www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. x No. x (2017), pp. xx-xx.

www.ui.ac.ir

DETECTING THE PRIME DIVISORS OF THE CHARACTER DEGREES AND THE CLASS SIZES BY A SUBGROUP GENERATED WITH FEW ELEMENTS

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Communicated by Patrizia Longobardi

Abstract. We prove that every finite group G contains a three-generated subgroup H with the following property: a prime p divides the degree of an irreducible character of G if and only if it divides the degree of an irreducible character of H. There is no analogous result for the prime divisors of the sizes of the conjugacy classes.

1. **Introduction**

Let G be a finite group and denote by $\pi(G)$ the set of the primes dividing the order of G. In [8] the authors prove that every finite group G contains a two-generated subgroup H such that $\pi(H) = \pi(G)$. A natural question is whether similar results can be proved considering instead of $\pi(G)$ the set $\pi_{cd}(G)$ of the prime divisors of the degrees of the irreducible complex characters of G or the set $\pi_{cs}(G)$ of the prime divisors of the sizes of the conjugacy classes of G. In other words, we ask whether there exists a positive integer d that that every finite group G contains a d-generated subgroup H such that $\pi_{cd}(H) = \pi_{cd}(G)$ or respectively $\pi_{cs}(H) = \pi_{cs}(G)$. Several results in the literature goes in the direction of showing that the influence of irreducible character degrees and conjugacy class sizes on the structure of finite groups is analogous: it seems that there is a "parallel" relation between them. This is not the case with our question. Indeed it has a positive answer in the case of the character degrees, but a negative one in the case of the class sizes.

Theorem 1. *Every finite group G contains a three-generated subgroup H such that $\pi_{cd}(H) = \pi_{cd}(G)$.*

MSC(2010): Primary: 20D60; Secondary: 20C15, 20E45.

Keywords: Character degrees; class sizes.

Received: 18 July 2016, Accepted: 13 October 2016.

Theorem 2. *For every positive integer d there exists a finite group G with the property that $\pi_{cs}(H) \neq \pi_{cs}(G)$ whenever H is a d-generated subgroup of G.*

The statement of Theorem 1 cannot be improved: it is not in general true that a finite group G contains a two-generated subgroup H such that $\pi_{cd}(H) = \pi_{cd}(G)$. The nonabelian group P of order 27 and exponent 3 admits an automorphism α of order 2, acting on $P/\operatorname{Frat}(P)$ as the inverting automorphism. Let $G = P \rtimes \langle \alpha \rangle$. The character degrees of G are (1, 1, 2, 2, 2, 2, 3, 3, 3, 3) and $\pi_{cd}(G) = \{2, 3\}$. It is easy to prove that G is three-generated but not two-generated. Consider now a proper subgroup H of G. If $H \leq P$, then $\pi_{cd}(H) \subseteq \{3\}$. Otherwise $|H \cap P| \leq 9$, hence $H \cap P$ is a normal and abelian Sylow 3-subgroup of H and, by the Ito's Theorem (see for example [3, 6.15]), $\pi_{cd}(H) \subseteq \{2\}$.

If x is a positive integer, we use $\pi(x)$ to denote the set of the prime divisors of x. To every set X of positive integers a graph Γ_X can be associated, called the prime vertex graph of X. The vertex set of Γ_X is the union $\cup_x \pi(x)$ where x runs through the elements of X, and there is an edge between two distinct vertices p and q if $p \cdot q$ divides x for some integer $x \in X$. In particular, if we consider the set X of the orders of the elements of a finite group G, the corresponding prime vertex graph is called the prime graph $\Gamma(G)$ of G: it has been introduced by Gruenberg and Kegel in the 1970s and studied extensively in recent years (see for examples [5], [10], [11]). For a finite group G, let $X_{cd}(G)$ be the set of the degrees of the irreducible complex characters and let $X_{sc}(G)$ be set of the sizes of the conjugacy classes. The corresponding prime vertex graphs are called, respectively, the character degree graph and the conjugacy class graph (see for example [6] for more information). We will denote these graphs by $\Gamma_{cd}(G)$ and $\Gamma_{cs}(G)$. Notice that the vertex set of $\Gamma(G)$, $\Gamma_{cd}(G)$ and $\Gamma_{cs}(G)$ is, respectively, $\pi(G)$, $\pi_{cd}(G)$ and $\pi_{cs}(G)$. As we recalled above, a finite group G contains a two-generated subgroup H with $\pi(G) = \pi(H)$. Not only its vertex set $\pi(G)$, but also the prime graph $\Gamma(G)$ itself can be recognized by a subgroup H generated by few elements: indeed every finite group G contains a three-generated subgroup H such that $\Gamma(H) = \Gamma(G)$ (see [8, Theorem C]). A natural question, arising from Theorem 1, is whether a similar result can be proved for the character degree graph.

Question 1. *Does there exist a positive integer d such that every finite group G contains a d-generated subgroup H with the property that $\Gamma_{cd}(H) = \Gamma_{cd}(G)$?*

I proposed this question to several experts in combinatorial problems connected with the behaviour of the character degrees. It seems that is it quite difficult to find the answer. One of the purpose of the present note is to draw the attention on this open problem.

2. Proof of Theorem 1

The proof of Theorem 1 combines the arguments used in [8] with the information about $\pi_{cd}(G)$ provided by the Ito-Michler Theorem (see [4] and [9]), which asserts that a prime p does not divide the degree of any irreducible character of a finite group G if and only if G has a normal abelian Sylow

p-subgroup. In other words, we have that

$$\pi_{cd}(G) = \pi(G) \setminus \{p \mid \text{the Sylow } p\text{-subgroup of } G \text{ is abelian and normal in } G\}.$$

Denote by $d(G)$ the smallest cardinality of a generating set of G. We deduce Theorem 1 as a corollary of the following result:

Theorem 3. *Let G be a finite group such that $\pi_{cd}(G) \neq \pi_{cd}(H)$ for each $H < G$. Then $d(G) \leq 3$.*

Proof. There exists a normal subgroup Y of G such that $d(G) = d(G/Y)$ but $d(G/Y^*) < d(G)$ for each $Y < Y^* \trianglelefteq G$. Information about the structure of Y can be deduced from [1, Theorem 1.4 and Theorem 2.7]: there exist a positive integer t and a monolithic primitive group L (with socle N) such that

$$G/Y \cong L_t = \{(l_1, \dots, l_t) \in L^t \mid l_1N = \dots = l_tN\}.$$

Let $\phi : G \to L_t$ be a group epimorphism with $\ker \phi = Y$ and let $X = \phi^{-1}(\mathrm{soc}(L_t))$. Since $\mathrm{soc}(L_t) = N^t$, there exists t normal subgroups $X_1, \dots, X_t$ of G such that $\phi(X) = \phi(X_1) \times \dots \times \phi(X_t)$ and $\phi(X_i) \cong N$. Moreover let $K = \phi^{-1}(\{(l, \dots, l) \mid l \in L\})$. Since $K/Y \cong L$, $G/Y \cong L_t$ and $\pi(L) = \pi(L_t)$, it must be $\pi(G) = \pi(K)$. We have two possibilities:
1) $K = G$. In this case $t = 1$ and, by the main theorem in [7],

$$d(G) = d(G/Y) = d(L) \leq \max(d(L/N), 2) \leq \max(d(G) - 1, 2),$$

hence $d(G) \leq 2$.
2) $K \neq G$. In this case $\pi_{cd}(K) \neq \pi_{cd}(G)$ and consequently, by the Ito-Micher theorem, there exists a prime p such that the Sylow p-subgroup of K is abelian and normal in K, while the Sylow p-subgroup of G is not. Let P be the Sylow p-subgroup of K. If $P \leq Y$ then p does not divide $|K/Y| = |L|$, and consequently does not divide $|G/Y| = |L_t|$ so P is also a Sylow p-subgroup of G; in this case G would have an abelian normal Sylow p-subgroup against our assumption. So PY/Y is a nontrivial abelian normal subgroup of $K/Y \cong L$ and this is possible only if $N \cong PY/Y$ is an elementary abelian p-group. In this case $N = \mathrm{soc}(L)$ has a complement, say T, in L. In particular $\{(t, \dots, t) \in L^t \mid t \in T\}$ is a complement of N^t in L_t and all the minimal normal subgroups of L_t are T-isomorphic to N. This implies that there exists a complement C/Y of X/Y in G/Y and that $X_1/Y, \dots, X_t/Y$ are C-isomorphic irreducible C-module. We must have $|C/Y| = |K/Y : PY/Y| = |T|$, hence $|C/Y|$ is not divisible by p. For $1 \leq i < j \leq t$ let $K_{ij} = X_iX_jC$ and let P_{ij} be a Sylow p-subgroup of K_{ij}. Since p does not divide $|C/Y|$, we have $P_{ij} \leq X_iX_j$. We claim that there exist $i < j$ such that P_{ij} is not a normal abelian subgroup of K_{ij}. If not, the Sylow p-subgroup P_i of X_i is normal in X_i for every i and $P_{ij} = P_iP_j$ is abelian for every $i < j$, hence $P_1, \dots, P_t$ are normal, abelian and pairwise commuting. It follows that $P = P_1 \cdots P_t$ is a normal Sylow p-subgroup of G and is abelian, against our assumption. Now we choose $i < j$ so that the Sylow p-subgroup of K_{ij} is not an abelian normal subgroup of K_{ij}. Let $r \in \pi(G)$. Assume $r \neq p$ and let R be a Sylow r-subgroup of K_{ij}. Since $|G : K_{ij}| = |N|^{t-2}$, R is also a Sylow subgroup of G. We claim that if R is normal in K_{ij}, then R is also normal in G. Indeed $R \trianglelefteq K_{ij}$ implies that R is a normal subgroup of C and

that YR/Y centralizes X_i/Y and X_j/Y. Since $X_k/Y \cong_C X_i/Y$ for every $k \in \{1,\dots,t\}$, we get that YR/Y centralizes X/Y, hence $YR \trianglelefteq CX = G$. Being R a characteristic subgroup of YR, we conclude $Y \trianglelefteq G$. Therefore if $r \neq p$, then K_{ij} contains an abelian normal Sylow r-subgroup if and only if G does. On the other hand neither G nor K_{ij} contains an abelian and normal Sylow p-subgroup. But then we deduce from the Ito-Michler Theorem that $\pi_{cd}(G) = \pi_{cd}(K_{ij})$, hence $G = K_{ij}$ and consequently $t = 2$. It is known (see [2, Proposition 6]) that if $N = \mathrm{soc}(L)$ is abelian and $F = \mathrm{End}_L(N)$, then $d(L_t) = \max(d(L/N), \theta + \lceil (t+s)/r \rceil)$, where $r = \dim_F N$, $s = \dim_F \mathrm{H}^1(L/N, N)$, $\theta = 0$ or 1 according to whether N is a trivial L/N-module or not and where $\lceil x \rceil$ denotes the smallest integer greater or equal to x. In our case, since $L/N \cong T$ and N have coprime orders, we have that $H^1(L/N, N) = 0$, hence $d(G) = d(L_2) \leq \max(d(L/N), 1 + 2) \leq \max(d(G) - 1, 3)$, which implies $d(G) \leq 3$. □

3. **Proof of Theorem 2**

Let $\Omega = \{1, \dots, m\}$ and let $\mathcal{P}_2(\Omega)$ be the set of the 2-subsets of Ω. To each $\sigma \in \mathcal{P}_2(\Omega)$ we associate a different prime p_σ. Let A_σ be a cyclic group of order p_σ and $A = \prod_\sigma A_\sigma$. For $1 \leq i \leq m$, let $C_i = \langle x_i \rangle$ be a cyclic group of order 2 and let $C = \prod_i C_i$. We define an action of C on A as follows: x_i centralizes A_σ if $i \notin \sigma$, x_i acts of A_σ as the inverting automorphism otherwise. Consider the semidirect product $G = A \rtimes C$. No Sylow subgroup of G is central, so $\pi_{cs}(G) = \pi(G) = \{2\} \cup \{p_\sigma \mid \sigma \in \mathcal{P}_2(\Omega)\}$.

Lemma 4. *Let H be a subgroup of G. If $\pi_{cs}(G) = \pi_{cs}(H)$, then $d(H) \geq \log_2(m)$.*

Proof. First we prove, by induction on t, the following claim: $(*)$ let t be a positive integer and let $D = \langle c_1, \dots, c_t \rangle$ be a t-generated subgroup of C; if $t \leq \log_2 m$, then there exists $\Omega^* \subseteq \Omega$ with $|\Omega^*| \geq m/2^t$ such that $D \leq C_G(A_\sigma)$ (and consequently $p_\sigma \notin \pi_{cs}(AD)$) for every $\sigma \in \mathcal{P}_2(\Omega^*)$. First assume $t = 1$. Let $c_1 = (y_1, \dots, y_m)$ and let $\Omega_1 = \{i \in \Omega \mid y_i = 1\}$ and $\Omega_2 = \{i \in \Omega \mid y_i = x_i\}$. If $\sigma = (i_1, i_2)$ and $a \in A_\sigma$, then $a^{c_1} = a^{y_{i_1} y_{i_2}}$: this implies that c_1 centralizes A_σ for every $\sigma \in \mathcal{P}_2(\Omega_1) \cup \mathcal{P}_2(\Omega_2)$. Clearly there exists $j \in \{1, 2\}$ with $|\Omega_j| \geq m/2$ and we can take $\Omega^* = \Omega_j$. Now assume $t > 1$ and let $E = \langle c_1, \dots, c_{t-1} \rangle$. By induction there exists $\Omega^{**} \subseteq \Omega$ such that $|\Omega^{**}| \geq m/2^{t-1}$ and E centralizes A_σ for every $p_\sigma \in \mathcal{P}_2(\Omega^{**})$. Let $c_t = (z_1, \dots, z_m)$, $\Omega_1^* = \{i \in \Omega^{**} \mid z_i = 1\}$ and $\Omega_2^* = \{i \in \Omega^{**} \mid z_i = x_i\}$. Notice that c_t centralizes A_σ for every $\sigma \in \mathcal{P}_2(\Omega_1^*) \cup \mathcal{P}_2(\Omega_2^*)$. Again, there exists $j \in \{1, 2\}$ with $|\Omega_j^*| \geq |\Omega^{**}|/2 \geq m/2^t$ and we can take $\Omega^* = \Omega_j^*$.

We can now complete the proof of our statement. Suppose $\pi_{cs}(G) = \pi_{cs}(H)$. This implies that p_σ divides $|H|$ for every $\sigma \in \mathcal{P}_2(\Omega)$, and consequently $A \leq H$. More precisely it must be $H = AD$, for some subgroup D of C. By $(*)$ we must have $d(H) \geq d(D) \geq \log_2(m)$. □

Proof of Theorem 2. Take $m = 2^d + 1$ and consider the group G described at the beginning of this section. Since $d < \log_2(m)$, we deduce from Lemma 4 that $\pi_{cs}(G) \neq \pi_{cs}(H)$ for every d-generated subgroup of G.

□

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