R) = \mathbf{J}(\mathcal{L}_R)$ . When  $\text{char } \mathbb{F} = p > 0$ , following Isaacs' terminology [5], this shows that any regular subgroup  $R \in \Delta_n(\mathbb{F})$  is an  $\mathbb{F}$ -algebra group.

Extending the results of [1], where the authors considered the commutative case, we can prove the following.

**Proposition 2.2.** *There exists a bijection between the set of conjugacy classes of regular subgroups  $R$  of the affine group  $\text{AGL}_n(\mathbb{F})$  with linear  $\delta_R$  and the set of isomorphism classes of associative nilpotent  $\mathbb{F}$ -algebras of dimension  $n$ .*

*Proof.* Consider the maps  $\Phi : \mathcal{N}_n(\mathbb{F}) \rightarrow \Delta_n(\mathbb{F})$  and  $\Psi : \Delta_n(\mathbb{F}) \rightarrow \mathcal{N}_n(\mathbb{F})$  previously defined. By [7, Proposition 3.4] two regular subgroups  $R_1, R_2 \in \Delta_n(\mathbb{F})$  are conjugate in  $\text{AGL}_n(\mathbb{F})$  if and only if the corresponding split local algebras  $\mathcal{L}_{R_1}, \mathcal{L}_{R_2}$  are isomorphic. In particular, this holds if and only if the nilpotent algebras  $\mathbf{J}(\mathcal{L}_{R_1})$  and  $\mathbf{J}(\mathcal{L}_{R_2})$  are isomorphic. So, if  $R_1$  and  $R_2$  are conjugate, then  $\Psi(R_1) \cong \Psi(R_2)$ . Conversely, if  $N_1, N_2 \in \mathcal{N}_n(\mathbb{F})$  are isomorphic, also the corresponding split local algebras  $\mathcal{L}_1$  and  $\mathcal{L}_2$  constructed as in (2.1) are isomorphic. Hence, the subgroups  $\Phi(N_1)$  and  $\Phi(N_2)$  are conjugate. □

### 3. Regular subgroups and projectively congruent matrices

We want to describe a method for constructing regular subgroups of  $AGL_n(\mathbb{F})$  starting from subgroups  $R \in \Delta_{n-1}(\mathbb{F})$ . First of all, take any matrix  $D \in \text{Mat}_{n-1}(\mathbb{F})$  and define

$$\mathcal{R}(R, D) = \left\{ \begin{pmatrix} 1 & X & x_n \\ 0 & I_{n-1} + \delta_R(X) & DX^T \\ 0 & 0 & 1 \end{pmatrix} : X \in \mathbb{F}^{n-1}, x_n \in \mathbb{F} \right\}.$$

It can be easily proved that  $\mathcal{R}(R, D) \in \Delta_n(\mathbb{F})$  if and only if

$$(3.1) \quad D\delta_R(e_i)^T e_j^T = \delta_R(e_j) D e_i^T \quad \text{for all } i, j = 1, \dots, n-1,$$

where  $\{e_1, \dots, e_{n-1}\}$  is the canonical basis of  $\mathbb{F}^{n-1}$  (by [7, Lemma 2.1] it suffices to study when  $\mathcal{R}(R, D)$  is closed with respect to multiplication).

Conversely, we want to prove that every regular subgroup  $R \in \Delta_n(\mathbb{F})$  can be written as  $\mathcal{R}(\tilde{R}, D)$  for some  $\tilde{R} \in \Delta_{n-1}(\mathbb{F})$  and some matrix  $D \in \text{Mat}_{n-1}(\mathbb{F})$  satisfying (3.1). We start with the following result.

**Proposition 3.1.** *Let  $R \in \Delta_n(\mathbb{F})$  and  $m = \dim \text{Ker}(\delta_R)$ . Then, up to conjugation,  $R$  is upper unitriangular with  $\delta_R(e_i) = 0$  for all  $i \in I = \{n-m+1, \dots, n\}$ .*

*Proof.* First we recall that, by Proposition 2.1,  $m \geq 1$  and so  $I \neq \emptyset$ . Up to conjugation we may suppose that  $\delta_R(e_i) = 0$  for all  $i \in I$ . Now, for all  $v \in \mathbb{F}^n$  and for all  $i \in I$  we have  $\delta_R(e_i \delta_R(v)) = \delta_R(e_i) \delta_R(v) = 0$ . Hence,  $e_i \delta_R(v) \in \text{Ker}(\delta_R) = \langle e_j : j \in I \rangle$ . This means that for all  $v \in \mathbb{F}^n$

we have  $\mu_R(v) = \begin{pmatrix} 1 & X & \tilde{X} \\ 0 & f_1(X) & f_2(X) \\ 0 & 0 & f_3(X) \end{pmatrix}$ , where  $X \in \mathbb{F}^{n-m}$ ,  $\tilde{X} \in \mathbb{F}^m$ ,  $f_1 : \mathbb{F}^{n-m} \rightarrow \text{GL}_{n-m}(\mathbb{F})$ ,

$f_2 : \mathbb{F}^{n-m} \rightarrow \text{Mat}_{n-m,m}(\mathbb{F})$  and  $f_3 : \mathbb{F}^{n-m} \rightarrow \text{GL}_m(\mathbb{F})$ . Note that the sets  $\{f_1(X) : X \in \mathbb{F}^{n-m}\}$  and  $\{f_3(X) : X \in \mathbb{F}^{n-m}\}$  are both unipotent subgroups, so there exist  $N \in \text{GL}_{n-m}(\mathbb{F})$  and  $M \in \text{GL}_m(\mathbb{F})$  such that both  $N^{-1}f_1(X)N$  and  $M^{-1}f_3(X)M$  are upper unitriangular for all  $X \in \mathbb{F}^{n-m}$ . Hence, conjugating by  $g = \text{diag}(1, N, M)$ , we obtain that  $\mu_R(v)$  is unitriangular for all  $v \in \mathbb{F}^n$  with  $\delta_R(e_i) = 0$  for all  $i \in I$  (since the set  $\{e_i : i \in I\}$  is fixed by  $g$ ). □

By Proposition 3.1, up to conjugation,  $R \in \Delta_n(\mathbb{F})$  can be written as the subgroup

$$R = \left\{ \begin{pmatrix} 1 & X & x_n \\ 0 & f_1(X) & f_2(X)^T \\ 0 & 0 & 1 \end{pmatrix} : X \in \mathbb{F}^{n-1}, x_n \in \mathbb{F} \right\},$$

where  $f_1 : \mathbb{F}^{n-1} \rightarrow \text{GL}_{n-1}(\mathbb{F})$  and  $f_2 : \mathbb{F}^{n-1} \rightarrow \mathbb{F}^{n-1}$  are such that  $f_1(Xf_1(Y) + Y) = f_1(X)f_1(Y)$  and  $f_2(Xf_1(Y) + Y)^T = f_2(X)^T + f_1(X)f_2(Y)^T$  for all  $X, Y \in \mathbb{F}^{n-1}$ . Since  $\delta_R$  is a linear function,  $f_1(X) = I_{n-1} + \delta(X)$  and  $f_2(X)^T = DX^T$  for some linear function  $\delta$  and some matrix  $D \in \text{Mat}_{n-1}(\mathbb{F})$  satisfying

(3.1). It follows that  $\tilde{R} = \left\{ \begin{pmatrix} 1 & X \\ 0 & I_{n-1} + \delta(X) \end{pmatrix} : X \in \mathbb{F}^{n-1} \right\} \in \Delta_{n-1}(\mathbb{F})$  and so  $R = \mathcal{R}(\tilde{R}, D)$ .

We now consider the special case of  $\mathcal{R}(\mathcal{T}_{n-1}, D) \in \Delta_n(\mathbb{F})$ , that we simply denote by  $\mathcal{R}_D$ . Notice that  $\delta_{\mathcal{T}_{n-1}} = 0$  and so any matrix  $D \in \text{Mat}_{n-1}(\mathbb{F})$  satisfies (3.1). So,

$$\mathcal{R}_D = \left\{ \begin{pmatrix} 1 & X & x_n \\ 0 & I_{n-1} & DX^T \\ 0 & 0 & 1 \end{pmatrix} : X \in \mathbb{F}^{n-1}, x_n \in \mathbb{F} \right\}.$$

We list here some easy properties of the subgroups  $\mathcal{R}_D$ . First of all, observe that if  $D = 0$  then  $\mathcal{R}_D$  coincides with the translation subgroup  $\mathcal{T}_n$ . Furthermore,  $\mathcal{R}_D$  is abelian if and only if  $D = D^T$ . Finally, using the fact the  $\mathcal{R}_D$  is unipotent, we set

$$\begin{aligned} d(\mathcal{R}_D) &= \max\{\text{deg min}_{\mathbb{F}}(r - I_{n+1}) : r \in \mathcal{R}_D\}, \\ r(\mathcal{R}_D) &= \max\{\text{rk}(r - I_{n+1}) : r \in \mathcal{R}_D\}, \\ k(\mathcal{R}_D) &= \dim \text{Ker}(\delta_{\mathcal{R}_D}) \end{aligned}$$

( $\text{min}_{\mathbb{F}}(g)$  denotes the minimum polynomial of  $g$  over  $\mathbb{F}$ ). Then, we have

$$\begin{aligned} d(\mathcal{R}_D) &= \begin{cases} 2 & \text{if } D^T = -D \text{ and } D \text{ has zero diagonal,} \\ 3 & \text{otherwise,} \end{cases} \\ r(\mathcal{R}_D) &= \begin{cases} 2 & \text{if } D \neq 0, \\ 1 & \text{if } D = 0, \end{cases} \\ k(\mathcal{R}_D) &= n - \text{rk}(D). \end{aligned}$$

We show that there exists a bijection between conjugacy classes of regular subgroups  $\mathcal{R}_D$  of  $\text{AGL}_n(\mathbb{F})$  and projective congruent classes of matrices of  $\text{Mat}_{n-1}(\mathbb{F})$ . Given two matrices  $A, B \in \text{Mat}_n(\mathbb{F})$ , we say that  $A$  and  $B$  are *projectively congruent* if  $PAP^T = \lambda B$ , for some non-zero element  $\lambda \in \mathbb{F}^*$  and some invertible matrix  $P \in \text{GL}_n(\mathbb{F})$  (see for instance, [10, 11]). Clearly, when  $\mathbb{F}$  is algebraically closed two square matrices are projectively congruent if and only if they are congruent.

**Lemma 3.2.** *Given two matrices  $A, B \in \text{Mat}_{n-1}(\mathbb{F})$ , the subgroups  $\mathcal{R}_A$  and  $\mathcal{R}_B$  are conjugate in  $\text{AGL}_n(\mathbb{F})$  if and only if  $A$  and  $B$  are projectively congruent.*

*Proof.* If  $A = 0$  then  $\mathcal{R}_A = \mathcal{T}_n$  which is normal in  $\text{AGL}_n(\mathbb{F})$ . Hence  $\mathcal{R}_B$  is conjugate to  $\mathcal{R}_A$  if and only if  $B = 0$ . Now, assume  $A \neq 0$ . By [7, Proposition 3.4] the regular subgroups  $\mathcal{R}_A$  and  $\mathcal{R}_B$  are conjugate in  $\text{AGL}_n(\mathbb{F})$  if and only if there exists an invertible matrix

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & P & M^T \\ 0 & N & \lambda \end{pmatrix} \in \text{GL}_{n+1}(\mathbb{F}),$$

with  $P \in \text{Mat}_{n-1}(\mathbb{F})$ ,  $M, N \in \mathbb{F}^{n-1}$ ,  $\lambda \in \mathbb{F}$ , such that  $\mathcal{R}_A \cdot g = g \cdot \mathcal{R}_B$ . This holds if and only if for any  $X \in \mathbb{F}^{n-1}$  and any  $x_n \in \mathbb{F}$  we have

$$AX^T N = 0, \quad NBY^T = 0 \quad \text{and} \quad \lambda AX^T = PBY^T,$$

where  $Y = XP + x_n N$ . Since  $A \neq 0$ , the condition  $AX^T N = 0$  for all  $X \in \mathbb{F}^{n-1}$  implies  $N = 0$ . It follows that  $\mathcal{R}_A$  and  $\mathcal{R}_B$  are conjugate in  $\text{AGL}_n(\mathbb{F})$  if and only if  $\lambda A = PBP^T$  for some  $\lambda \in \mathbb{F}^*$  and some  $P \in \text{GL}_{n-1}(\mathbb{F})$ , that is, if and only if  $A$  and  $B$  are projectively congruent.  $\square$

**Example 3.3.** Using Lemma 3.2 we can obtain the classification of the projective congruence classes of matrices in  $\text{Mat}_2(\mathbb{F})$ , for any field  $\mathbb{F}$ . By [7, Lemmas 5.3, 5.4 and 7.1], a complete set of representatives of such classes is given, for instance, by the following  $3 + |\mathbb{F}| + |\mathbb{F}^*| : (\mathbb{F}^*)^2$  matrices:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & \lambda \end{pmatrix},$$

where  $\rho \in \mathbb{F}^*/(\mathbb{F}^*)^2$  and  $\lambda \in \mathbb{F}^*$ .

Since the matrices  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  are projectively congruent, respectively, to  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ , this approach gives an alternative proof for the classification given in [10].

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