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ON GROUPS WITH TWO ISOMORPHISM CLASSES OF CENTRAL FACTORS

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Abstract. The structure of groups which have at most two isomorphism classes of central factors (B_2-groups) are investigated. A complete description of B_2-groups is obtained in the locally finite case and in the nilpotent case. In addition detailed information is obtained about soluble B_2-groups. Also structural information about insoluble B_2-groups is given, in particular when such a group has the derived subgroup satisfying the minimal condition.

1. Introduction

Given a group G, a subgroup K of G is said to be a *derived subgroup* or a *commutator subgroup* in G if $K = H'$ for some subgroup H of G, where H' denotes the derived subgroup of H.

Let $C(G)$ denote the set of all derived subgroups in G:

$$C(G) = \{H' | H \leq G\}.$$

The influence of $C(G)$ on the structure of the group G has been studied by many authors. For example, F. de Giovanni and D. J. S. Robinson in [1] and M. Herzog, P. Longobardi and M. Maj in [2], have investigated the case $C(G)$ finite. In particular, they proved that if G is locally graded, $C(G)$ is finite if and only if G' is finite.

Let n be a positive integer and let D_n denote the class of all groups with at most n isomorphism types of derived subgroups. Clearly D_1 is the class of abelian groups and a non-abelian group G belongs to D_2 if and only if $H' \simeq G'$ whenever H is a non-abelian subgroup of G. P. Longobardi, M.

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Maj, D. J. S. Robinson and H. Smith in [4] focused their attention on groups in D_2 and described in a precise way some large classes of D_2-groups.

Some additional information about these classes of groups can be founded in [5], [6].

In this paper we are concerned with groups G for which the set of isomorphism types of elements in $\{\frac{H}{Z(H)} | H \leq G\}$ is very small.

If n is a positive integer, let B_n denote the class of groups G such that the factor groups in $\{\frac{H}{Z(H)} | H \leq G\}$ fall into at most n isomorphism classes.

Of course, B_1 is the class of abelian groups, while a non-abelian group G belongs to B_2 if and only if $\frac{H}{Z(H)} \simeq \frac{G}{Z(G)}$ whenever H is a non-abelian subgroup of G.

We give a characterization of nilpotent B_2-groups. In particular we prove, for a non-abelian group G, that G is nilpotent and belongs to B_2, if and only if either $\frac{G}{Z(G)}$ is elementary abelian of order p^2 (p a prime) or $\frac{G}{Z(G)} \simeq \mathbb{Z} \times \mathbb{Z}$.

In addition, we show that if G is a locally finite group, then $G \in B_2$ if and only if $G = Z(G)H$, where H is a finite minimal non-abelian subgroup of G.

In the soluble case we prove that, if G is a soluble non-nilpotent B_2-group, then

i) $Z(\frac{G}{Z(G)}) = 1$.
ii) $G = A < x >$, where A is a normal abelian sugroup of G.
iii) Every non-abelian subgroup of $\frac{G}{Z(G)}$ is isomorphic to $\frac{G}{Z(G)}$.

Moreover we show that locally graded B_2-groups are soluble.

Finally we analyze the insoluble case and we prove that if G is an insoluble B_2-group, then G cannot satisfy the so-called Tits alternative. Moreover, if G' satisfies the minimal condition, then $\frac{G}{Z(G)}$ is a Tarski group.

2. **Elementary results**

If G is a minimal non-abelian group, then obviously G is in B_2.

The following proposition gives more examples of groups in B_2.

Proposition 2.1. *Let G be a group such that $G = TZ(G)$, where $T \leq G$ is minimal non-abelian. Then $G \in B_2$.*

Proof. Assume that $G = TZ(G)$. Then $Z(T) = T \cap Z(G)$. Let $H \leq G$, H non abelian. Thus $HZ(G) = HZ(G) \cap G = Z(G)(T \cap HZ(G))$. Suppose that $T \cap HZ(G) < T$. Since T is minimal non abelian, $T \cap HZ(G)$ is abelian, so $Z(G)(T \cap HZ(G))$ is also abelian. Hence $HZ(G)$ is abelian, which gives the contradiction H abelian. Thus $T \cap HZ(G) = T$, so that $T \subseteq HZ(G)$ and $TZ(G) \subseteq HZ(G) \subseteq G$. Then $HZ(G) = G$ and so $Z(H) = H \cap Z(G)$. Therefore $\frac{G}{Z(G)} = \frac{HZ(G)}{Z(G)} \simeq \frac{H}{H \cap Z(G)} = \frac{H}{Z(H)}$, as required. □

Proposition 2.2. *Let G be a group and suppose that either $\frac{G}{Z(G)}$ is elementary abelian of order p^2 (p a prime) or $\frac{G}{Z(G)} \simeq \mathbb{Z} \times \mathbb{Z}$. Then $G \in B_2$.*

Proof. First suppose that $\frac{G}{Z(G)}$ is elementary abelian, with $\left|\frac{G}{Z(G)}\right| = p^2$, p a prime. Let H be a non-abelian subgroup of G. Then $\frac{HZ(G)}{Z(G)} \leq \frac{G}{Z(G)}$, where $\frac{HZ(G)}{Z(G)} \simeq \frac{H}{H \cap Z(G)}$. If $\frac{HZ(G)}{Z(G)} < \frac{G}{Z(G)}$ then from $\frac{H}{H\cap Z(G)}$ cyclic it follows that H is abelian, a contradiction. Then we have $\frac{HZ(G)}{Z(G)} = \frac{G}{Z(G)}$, so $G = HZ(G)$; in particular $Z(H) \leq H \cap Z(G)$, and $\frac{H}{Z(H)} = \frac{H}{H\cap Z(G)} \simeq \frac{HZ(G)}{Z(G)} = \frac{G}{Z(G)}$, as required.

Now suppose that $\frac{G}{Z(G)} \simeq \mathbb{Z} \times \mathbb{Z}$. Then G is nilpotent of class 2 and $G = Z(G) < x, y >$ for some $x, y \in G \setminus Z(G)$. Obviously $G' =< x, y >'=< [x, y] >$. If $o([x, y]) = n$, then $[x^n, y] = [x, y]^n = 1$ and $x^n \in Z(G)$, a contradiction. Therefore G' is an infinite cyclic group. If H is a non-abelian subgroup of G, then $\frac{H}{Z(H)} \simeq \frac{\frac{H}{Z(G)\cap H}}{\frac{Z(H)}{Z(G)\cap H}}$ cannot be cyclic, therefore it is 2-generated being a quotient of $\frac{H}{Z(G)\cap H} \simeq \frac{Z(G)H}{Z(G)} \leq \frac{G}{Z(G)}$. Moreover it is torsion-free, in fact if $h^n \in Z(H)$ for some $h \in H, n > 0$, then $[h, k]^n = [h^n, k] = 1$ for every $k \in H$, then $h \in Z(H)$ since G' is torsion-free. Hence $\frac{H}{Z(H)} \simeq \mathbb{Z} \times \mathbb{Z} \simeq \frac{G}{Z(G)}$, as required. □

We continue by assembling some elementary facts about the class B_2.

Lemma 2.3. i) *The class B_2 is subgroup closed.*

ii) *If $G \in B_2$, then $\frac{G}{Z(G)}$ is 2-generated.*

iii) *If G is a nilpotent group and $G \in B_2$, then $\frac{G}{Z(G)}$ is abelian.*

iv) *If G is a non-nilpotent group in B_2, then every locally nilpotent subgroup of G is abelian.*

v) *If G is soluble non-nilpotent group in B_2, then G is metabelian.*

vi) *If G is a non soluble group in B_2, then every soluble subgroup of G is abelian.*

vii) *If G is non soluble group in B_2, then every normal soluble subgroup of G is contained in $Z(G)$.*

Proof. The first statement is obvious. In order to prove *ii)* consider $a, b \in G$, with $[a, b] \neq 1$, then $\frac{G}{Z(G)} \simeq \frac{<a,b>}{Z(<a,b>)}$ as required. Now assume G nilpotent non-abelian in B_2 and let $x \in Z_2(G) \setminus Z(G)$. Then $[x, g] \neq 1$ for some $g \in G$ and we have $< x, g >$ nilpotent of class 2 since $[x, g] \in Z(G)$; thus $\frac{G}{Z(G)} \simeq \frac{<x,g>}{Z(<x,g>)}$ is abelian and *iii)* holds. In order to prove *iv)*, assume G soluble non-nilpotent in B_2 and consider a locally nilpotent subgroup F of G. Let $a, b \in F$, with $[a, b] \neq 1$. Then $\frac{G}{Z(G)} \simeq \frac{<a,b>}{Z(<a,b>)}$, thus $\frac{G}{Z(G)}$ is nilpotent and G is nilpotent, a contradiction. Therefore F is abelian. In order to prove *v)*, suppose that G is a soluble non-nilpotent group in B_2. Write $F = FittG$, the Fitting subgroup of G. Then F is abelian by *iv)*. Moreover $C_G(F) \subseteq F$, (see for instance 5.4.4(ii) in [8]). Let $x \in G \setminus F$ and write $H = F < x >$. Then H is not abelian and $H' \leq F$ is abelian. Therefore $\frac{G}{Z(G)} \simeq \frac{H}{Z(H)}$ is metabelian. In addition $\frac{G'}{G'\cap Z(G)} \simeq \frac{G'Z(G)}{Z(G)} = (\frac{G}{Z(G)})' \simeq (\frac{H}{Z(H)})'$ is abelian, hence G' is nilpotent and $G' \leq F$. But F is abelian, thus G' is abelian and G is metabelian. Therefore *v)* holds. If G is non soluble in B_2 and S is a soluble subgroup of G, then S is abelian, otherwise $\frac{G}{Z(G)} \simeq \frac{S}{Z(S)}$ is soluble and so is G. Therefore *vi)* holds. Finally if $G \in B_2$ is non soluble and $N \trianglelefteq G$ is soluble, then N is abelian by *vi)* and $N < g >$ is soluble, hence abelian, for every $g \in G$. Then $N \leq Z(G)$ and *vii)* holds. □

As we will see, the class B_2 is not closed under homomorphic images, but we have the following useful result.

Proposition 2.4. *Let G be a non-nilpotent group in B_2. If $S \leq Z(G)$, then $\frac{G}{S} \in B_2$.*

Proof. Let $\frac{H}{S} \leq \frac{G}{S}$. First we show that $Z(\frac{H}{S}) = \frac{Z(H)}{S}$. In fact obviously $\frac{Z(H)}{S} \leq Z(\frac{H}{S})$. Write $\frac{V}{S} = Z(\frac{H}{S})$. Then $V \leq Z_2(H)$. If $V \nleq Z(H)$, then there exists $h \in H$ such that $V \nsubseteq C_G(h)$. Then the subgroup $V < h >$ is nilpotent and non-abelian, a contradiction by Lemma 2.3 *iv*). Therefore $Z(\frac{H}{S}) = \frac{Z(H)}{S}$ for every non-abelian subgroup $\frac{H}{S}$ of $\frac{G}{S}$. In particular we have $Z(\frac{G}{S}) = \frac{Z(G)}{S}$. Hence, for every non-abelian subgroup $\frac{H}{S}$ of $\frac{G}{S}$, we have $\frac{\frac{G}{S}}{Z(\frac{G}{S})} = \frac{\frac{G}{S}}{\frac{Z(G)}{S}} \simeq \frac{G}{Z(G)} \simeq \frac{H}{Z(H)} \simeq \frac{\frac{H}{S}}{\frac{Z(H)}{S}} = \frac{\frac{H}{S}}{Z(\frac{H}{S})}$. Therefore $\frac{G}{S} \in B_2$. □

Of course our aim is to study non-abelian B_2-groups, and it is natural to look first at nilpotent B_2-groups: these admit a very easy description.

Theorem 2.5. *Let G be a non-abelian group. Then G is nilpotent and belongs to B_2, if and only if either $\frac{G}{Z(G)}$ is elementary abelian of order p^2 (p a prime) or $\frac{G}{Z(G)} \simeq \mathbb{Z} \times \mathbb{Z}$.*

Proof. Assume that either $\frac{G}{Z(G)}$ is elementary abelian of order p^2 (p a prime) or $\frac{G}{Z(G)} \simeq \mathbb{Z} \times \mathbb{Z}$. Then G is obviously nilpotent and $G \in B_2$ by Proposition 2.2.

Now assume that $G \in B_2$ is nilpotent and put $Z_i = Z_i(G)$. Then $\frac{G}{Z_1}$ is 2-generated and abelian by Lemma 2.3. There exist $a \in Z_2 \setminus Z_1$ and $b \in G$ such that $[a, b] \neq 1$.

Put $H =< a, b >$, then $H' =< [a, b] >$ and $\frac{H}{Z(H)} \simeq \frac{G}{Z(G)}$.

If $[a, b]$ is torsion-free, then $\frac{H}{Z(H)} \simeq \mathbb{Z} \times \mathbb{Z}$, as required. Assume $[a, b]$ periodic, then H' is finite. Since H is finitely generated, we have $\frac{H}{Z(H)}$ finite. Let $cZ(H) \in \frac{H}{Z(H)}$, $c \notin Z(H)$, of order p for some prime p. There exists $x \in H$ such that $[c, x] \neq 1$ but $[c, x]^p = [c^p, x] = 1$. Now it is easy to see that $\frac{G}{Z(G)}$ has order p^2, as claimed. □

Using Theorem 2.5, it is now possible to show that the class B_2 is not closed under homomorphic images.

For, let G be the free 2-generated nilpotent of class 2 group. Then $G \in B_2$. Let A be a 2-generated nilpotent p-group of class 2 and let B be a 2-generated q-group nilpotent of class 2, where p, q are distinct primes. Finally, put $H = A \times B$. There exists $N \trianglelefteq G$ such that $\frac{G}{N} \simeq H$ but $H \notin B_2$.

3. Locally finite B_2-groups

In this section we will classify all locally finite B_2-groups.

Theorem 3.1. *Let G be a finite group. Then $G \in B_2$ if and only if $G = Z(G)H$, where H is minimal non-abelian.*

Proof. Assume that $G = Z(G)H$ where H is minimal non-abelian. Then $G \in B_2$ by Proposition .

Now let $G \in B_2$. Consider $H \leq G$, with H non-abelian of minimal order. Then $\frac{G}{Z(G)} \simeq \frac{H}{Z(H)} \simeq \frac{\frac{H}{H \cap Z(G)}}{\frac{Z(H)}{H \cap Z(G)}}$, thus $\left|\frac{G}{Z(G)}\right| \leq \left|\frac{H}{H \cap Z(G)}\right| = \left|\frac{HZ(G)}{Z(G)}\right| \leq \left|\frac{G}{Z(G)}\right|$. Then $HZ(G) = G$ as claimed. □

Corollary 3.2. *Let G be a locally finite group. Then $G \in B_2$ if and only if $G = Z(G)H$, where H is a finite minimal non-abelian subgroup of G.*

Proof. Suppose that G is a locally finite B_2-group. Then there exist a, $b \in G$ such that $\frac{G}{Z(G)} =< aZ(G), bZ(G) >$ and so $G =< a, b > Z(G)$. Since G is locally finite, $< a, b >$ is finite. By Theorem 3.1

we have $< a, b >= Z(< a, b >)H$, where H is minimal non-abelian and so, since $Z(< a, b >) \leq Z(G)$, $G = Z(G) < a, b >= Z(G)H$.

Now suppose that $G = Z(G)H$, where H is finite and minimal non-abelian, then $G \in B_2$ by Proposition . □

Corollary 3.3. *Let G be a B_2-group. Then G is locally finite if and only if G is a soluble torsion group.*

Proof. Suppose that G is a soluble torsion group. Then G is locally finite (see for instance Proposition 5.4.11 in [8]).

Now suppose that G is a locally finite B_2-group. By Corollary , there exists $H \leq G$ finite and minimal non-abelian such that $G = Z(G)H$. Then H is soluble by a classical theorem of Miller and Moreno [7] and so G is soluble and torsion, as required. □

4. Soluble B_2-groups

In this section we will analyze the structure of infinite soluble B_2-groups.

Every soluble non-nilpotent B_2 group is metabelian, by Lemma 2.3 *v*).

Moreover $\frac{G}{Z(G)} \in B_2$ by Proposition . More information is collected in the following theorem.

Theorem 4.1. *Let G be a soluble non-nilpotent B_2-group. Then*

i) $Z(\frac{G}{Z(G)}) = 1$.

ii) *$G = A < x >$, where A is a normal abelian sugroup of G.*

iii) *Every non-abelian subgroup of $\frac{G}{Z(G)}$ is isomorphic to $\frac{G}{Z(G)}$.*

Proof. i) Write as usual $\frac{Z_2(G)}{Z(G)} = Z(\frac{G}{Z(G)})$. For every $g \in G$, the group $Z_2(G) < g >$ is nilpotent and so it is abelian by Lemma 2.3 *iv*). Then $Z_2(G) \subseteq C_G(g)$ for every $g \in G$ and $Z_2(G) \leq Z(G)$. Thus $Z_2(G) = Z(G)$.

ii) By Lemma 2.3 *v*), G is metabelian. Let B be a maximal normal abelian subgroup of G such that $G' \subseteq B$. If $B \leq Z(G)$ then $G' \subseteq B \subseteq Z(G)$ and so G is nilpotent of class 2, a contradiction. Therefore there exists $g \in G$ such that $B \nsubseteq C_G(g)$. Now consider $H = B < g >$. Since H is non-abelian, it follows that $\frac{H}{Z(H)} \simeq \frac{G}{Z(G)}$ and so $\frac{G}{Z(G)}$ is abelian-by-cyclic. Then there exists $\frac{A}{Z(G)} \trianglelefteq \frac{G}{Z(G)}$ such that $\frac{A}{Z(G)}$ is abelian and $\frac{G}{A}$ is cyclic. Thus A is nilpotent. By Lemma 2.3 *iv*), A is abelian and G is abelian-by-cyclic.

iii) Let $\frac{H}{Z(G)}$ be a non-abelian subgroup of $\frac{G}{Z(G)}$. Then H is non-abelian, thus $\frac{G}{Z(G)} \simeq \frac{H}{Z(H)} \simeq \frac{\frac{H}{Z(G)}}{\frac{Z(H)}{Z(G)}}$ so it suffices to prove that $Z(H) \subseteq Z(G)$. Now $G = A < x >$, where A is a normal abelian subgroup of G by *ii*). Obviously we can suppose that A is maximal for these conditions.

Firstly suppose that $\frac{G}{A}$ is finite. Consider $yA \in \frac{G}{A}$ of order p, a prime. Then $A < y >$ is non-abelian and $\frac{A<y>}{Z(A<y>)} \simeq \frac{G}{Z(G)}$ is abelian-by-prime order. Therefore there exists $\frac{B}{Z(G)} \trianglelefteq \frac{G}{Z(G)}$ such that $\frac{B}{Z(G)}$ is abelian and $|\frac{G}{B}| = p$ and so B is nilpotent and then abelian by Lemma 2.3 *iv*). So $|\frac{G}{A}| = p$, where p is a prime. Therefore $x^p \in A$. Suppose that there exists an element $h = ax^r \in Z(H)$ with $a \in A$ and r, p coprime. Then $G = A < ax^r >$ and so $H =< ax^r > (A \cap H)$ which is abelian, a contradiction.

Thus $Z(H) \subseteq A$. Since H is non-abelian, there exists an element $cx^s \in H$ where s and p are coprime, $c \in A$. It follows that $G = A < cx^s >$ and then $Z(H) \subseteq Z(G)$.

Now suppose that $G = A < x >$, with $\frac{G}{A}$ infinite. Use the bar notation to denote elements and subgroups of $G/Z(G)$. Suppose that $\overline{D} = C_{\overline{A}}(\bar{x}^r) \neq 1$, for some $r \neq 0$. Then $\overline{D} < \bar{x} >$ is non-abelian, since $Z(\overline{G}) = 1$ by $i)$, and $\bar{x}^r \in Z(\overline{D} < \bar{x} >)$. Therefore $\frac{\overline{D}<\bar{x}>}{Z(\overline{D}<\bar{x}>)} \simeq \overline{G}$ is abelian-by-finite, which is a contradiction since $\frac{\overline{G}}{\overline{A}}$ is infinite cyclic.

Obviously $\overline{H} \not\subseteq \overline{A}$ and $\overline{H} \not\subseteq < \bar{x} >$. Consider an element $h = \bar{a}\bar{x}^s \in \overline{H}$, where $s \neq 0$ and suppose that $Z(\overline{H}) \neq 1$. Thus there exists an element $\bar{b}\bar{x}^r$, with $bx^r \notin Z(G)$, which commutes with every element of $\overline{H} \cap \overline{A}$ which is non-trivial, since $\overline{H}$ is not cyclic. Therefore $C_{\overline{A}}(\bar{x}^r) \neq 1$. It follows that $r = 0$ and so $\bar{b}\bar{x}^r = \bar{b}$ commutes with $\bar{a}\bar{x}^s$. Then $\bar{x}^s$ commutes with $\bar{b}$ and so $s = 0$, the final contradiction. □

Notice that groups satisfying $iii)$ of Theorem 4.1, i.e. groups isomorphic to every non-abelian subgroup have been investigated by H. Smith and J. Wiegold in [10].

5. Insoluble B_2-groups

We start with the followingresult.

Theorem 5.1. *Let G be a group such that $\frac{G}{Z(G)}$ has a proper subgroup of finite index. If $G \in B_2$, then G is soluble.*

Proof. Suppose that G is non soluble. Then $\frac{G}{Z(G)}$ is infinite, since it is in B_2 by Proposition , and a finite group in B_2 is soluble by Corollary . Moreover it is 2-generated. We show that $\frac{N}{Z(G)} \simeq \frac{G}{Z(G)}$ for every non-trivial normal subgroup of $\frac{G}{Z(G)}$.

First notice that $M \cap Z(G) = Z(M)$ for every $M \trianglelefteq G$. In fact obviously $M \cap Z(G) \leq Z(M)$. Let $g \in G$, then $Z(M) < g >$ is soluble, therefore $Z(M) < g >$ is abelian by Lemma 2.3 $vi)$, thus $Z(M) \subseteq C_G(g)$; that holds for every $g \in G$, hence $Z(M) \leq Z(G)$.

Now suppose $\frac{N}{Z(G)} \trianglelefteq \frac{G}{Z(G)}$, $\frac{N}{Z(G)} \neq 1$. Then $N \trianglelefteq G$ and $Z(G) \leq N$, therefore $Z(G) = Z(G) \cap N = Z(N)$, by the previous remark. If N is abelian, then $N \leq Z(G)$ and $\frac{N}{Z(G)} = 1$, which is not the case. Then N is not abelian, therefore $\frac{N}{Z(G)} = \frac{N}{Z(N)} \simeq \frac{G}{Z(G)}$, as required.

Therefore $\frac{G}{Z(G)}$ is a finitely generated infinite group that is isomorphic to all its non-trivial normal subgroups and that contains a proper normal subgroup of finite index. Then, by a theorem in [3], $\frac{G}{Z(G)}$ is cyclic and G is soluble, a contradiction. □

Corollary 5.2. *Let $G \in B_2$ locally graded. Then G is soluble.*

Proof. The group $\frac{G}{Z(G)}$ is 2-generated by Lemma 2.3 $ii)$. It is also locally graded by [9]. Then it has a proper normal subgroup of finite index. By Theorem 5.1, G is soluble. □

Let $\mathcal{T}$ denote the class of groups that satisfy the *Tits alternative*, i.e., $G \in \mathcal{T}$ if and only if either G is soluble-by-finite or G contains a free subgroup of rank 2.

Theorem 5.3. *Let G be an insoluble B_2-group. Then G is not a $\mathcal{T}$-group.*

Proof. First assume that G has a free subgroup F of rank 2. Then $Z(F) = 1$. Moreover H is free for every non-abelian subgroup H of F. Then $F \simeq \frac{F}{Z(F)} \simeq \frac{H}{Z(H)} \simeq H$. This is impossible since a free group of rank 2 contains a free subgroup of infinite rank [8].

Now assume G soluble-by-finite. Then there exists $N \trianglelefteq G$, N soluble with $\frac{G}{N}$ finite. By Lemma 2.3 *vii*), $N \leq Z(G)$. Therefore $\frac{G}{Z(G)}$ is finite, so G' is finite by Schur's Lemma. Thus, by Corollary , G is soluble, a contradiction. □

Up to this point none of the special types of B_2-groups we have studied has involved a Tarski group, yet Tarski groups certainly belong to B_2. Our next result shows that every insoluble B_2 group whose derived subgroup satisfies the minimal condition has $\frac{G}{Z(G)}$ of Tarski type.

Theorem 5.4. *Let G be an insoluble B_2-group such that G' satisfies the minimal condition. Then G has the following properties:*

i) $\frac{G}{Z(G)}$ *is a simple, minimal non-abelian group.*
ii) *Soluble subgroups of G are abelian.*
iii) *If $N \triangleleft G$, then $N \leq Z(G)$ or $G' \leq N$.*

In particular, $\frac{G}{Z(G)}$ is a Tarski group.

Proof. i) If G' is soluble, then G is soluble, which is not the case. Then there exists a minimal non soluble subgroup $S \leq G'$. Then $\frac{G}{Z(G)} \simeq \frac{S}{Z(S)}$ since $G \in B_2$, thus $\frac{G}{Z(G)}$ is minimal non soluble. Let $\frac{H}{Z(G)} < \frac{G}{Z(G)}$, then $\frac{H}{Z(G)}$ is soluble. Therefore H is soluble. From Lemma 2.3 *vi*), H is abelian and hence $\frac{H}{Z(G)}$ is abelian.

Now we prove that $\frac{G}{Z(G)}$ is simple. Let $\frac{N}{Z(G)} \triangleleft \frac{G}{Z(G)}$, then N is abelian. Thus $N \leq Z(G)$, otherwise there exists $x \in G$ such that $\frac{N<x>}{Z(N<x>)} \simeq \frac{G}{Z(G)}$ so that G is soluble, a contradiction.

ii) It follows from Lemma 2.3 *vi*).

iii) If $N \triangleleft G$, from *i*) it follows that either $\frac{NZ(G)}{Z(G)} = 1$ or $\frac{NZ(G)}{Z(G)} = \frac{G}{Z(G)}$. Then either $N \leq Z(G)$ or $G' \leq N$. □

Conversely, we have:

Proposition 5.5. *Let G be an insoluble group such that every nilpotent subgroup is abelian and $\frac{G}{Z(G)}$ is simple, minimal non-abelian. Then G is a B_2-group.*

Proof. Let H be a non-abelian subgroup of G. Now consider $\frac{HZ(G)}{Z(G)} \leq \frac{G}{Z(G)}$. If $\frac{HZ(G)}{Z(G)} < \frac{G}{Z(G)}$, then $\frac{HZ(G)}{Z(G)}$ is abelian and hence H is nilpotent. Then H is abelian, which is a contradiction. Thus $\frac{HZ(G)}{Z(G)} = \frac{G}{Z(G)}$ and $\frac{G}{Z(G)} \simeq \frac{H}{H \cap Z(G)}$. Since $\frac{G}{Z(G)}$ is simple, $H \cap Z(G) = Z(H)$. Therefore $\frac{G}{Z(G)} \simeq \frac{H}{Z(H)}$, and we have the result. □

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