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REGULAR SUBGROUPS, NILPOTENT ALGEBRAS AND PROJECTIVELY CONGRUENT MATRICES

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Abstract. In this paper we highlight the connection between certain classes of regular subgroups of the affine group $\mathrm{AGL}_n(\mathbb{F})$, $\mathbb{F}$ a field, and associative nilpotent $\mathbb{F}$-algebras of dimension n. We also describe how the classification of projective congruence classes of square matrices is equivalent to the classification of regular subgroups of particular shape.

1. Introduction

Let $\mathbb{F}$ be a field. We can identify the affine group $\mathrm{AGL}_n(\mathbb{F})$ with the subgroup of $\mathrm{GL}_{n+1}(\mathbb{F})$ consisting of the matrices having $(1, 0, \ldots, 0)^{\mathrm{T}}$ as first column. It follows that the group $\mathrm{AGL}_n(\mathbb{F})$ acts on the right on the set $\mathcal{A} = \{(1, v) : v \in \mathbb{F}^n\}$ of affine points. A subgroup R of $\mathrm{AGL}_n(\mathbb{F})$ is called regular if it acts regularly on $\mathcal{A}$, namely if, for every $v \in \mathbb{F}^n$, there exists a unique element in R having $(1, v)$ as first row. For instance, the translation subgroup $\mathcal{T}_n$ of $\mathrm{AGL}_n(\mathbb{F})$ is a regular subgroup.

The problem of classifying, up to conjugation, the regular subgroups of $\mathrm{AGL}_n(\mathbb{F})$ attracted the interest of many authors. For instance, we recall the first systematic works of Caranti, Dalla Volta, Sala [1] and Tamburini [9] or the more recent paper of Catino, Colazzo and Stefanelli [2]. We recall also the work of Hegedűs [4], who constructed (nonabelian) regular subgroups containing no nontrivial translations. Recent generalizations of Hegedűs' examples have been obtained in [3, 8].

Following the notation of [7], we write every element r of a regular subgroup $R \leq \mathrm{AGL}_n(\mathbb{F})$ as

$$r = \begin{pmatrix} 1 & v \\ 0 & \pi_R(r) \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & I_n + \delta_R(v) \end{pmatrix} = \mu_R(v), \tag{1.1}$$

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considering the functions $\pi_R : R \to \mathrm{GL}_n(\mathbb{F})$, $\mu_R : \mathbb{F}^n \to R$, and $\delta_R : \mathbb{F}^n \to \mathrm{Mat}_n(\mathbb{F})$. In this paper, as done in [7], we focus our attention on the case where the function δ_R is linear. To simplify the notation, we denote by $\Delta_n(\mathbb{F})$ the set of the regular subgroups R of $\mathrm{AGL}_n(\mathbb{F})$ for which the function δ_R is linear.

In Section 2 we illustrate some of the properties of the subgroups $R \in \Delta_n(\mathbb{F})$. In particular, the linearity of δ_R allows us to highlight the connection between regular subgroups and finite dimensional associative nilpotent algebras. In Section 3 we show how the classification of the regular subgroups $R \in \Delta_n(\mathbb{F})$ can be obtained, in principle, working by induction on n. Furthermore, we construct particular regular subgroups $\mathcal{R}_D \in \Delta_n(\mathbb{F})$ associated to square matrices $D \in \mathrm{Mat}_{n-1}(\mathbb{F})$ and we prove that two subgroups $\mathcal{R}_{D_1}$ and $\mathcal{R}_{D_2}$ are conjugate in $\mathrm{AGL}_n(\mathbb{F})$ if and only if the corresponding matrices D_1 and D_2 are projectively congruent.

2. Regular subgroups and nilpotent algebras

A complete classification, up to conjugation, of the regular subgroups of $\mathrm{AGL}_n(\mathbb{F})$ seems to be a rather difficult problem. For instance, in [7, Example 2.5] it was shown that the group $\mathrm{AGL}_2(\mathbb{R})$ contains $2^{|\mathbb{R}|}$ conjugacy classes of regular subgroups. However, a regular subgroup in $\Delta_n(\mathbb{F})$ has interesting properties that should allow a classification (see Section 3 and [6, 7]): for instance, it is unipotent (but the converse is not true), see [7]. We recall that any regular abelian subgroup of $\mathrm{AGL}_n(\mathbb{F})$ is in $\Delta_n(\mathbb{F})$ (also here, for $n \geq 3$, the converse is not true).

Proposition 2.1. *Let $R \in \Delta_n(\mathbb{F})$. Then the following hold:*

(a) *the set $\{v \in \mathbb{F}^n : \mu_R(v) \in \mathbf{Z}(R)\}$ is a subspace of $\mathbb{F}^n$;*
(b) *the subspace $\{w \in \mathbb{F}^n : w\pi_R(r) = w, \forall\, r \in R\}$ coincides with $\mathrm{Ker}\,(\delta_R)$;*
(c) $\dim \mathrm{Ker}\,(\delta_R) \geq 1$.

Proof. (a) First, recall that $\mu_R(0) = I_{n+1}$ and that $\mu_R(v) \in \mathbf{Z}(R)$ if and only if $v\delta_R(w) = w\delta_R(v)$ for all $w \in \mathbb{F}^n$ (see [7, 9]). The result easily follows from the linearity of δ_R.
(b) It follows from $\pi_R(\mu_R(v)) = I_n + \delta_R(v)$.
(c) Since δ_R is linear, R is unipotent and so we may suppose that R is upper unitriangular. Furthermore, $\delta_R(v_1\delta_R(v_2)) = \delta_R(v_1)\delta_R(v_2)$ for all $v_1, v_2 \in \mathbb{F}^n$, see [7]. Now, let $m \geq 0$ be the maximum integer for which there exists $0 \neq v \in \mathbb{F}^n$ such that $\delta_R(v)$ has the last m rows equal to 0. By way of contradiction, suppose that $\mathrm{Ker}\,(\delta_R) = \{0\}$, that is $m < n$. Let $0 \neq w \in \mathbb{F}^n$ be such that $\delta_R(w)$ has the last m rows equal to 0 and set $v = e_{n-m}\delta_R(w)$, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{F}^n$. Since R is upper unitriangular, we obtain that $\delta_R(v) = \delta_R(e_{n-m})\delta_R(w)$ has the last $m + 1$ rows equal to 0, in contradiction with the maximality of m. □

Item (c) of previous proposition shows that any regular subgroup $R \in \Delta_n(\mathbb{F})$ contains nontrivial translations. In fact, the examples constructed in [4] and in [3, 8] make use of quadratic functions, instead of simpler linear functions.

We now illustrate the connection between regular subgroups and finite dimensional associative nilpotent algebras. Denote by $\mathcal{N}_n(\mathbb{F})$ the set of the associative nilpotent $\mathbb{F}$-algebras of dimension n (as $\mathbb{F}$-spaces). Following the proof of [7, Theorem 3.1], we can embed a given algebra $N \in \mathcal{N}_n(\mathbb{F})$ into $\mathrm{Mat}_{n+1}(\mathbb{F})$ via

$$m \mapsto \begin{pmatrix} 0 & m_{\mathcal{B}} \\ 0 & \delta(m) \end{pmatrix},$$

where, for any $m \in N$, $m_{\mathcal{B}}$ and $\delta(m)$ denote, respectively, the coordinate row vector of m and the matrix of the right multiplication by m with respect to a fixed basis $\mathcal{B}$ of N over $\mathbb{F}$. Identifying N with its image, we have that

$$\mathcal{L} = \mathbb{F}I_{n+1} + N \tag{2.1}$$

is a split local subalgebra of $\mathrm{Mat}_{n+1}(\mathbb{F})$, with Jacobson radical $\mathbf{J}(\mathcal{L}) = N$. Clearly, the subset $R = \{I_{n+1} + m : m \in N\} \subseteq \mathcal{L}$ consists of invertible matrices and is closed under multiplication, since $\delta(m_1\delta(m_2)) = \delta(m_1)\delta(m_2)$ for all $m_1, m_2 \in N$. By [7, Lemma 2.1] R is a regular subgroup, lying in $\Delta_n(\mathbb{F})$. This allows us to define a function $\Phi : \mathcal{N}_n(\mathbb{F}) \to \Delta_n(\mathbb{F})$ setting $\Phi(N) = R$ as described before.

Conversely, given a regular subgroup $R \in \Delta_n(\mathbb{F})$, we may consider the set

$$N = R - I_{n+1} = \left\{ \begin{pmatrix} 0 & v \\ 0 & \delta_R(v) \end{pmatrix} : v \in \mathbb{F}^n \right\}.$$

We have that $\mathcal{L}_R = \mathbb{F}I_{n+1} + N$ is a split local $\mathbb{F}$-algebra of dimension n, by [7, Theorem 3.3]. Notice that $N = \mathbf{J}(\mathcal{L}_R) \in \mathcal{N}_n(\mathbb{F})$. Hence, we can consider the function $\Psi : \Delta_n(\mathbb{F}) \to \mathcal{N}_n(\mathbb{F})$ defined by $\Psi(R) = \mathbf{J}(\mathcal{L}_R)$. When $\mathrm{char}\,\mathbb{F} = p > 0$, following Isaacs' terminology [5], this shows that any regular subgroup $R \in \Delta_n(\mathbb{F})$ is an $\mathbb{F}$-algebra group.

Extending the results of [1], where the authors considered the commutative case, we can prove the following.

Proposition 2.2. *There exists a bijection between the set of conjugacy classes of regular subgroups R of the affine group $\mathrm{AGL}_n(\mathbb{F})$ with linear δ_R and the set of isomorphism classes of associative nilpotent $\mathbb{F}$-algebras of dimension n.*

Proof. Consider the maps $\Phi : \mathcal{N}_n(\mathbb{F}) \to \Delta_n(\mathbb{F})$ and $\Psi : \Delta_n(\mathbb{F}) \to \mathcal{N}_n(\mathbb{F})$ previously defined. By [7, Proposition 3.4] two regular subgroups $R_1, R_2 \in \Delta_n(\mathbb{F})$ are conjugate in $\mathrm{AGL}_n(\mathbb{F})$ if and only if the corresponding split local algebras $\mathcal{L}_{R_1}, \mathcal{L}_{R_2}$ are isomorphic. In particular, this holds if and only if the nilpotent algebras $\mathbf{J}(\mathcal{L}_{R_1})$ and $\mathbf{J}(\mathcal{L}_{R_2})$ are isomorphic. So, if R_1 and R_2 are conjugate, then $\Psi(R_1) \cong \Psi(R_2)$. Conversely, if $N_1, N_2 \in \mathcal{N}_n(\mathbb{F})$ are isomorphic, also the corresponding split local algebras $\mathcal{L}_1$ and $\mathcal{L}_2$ constructed as in (2.1) are isomorphic. Hence, the subgroups $\Phi(N_1)$ and $\Phi(N_2)$ are conjugate. □

3. Regular subgroups and projectively congruent matrices

We want to describe a method for constructing regular subgroups of $\mathrm{AGL}_n(\mathbb{F})$ starting from subgroups $R \in \Delta_{n-1}(\mathbb{F})$. First of all, take any matrix $D \in \mathrm{Mat}_{n-1}(\mathbb{F})$ and define

$$\mathcal{R}(R,D) = \left\{ \begin{pmatrix} 1 & X & x_n \\ 0 & I_{n-1}+\delta_R(X) & DX^{\mathrm{T}} \\ 0 & 0 & 1 \end{pmatrix} : X \in \mathbb{F}^{n-1},\ x_n \in \mathbb{F} \right\}.$$

It can be easily proved that $\mathcal{R}(R,D) \in \Delta_n(\mathbb{F})$ if and only if

$$D\delta_R(e_i)^{\mathrm{T}} e_j^{\mathrm{T}} = \delta_R(e_j) D e_i^{\mathrm{T}} \qquad \text{for all } i,j = 1,\ldots,n-1, \tag{3.1}$$

where $\{e_1,\ldots,e_{n-1}\}$ is the canonical basis of $\mathbb{F}^{n-1}$ (by [7, Lemma 2.1] it suffices to study when $\mathcal{R}(R,D)$ is closed with respect to multiplication).

Conversely, we want to prove that every regular subgroup $R \in \Delta_n(\mathbb{F})$ can be written as $\mathcal{R}(\tilde{R},D)$ for some $\tilde{R} \in \Delta_{n-1}(\mathbb{F})$ and some matrix $D \in \mathrm{Mat}_{n-1}(\mathbb{F})$ satisfying (3.1). We start with the following result.

Proposition 3.1. *Let $R \in \Delta_n(\mathbb{F})$ and $m = \dim \mathrm{Ker}\,(\delta_R)$. Then, up to conjugation, R is upper unitriangular with $\delta_R(e_i) = 0$ for all $i \in I = \{n-m+1,\ldots,n\}$.*

Proof. First we recall that, by Proposition 2.1, $m \geq 1$ and so $I \neq \emptyset$. Up to conjugation we may suppose that $\delta_R(e_i) = 0$ for all $i \in I$. Now, for all $v \in \mathbb{F}^n$ and for all $i \in I$ we have $\delta_R(e_i\delta_R(v)) = \delta_R(e_i)\delta_R(v) = 0$. Hence, $e_i\delta_R(v) \in \mathrm{Ker}\,(\delta_R) = \langle e_j : j \in I\rangle$. This means that for all $v \in \mathbb{F}^n$ we have $\mu_R(v) = \begin{pmatrix} 1 & X & \tilde{X} \\ 0 & f_1(X) & f_2(X) \\ 0 & 0 & f_3(X) \end{pmatrix}$, where $X \in \mathbb{F}^{n-m}$, $\tilde{X} \in \mathbb{F}^m$, $f_1 : \mathbb{F}^{n-m} \to \mathrm{GL}_{n-m}(\mathbb{F})$, $f_2 : \mathbb{F}^{n-m} \to \mathrm{Mat}_{n-m,m}(\mathbb{F})$ and $f_3 : \mathbb{F}^{n-m} \to \mathrm{GL}_m(\mathbb{F})$. Note that the sets $\{f_1(X) : X \in \mathbb{F}^{n-m}\}$ and $\{f_3(X) : X \in \mathbb{F}^{n-m}\}$ are both unipotent subgroups, so there exist $N \in \mathrm{GL}_{n-m}(\mathbb{F})$ and $M \in \mathrm{GL}_m(\mathbb{F})$ such that both $N^{-1}f_1(X)N$ and $M^{-1}f_2(X)M$ are upper unitriangular for all $X \in \mathbb{F}^{n-m}$. Hence, conjugating by $g = \mathrm{diag}(1,N,M)$, we obtain that $\mu_R(v)$ is unitriangular for all $v \in \mathbb{F}^n$ with $\delta_R(e_i) = 0$ for all $i \in I$ (since the set $\{e_i : i \in I\}$ is fixed by g). $\square$

By Proposition 3.1, up to conjugation, $R \in \Delta_n(\mathbb{F})$ can be written as the subgroup

$$R = \left\{ \begin{pmatrix} 1 & X & x_n \\ 0 & f_1(X) & f_2(X)^{\mathrm{T}} \\ 0 & 0 & 1 \end{pmatrix} : X \in \mathbb{F}^{n-1},\ x_n \in \mathbb{F} \right\},$$

where $f_1 : \mathbb{F}^{n-1} \to \mathrm{GL}_{n-1}(\mathbb{F})$ and $f_2 : \mathbb{F}^{n-1} \to \mathbb{F}^{n-1}$ are such that $f_1(Xf_1(Y)+Y) = f_1(X)f_1(Y)$ and $f_2(Xf_1(Y)+Y)^{\mathrm{T}} = f_2(X)^{\mathrm{T}} + f_1(X)f_2(Y)^{\mathrm{T}}$ for all $X,Y \in \mathbb{F}^{n-1}$. Since δ_R is a linear function, $f_1(X) = I_{n-1}+\delta(X)$ and $f_2(X)^{\mathrm{T}} = DX^{\mathrm{T}}$ for some linear function δ and some matrix $D \in \mathrm{Mat}_{n-1}(\mathbb{F})$ satisfying (3.1). It follows that $\tilde{R} = \left\{ \begin{pmatrix} 1 & X \\ 0 & I_{n-1}+\delta(X) \end{pmatrix} : X \in \mathbb{F}^{n-1} \right\} \in \Delta_{n-1}(\mathbb{F})$ and so $R = \mathcal{R}(\tilde{R},D)$.

We now consider the special case of $\mathcal{R}(\mathcal{T}_{n-1}, D) \in \Delta_n(\mathbb{F})$, that we simply denote by $\mathcal{R}_D$. Notice that $\delta_{\mathcal{T}_{n-1}} = 0$ and so any matrix $D \in \mathrm{Mat}_{n-1}(\mathbb{F})$ satisfies (3.1). So,

$$\mathcal{R}_D = \left\{ \begin{pmatrix} 1 & X & x_n \\ 0 & I_{n-1} & DX^{\mathrm{T}} \\ 0 & 0 & 1 \end{pmatrix} : X \in \mathbb{F}^{n-1},\ x_n \in \mathbb{F} \right\}.$$

We list here some easy properties of the subgroups $\mathcal{R}_D$. First of all, observe that if $D = 0$ then $\mathcal{R}_D$ coincides with the translation subgroup $\mathcal{T}_n$. Furthermore, $\mathcal{R}_D$ is abelian if and only if $D = D^{\mathrm{T}}$. Finally, using the fact the $\mathcal{R}_D$ is unipotent, we set

$$\begin{aligned} \mathsf{d}(\mathcal{R}_D) &= \max\{\deg \min_{\mathbb{F}}(r - I_{n+1}) : r \in \mathcal{R}_D\}, \\ \mathsf{r}(\mathcal{R}_D) &= \max\{\mathrm{rk}(r - I_{n+1}) : r \in \mathcal{R}_D\}, \\ \mathsf{k}(\mathcal{R}_D) &= \dim \mathrm{Ker}\,(\delta_{\mathcal{R}_D}) \end{aligned}$$

($\min_{\mathbb{F}}(g)$ denotes the minimum polynomial of g over $\mathbb{F}$). Then, we have

$$\begin{aligned} \mathsf{d}(\mathcal{R}_D) &= \begin{cases} 2 & \text{if } D^{\mathrm{T}} = -D \text{ and } D \text{ has zero diagonal}, \\ 3 & \text{otherwise}, \end{cases} \\ \mathsf{r}(\mathcal{R}_D) &= \begin{cases} 2 & \text{if } D \neq 0, \\ 1 & \text{if } D = 0, \end{cases} \\ \mathsf{k}(\mathcal{R}_D) &= n - \mathrm{rk}(D). \end{aligned}$$

We show that there exists a bijection between conjugacy classes of regular subgroups $\mathcal{R}_D$ of $\mathrm{AGL}_n(\mathbb{F})$ and projective congruent classes of matrices of $\mathrm{Mat}_{n-1}(\mathbb{F})$. Given two matrices $A, B \in \mathrm{Mat}_n(\mathbb{F})$, we say that A and B are *projectively congruent* if $PAP^{\mathrm{T}} = \lambda B$, for some non-zero element $\lambda \in \mathbb{F}^*$ and some invertible matrix $P \in \mathrm{GL}_n(\mathbb{F})$ (see for instance, [10, 11]). Clearly, when $\mathbb{F}$ is algebraically closed two square matrices are projectively congruent if and only if they are congruent.

Lemma 3.2. *Given two matrices $A, B \in \mathrm{Mat}_{n-1}(\mathbb{F})$, the subgroups $\mathcal{R}_A$ and $\mathcal{R}_B$ are conjugate in $\mathrm{AGL}_n(\mathbb{F})$ if and only if A and B are projectively congruent.*

Proof. If $A = 0$ then $\mathcal{R}_A = \mathcal{T}_n$ which is normal in $\mathrm{AGL}_n(\mathbb{F})$. Hence $\mathcal{R}_B$ is conjugate to $\mathcal{R}_A$ if and only if $B = 0$. Now, assume $A \neq 0$. By [7, Proposition 3.4] the regular subgroups $\mathcal{R}_A$ and $\mathcal{R}_B$ are conjugate in $\mathrm{AGL}_n(\mathbb{F})$ if and only if there exists an invertible matrix

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & P & M^{\mathrm{T}} \\ 0 & N & \lambda \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{F}),$$

with $P \in \mathrm{Mat}_{n-1}(\mathbb{F})$, $M, N \in \mathbb{F}^{n-1}$, $\lambda \in \mathbb{F}$, such that $\mathcal{R}_A \cdot g = g \cdot \mathcal{R}_B$. This holds if and only if for any $X \in \mathbb{F}^{n-1}$ and any $x_n \in \mathbb{F}$ we have

$$AX^{\mathrm{T}}N = 0, \quad NBY^{\mathrm{T}} = 0 \quad \text{and} \quad \lambda AX^{\mathrm{T}} = PBY^{\mathrm{T}},$$

where $Y = XP + x_nN$. Since $A \neq 0$, the condition $AX^{\mathrm{T}}N = 0$ for all $X \in \mathbb{F}^{n-1}$ implies $N = 0$. It follows that $\mathcal{R}_A$ and $\mathcal{R}_B$ are conjugate in $\mathrm{AGL}_n(\mathbb{F})$ if and only if $\lambda A = PBP^{\mathrm{T}}$ for some $\lambda \in \mathbb{F}^*$ and some $P \in \mathrm{GL}_{n-1}(\mathbb{F})$, that is, if and only if A and B are projectively congruent. □

Example 3.3. *Using Lemma 3.2 we can obtain the classification of the projective congruence classes of matrices in* $\mathrm{Mat}_2(\mathbb{F})$*, for any field* $\mathbb{F}$*. By* [7, Lemmas 5.3, 5.4 and 7.1]*, a complete set of representatives of such classes is given, for instance, by the following* $3 + |\mathbb{F}| + |\mathbb{F}^* : (\mathbb{F}^*)^2|$ *matrices:*

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & \lambda \end{pmatrix},$$

where $\rho \in \mathbb{F}^*/(\mathbb{F}^*)^2$ *and* $\lambda \in \mathbb{F}^*$.

Since the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *are projectively congruent, respectively, to* $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$*, this approach gives an alternative proof for the classification given in* [10].

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