FINITE NON-NILPOTENT GROUPS WITH FEW NON-NORMAL NON-CYCLIC SUBGROUPS

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Communicated by Attila Maroti

Abstract. For a finite group $G$, let $\nu_{nc}(G)$ denote the number of conjugacy classes of non-normal non-cyclic subgroups of $G$. We characterize the finite non-nilpotent groups whose all non-normal non-cyclic subgroups are conjugate.

1. Introduction

Let $G$ be a finite group. We denote by $\nu(G)$, $\nu_c(G)$ and $\nu_{nc}(G)$ the number of conjugacy classes of non-normal subgroups, non-normal cyclic subgroups and non-normal non-cyclic subgroups of $G$, respectively. Obviously, $\nu(G) = \nu_c(G) + \nu_{nc}(G)$. Many works have been done on $\nu(G)$; e.g. Dedekind [3] in (1978) classified all the finite groups with $\nu(G) = 0$. Groups with $\nu(G) = 1, 2, 3$ and $4$ are classified in [1, 6, 9, 10, 11] and finite $p$-groups with $\nu(G) = p, p + 1$ are classified in [2, 4]. In [12, 13], the authors classified the finite groups with $\nu_c(G) = 1, 2$. In [12], the authors prove that $\nu_c(G) = 1$ if and only if $\nu(G) = 1$, hence, if $\nu_{nc}(G) \neq 0$ then $\nu(G) \geq 3$. But the only research on $\nu_{nc}(G)$ is a relationship between $\nu_{nc}(G)$ and solvability: J. Lu and W. Meng [8] proved that if $\nu_{nc}(G) \leq |\pi(G)|$ then $G$ is solvable.

The purpose of this paper is to consider the influence of the value of $\nu_{nc}(G)$ on the structure of $G$ and classify the finite non-nilpotent groups $G$ for which $\nu_{nc}(G) = 1$ holds.

Our notation is standard and can be found in e.g. [5]. Throughout this paper, $\Phi(G), \pi(G)$ and $Z(G)$ are the Frattini subgroup of $G$, the set of primes dividing the $|G|$ and the center of $G$, respectively; also $Q_8, A_4$ and

$$M_{p^{n+1}} = \langle x, y \mid x^{p^n}, y^p, x^y = x^{1+p^{n-1}} \rangle,$$

are the quaternion group of order 8, the alternating group of degree 4 and modular $p$-group of order $p^{n+1}$, respectively.

Keywords: Non-normal subgroups; Conjugacy classes of non-normal subgroups; Non-nilpotent groups.
Received: 02 November 2016, Accepted: 12 January 2017.
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2. The Main Results

Let $G$ be a finite non-nilpotent group. Thus there is some $P \in \text{Syl}_p(G)$ such that $P \not\cong G$. Hence $N_G(P) < G$, and so $N_G(P) \leq M$, where $M$ is a maximal subgroup of $G$. Clearly, $M \not\cong G$.

We consider the above notations fixed in the following Theorems.

Since $\nu_{nc}(G) = 1 \leq |\pi(G)|$, by [8, Theorem 1.2], any group $G$ with $\nu_{nc}(G) = 1$ is solvable. Let us present the following alternative proof which is simpler.

**Proposition 2.1.** Any finite non-nilpotent group $G$ with $\nu_{nc}(G) = 1$ is solvable.

**Proof.** Assume, for the sake of a contradiction, that $G$ is not solvable. Then the Sylow 2-subgroups of $G$ are non-normal and non-cyclic, and so they are maximal in $G$ because $\nu_{nc}(G) = 1$. It follows that the Sylow subgroups of $G$ of odd order are non-normal in $G$. Thus all of their normalizers are cyclic.

By Burnside’s theorem, $G$ has a normal $q$-complement for each odd prime $q \in \pi(G)$. The intersection of all of these normal $q$-complements is a Sylow 2-subgroup of $G$, a contradiction. □

**Theorem 2.2.** Let $G$ be a finite non-nilpotent group with $\nu_{nc}(G) = 1$. If $M$ is abelian, then $G$ is isomorphic to one of the following groups:

1. $(\mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_p$, where $p = q^2 + q + 1$, is prime;
2. $(\mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_{2^\ell+1}$, where $2^\ell = (q + 1)|Z(G)|$ and $q$ is prime;
3. $\mathbb{Z}_p \times (\mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_p$, where $p \mid q + 1$ and $p \nmid q - 1$;
4. $\mathbb{Z}_p \rtimes \mathbb{Z}_q \times \mathbb{Z}_p$, and $\mathbb{Z}(G) \cong \mathbb{Z}_p \rtimes \mathbb{Z}_p$, where $p, q$ are distinct primes with $p \mid q - 1$;
5. $\mathbb{Z}_p \times (\mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_p$, where $p, q$ are distinct primes with $p \mid q - 1$;
6. $(\mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_\ell$, where $\ell = -1 (\mod 3)$ is a prime and $\ell = q^2 + q + 1$, is not a prime.

**Proof.** Since $M$ is abelian, $C_G(P) = M$, and so $C_G(P) = N_G(P)$. By Burnside’s Theorem, $G$ has a normal $p$-complement; let $N$ be one of them. We consider two cases:

**Case 1.** $P = M$.

Thus $N$ is an elementary abelian $q$-group, for some prime $q \neq p$. Also we have $L := C_P(N) = Z(G)$ and $G' = N$. Clearly, every proper subgroup of $N$ is non-normal in $G$. So if $|N| \geq q^4$, then there are two non-conjugate proper subgroups of $N$, which are non-normal and non-cyclic in $G$; a contradiction with $\nu_{nc}(G) = 1$. Therefore, $|N| \leq q^3$ and so we have three cases as follows.

(i) $N \cong \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q$: We see that $P$ must be cyclic. Since $\nu_{nc}(G) = 1$, $N$ has no proper subgroup which is invariant under a subgroup of $P$, so $C_P(N) = 1$ and $Z(G) = 1$. Thus since $N$ has $q^2 + q + 1$ subgroups of order $q^2$, all of them must be conjugate, and so $|P| = q^2 + q + 1$. Now from [7, Theorem A] we have $|P| = p$. It follows that $G \cong (\mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_p$, where $q^2 + q + 1 = p$ is prime.

(ii) $N \cong \mathbb{Z}_q \times \mathbb{Z}_q$: We set $L := C_P(N)$. First suppose that $P$ is cyclic, then $L = Z(G)$. Let $S$ be a non-normal non-cyclic subgroup of $G$. Thus, $S = H \rtimes K$ for some $K < P$ and $H < Q$ with $|H| = q$. Since $S$ is non-cyclic, the action of $K$ on $H$ is non-trivial, and so $|K/L| \mid q - 1$. Therefore, $q \neq 2$. We conclude from $\nu_{nc}(G) = 1$ that $P/K$ acts transitively on the set of proper subgroups of $N$ by conjugation, hence $|P/K| = q + 1$, and finally that $p \mid (q - 1, q + 1)$. Thus, $p = 2$ and $|P| = (q + 1)|K|$. Now we can choose $K$ such that $|K/L| = 2$ and since $HK$ is non-abelian, $HK/L \cong D_{2q}$ (the dihedral group of order $2q$). Thus, $|P| = 2(q + 1)|L|$ and $G \cong (\mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_{2^\ell+1}$ where $2^\ell = (q + 1)|Z(G)|$. 


Now suppose that $P$ is non-cyclic. If $L$ contains a subgroup $T$ of type $(p, p)$, then for any non-trivial proper subgroup $H$ of $N$, HT is non-normal and non-cyclic; which contradicts $\nu_{nc}(G) = 1$. Therefore, $L$ is cyclic. Assume that $K \leq P$ is of type $(p, p)$. Since $K \not\trianglelefteq G$ then $P = K$ and $|L| = p$. Now since $P/L$ acts on the set of proper subgroups of $N$ by conjugation, so $p \mid q + 1$. If $p \mid q - 1$, then $p = 2$ and $G$ contains a normal subgroup of order $q$, a contradiction. Therefore, $G \cong \mathbb{Z}_p \times (\mathbb{Z}_q \times \mathbb{Z}_q) \times \mathbb{Z}_p$, where $p \mid q + 1$ and $p \mid q - 1$.

(iii) $N \cong \mathbb{Z}_q$: Then clearly $P$ is non-cyclic and $P/L$ is cyclic. Since $P \not\trianglelefteq G$ there is at most one maximal subgroup which is normal in $G$. If all maximal subgroups of $P$ are non-normal in $G$, then they have to be cyclic. Therefore, $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$, which is impossible since $(1 \neq L \leq G)$. Thus, there exists a maximal subgroup, say $K$, of $P$, which is normal in $G$. Hence, $[N, K] = 1$ and so $K = L = Z(G)$. Let $a \in P \setminus L$. Since $P$ is non-cyclic, there exists an element $b \in P \setminus \langle a \rangle$ of order $p$. It follows that $\langle a, b \rangle \not\trianglelefteq G$, and so $P = \langle a, b \rangle$. As $L$ is maximal in $P$ we may assume that $b \in L$. Thus, $L = \langle a^p, b \rangle$ and $G \cong \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_{p^c}$, where $p^c = |a|$, $p \mid q - 1$ and $Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_{p^c}$.

**Case 2.** $P < M$.

If $P$ is non-cyclic then $M$ must be non-cyclic, which contradicts $\nu_{nc}(G) = 1$. So, $P$ is cyclic. Since $G$ is solvable, $[G : M]$ is a power of $q$ for some $(p \neq q)q \in \pi(G)$. Let $Q \in Syl_q(G)$. We show that $Q \leq G$. Assume, on the contrary, that $Q \not\trianglelefteq G$. If $M$ is non-cyclic, then $\mathcal{N}_G(Q)$ is cyclic, and so $G$ is $q$-nilpotent, a contradiction. Therefore, $M$ is cyclic. If $Q$ is cyclic, then $G'$ is cyclic and $Q \leq G'$. Thus, $Q \leq G$; a contradiction. Hence, $Q$ is non-cyclic, and so $Q = \mathcal{N}_G(Q) \leq N$, where $N$ is normal complement of $P$, which is impossible. Therefore, $Q \not\leq G$.

First suppose that $M$ is non-cyclic. Then for some prime number $(p \neq r)$, $M$ contains a non-cyclic Sylow $r$-subgroup $R$. Since $\nu_{nc}(G) = 1$ then $R \leq G$. Assume that $K$ is an abelian subgroup of type $(r, r)$ of $R$. Hence, $PK$ is a non-normal and non-cyclic subgroup of $G$, and so $M = PK$ (i.e. $R = K \cong \mathbb{Z}_r \times \mathbb{Z}_r$). Also we get $R \leq Z(G)$, thus $G' = [Q, M] = [Q, P]$. If $|[Q, P]| \geq q^2$, then assume that $H < [Q, P]$ is of order $q$. Therefore, $HR$ is a non-normal non-cyclic subgroup of $G$, a contradiction with $\nu_{nc}(G) = 1$. Hence, $|[Q, P]| = q$, and so $G \cong R \times [Q, P]$. On the other hand $Rb(P)$ is normal in $G$. Thus, $\Phi(P) \leq Z(G)$ as $\Phi(P) \not\leq G$. If $r = q$ then by the assumptions $P = \langle x \rangle$, $[Q, P] = \langle y \rangle$ and $R = \langle z, t \rangle$, we have $\langle yt \rangle \not\trianglelefteq G$ and $\langle z, yt \rangle$ is non-normal non-cyclic in $G$; a contradiction. So, $r \neq q$ and we have $G \cong \mathbb{Z}_r \times \mathbb{Z}_r \times \mathbb{Z}_q \times \mathbb{Z}_{p^c}$, where $|P| = p^c$ and $p \mid q - 1$.

Now assume that $M$ is cyclic. We claim that $Q \cap M = 1$ and so $Q$ is an elementary abelian $q$-group. Assume, contrary to our claim, that $Q \cap M \neq 1$. If $Q$ is cyclic, then $|Q/(Q \cap M)| = q$, by $[Q, M] < Q$ we have $[Q, M] \leq Q \cap M < M$ and $M \leq G$; a contradiction. Thus, $Q$ is non-cyclic.

If $Q \cap M$ is a maximal subgroup of $Q$, then $q \neq 2$ and $Q$ contains a maximal cyclic subgroup $H$ of order $q$. Since $G = \Omega_1(Q)M$, by Maschke’s Theorem, $G' = H$. Now let $S$ be a non-normal non-cyclic subgroup of $G$. Since every non-cyclic subgroups of $Q$ is normal in $G$, so $S \not\leq Q$ is non-abelian, then $G' = S'$, and $S \leq G$; a contradiction. Hence $Q \cap M$ is not maximal in $Q$. If $Q$ has a generalized quaternion structure, then $Q \cap M = \Phi(Q)$ and all the maximal subgroups of $Q$ are non-normal in $G$. Therefore $Q \cong Q_8$ (otherwise the maximal cyclic subgroup of $Q$ must be normal in $G$) and hence $|M : \mathcal{C}_M(Q)| = 3$. Assume that $S = HK \not\trianglelefteq G$ such that $H$ is maximal in $Q$ and $K \leq M$; thus $K \leq \mathcal{C}_M(Q)$ since $H$ is $K$-invariant. Hence $S$ is cyclic and $\nu_{nc}(G) = 0$; a contradiction. So $Q$ has an abelian subgroup $H$ of type $(q, q)$ such that $|H \cap M| = q$ (because the only subgroup $Q \cap M$ of order $q$ is central); hence $H \not\trianglelefteq G$. Since all the maximal subgroups of $Q$ which contain $Q \cap M$ are non-normal
in $G$ and $Q$ dose not contain any cyclic maximal subgroup, then $H$ is a maximal subgroup of $Q$ and contains $Q \cap M$. Thus $Q$ is of order $q^3$ and exponent $q$. Hence $Q \cap M = C_Q(P) \leq Z(G)$ is a subgroup of order $q$.

If $Q$ is non-abelian, then $q$ is odd also $Q_q^*(C_M(Q)) = 1$ (otherwise $H \nsubseteq HO_q^*(C_M(Q)) \nsubseteq G$). Hence $Q \cap M = C_M(Q)$ and $|M/(Q \cap M)| = q + 1$. Therefore, $M$ has a subgroup $K$ of order 2 under which a maximal subgroup, say $H_1$, of $H$ is an invariant. Since $H_1K \leq G$ we have the contradiction $H_1 \leq G$. Therefore $Q$ is abelian.

Since $[Q, M]$ is the only maximal subgroup of $Q$ that is normal in $G$ (otherwise $[Q, M]$ contains a normal subgroup in $G$ of order $q$), so $Q$ has $q^2 + q$ maximal normal subgroups in $G$ such that only $q + 1$ of them contains $C_M(Q)$. Hence we have the contradiction $\nu_{nc}(G) \geq 2$. Therefore $Q \cap M = C_M(Q) = 1$ and $G' = [Q, M] = Q$ is a minimal normal subgroup of $G$.

Assume that $HK$ is a non-normal non-cyclic subgroup of $G$, where $H < Q$ and $(1 \neq K)$ is the largest subgroup of $M$ in which $H$ is $K$-invariant. Let $L$ be a complement of $H$ in $Q$. Since $L$ is $K$-invariant, $LK$ and $HK$ are conjugate in $G$, hence $|Q| = q^2$. If $HK < N_G(HK)$, then $N_G(HK) \leq G$, and so the Sylow $q$-subgroup of $N_G(HK)$ must be normal in $G$. It follows that $Q \leq N_G(HK)$ and $N_G(HK) = QK$. Since $N_Q(K) = K$, by Frattini argument we get the contradiction $QK = (HK)N_Q(K) = HK$. Thus $HK = N_G(HK)$. We deduce that $HK$ and $LK$ are non-normal non-conjugate subgroups of $QK$ and both of them have $q$ conjugates in $QK$. Therefore $q$ is odd and $QK$ contains $q(q + 1)$ subgroups isomorphic to $HK$ all of which are conjugate in $G$. Since $HK$ has $q|M : K|$ conjugate in $G$, then $|M : K| = q + 1$ and $Q$ dose not contain any $M/K$-invariant maximal subgroup. On the other hand $q + 1$ is even, hence $M/K$ contains a subgroup $T/K$ of order 2 which is invariant in at least one maximal subgroup of $Q$; so $Q$ has a $T$-invariant maximal subgroup. This is contrary to the choice of $K$. It follows that $K = 1$, $|H| = q^2$ and $|Q| = q^3$. Also $C_M(Q) = 1$. Therefore $|M| = q^2 + q + 1$ is not a prime number.

If $q \equiv 1 \pmod{3}$, then $3 | |M|$. Assume that $y \in M$ is of order 3 and $x \in Q$ is of order $q$. Since $y$ acts fixed-point free on $Q$, $xy^2x^{-2} = 1$. Therefore $H = \langle x, x^y, x^{y^2} \rangle = \langle x, x^y \rangle$ is a $(y)$-invariant subgroup of $G$. Since $H(y) \nsubseteq G$, this is contrary to $\nu_{nc}(G) = 1$. So $q \equiv -1 \pmod{3}$.

We remark that the smallest order of groups (6) presented in previous Theorem, is 177023. This order occurs for $q = 11$ and $\ell = 11^2 + 11 + 1 = 133$ with the following presentation,

$$G = \langle a, b, c, x \mid R, C, W \rangle,$$

where:

$$R := \{a^{11}, b^{11}, c^{11}, x^{133}\}, \quad C := \{[a, b], [a, c], [b, c]\} \& \quad W := \{a^x = a^2bc^7, b^x = a^7b^3c^{-1}, c^x = a^7c^{-1}\}. $$

**Theorem 2.3.** Let $G$ be a finite non-nilpotent group with $\nu_{nc}(G) = 1$. If $M$ is non-abelian, then $G$ is isomorphic to one of the following groups:

1. $\mathbb{Z}_{q^2} \ltimes \mathbb{Z}_{p^n}$, where $p, q$ are distinct primes with $p \mid q - 1$ and $Z(G) = \Phi(P)$.
2. $\mathbb{Z}_q \rtimes \mathbb{Z}_8$, where $q$ is prime, with following presentation

$$G = \langle x, y, z \mid x^q = y^4 = z^4 = [x, y] = 1, y^2 = y^{-1}, y^2 = z^2, x^2 = x^{-1} \rangle.$$  

3. $\mathbb{Z}_q \times (\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_p)$, with following presentation

$$\langle x, y, z \mid x^q = y^{p^n} = z^p = [x, z] = 1, y^z = y^{1+p^{n+1}}, x^y = x^{i} \rangle.$$
where $p, q$ are distinct primes, $i \equiv 1 \pmod{q}$, $n \geq 2$ if $p$ is odd and $n \geq 3$ if $p = 2$.

(4) $Q_8 \times \mathbb{Z}_q \times \mathbb{Z}_p$ and $Z(G) \cong \Phi(P)\Phi(Q_8)$, where $p, q$ are distinct primes.

Proof. Suppose that $|G : M|$ is a power of $q$ for some $(p \neq q) \in \pi(G)$. Let $Q \in \text{Sy}_q(G)$. If $Q \not\in G$, then $N_G(Q)$ must be cyclic, and so $G$ is $q$-nilpotent; a contradiction. Thus $Q \leq G$. We conclude from $G = QM$ that $Q \cap M \leq G$, hence $|Q/(Q \cap M)| = q^2$. If $Q$ is non-cyclic and $|Q/(Q \cap M)| = q^2$, then $Q$ is isomorphic either to $Q_8 \times Q_8$ with $Q \cap M = 1$ or $Q_8$ with central subgroup $Q \cap M$ of order 2. We consider two cases as follows.

Case 1. $P = N_G(P)$.

First suppose that $Q \cap M \neq 1$. Since $Q \cap M$ is the only $p$-invariant subgroup of $Q$, then $\Phi(Q) \neq 1$ and so $M = P \cdot \Phi(Q)$. If $|Q/\Phi(Q)| = q^2$, then $Q \cong Q_8$. Hence, $\Phi(Q)$ is central, and since $P$ is cyclic, $M$ is abelian; a contradiction. Thus $Q$ must be cyclic. It follows that $|Q| = q^2$, because any subgroup of $Q$ is $p$-invariant. If $\Phi(P) \not\in M$, then $P = N_G(\Phi(P))$. Since $\Phi(Q)\Phi(P)$ is non-abelian, it is a normal subgroup of $G$. So, $G = P\Phi(Q)$; a contradiction. Hence, $\Phi(P) \leq M$ and $[\Phi(Q), \Phi(P)] = 1$. We deduce that $[Q, \Phi(P)] = 1$ because $Q$ is cyclic. Thus, $Z(G) = \Phi(P)$ and $G \cong \mathbb{Z}_{q^2} \rtimes \mathbb{Z}_{p^n}$ where $p | q - 1$ and $n \in \mathbb{N}$.

Now assume that $Q \cap M = 1$. Then $Q$ is an elementary abelian $q$-group of order at most $q^2$. If $P < M$, then $PQ$ is a non-normal non-cyclic subgroup of $G$, which can not be conjugate to $M$; a contradiction. Thus, $P = N_G(P) = M$.

We show that $|Q| = q$. Assume on the contrary that $|Q| = q^2$. Hence, for each proper subgroup $K$ of $P$, $KQ$ is normal in $G$, because $KQ$ is non-cyclic, and $K = KQ \cap P \leq P$. It follows that $P$ is a Dedekind group. Since $P$ is non-abelian, $P$ is a 2-group. Let $x$ be an element of $P$ of order 2. Thus there exists a proper subgroup, say $H$, of $Q$, which is invariant under $\langle x \rangle$. Therefore, $H(x)$ is cyclic, for it is non-normal in $G$ (i.e. $\langle x \rangle$ act trivially on $H$). Then by Maschke’s Theorem, $\langle x \rangle$ act trivially on $Q$. So, $Z(P) \leq G$. Now let $y$ be an element of $P$ of order 4. Thus, $\langle y \rangle$ act trivially on $Q$. We conclude similarly that $[Q, y] = 1$. Hence, $[Q, P] = 1$; a contradiction. So, $|Q| = q$.

If $|\Omega_1(P)| = p$, then $P \cong Q_8$. So, $G \cong \mathbb{Z}_q \rtimes Q_8$ which has the presentation (2) of the theorem. Assume that $|\Omega_1(P)| > p$. Since $P \not\in G$, it has at least one non-normal maximal subgroup which must be cyclic. Clearly, $P \not\cong Q_8$. Hence $P$ has exactly one non-cyclic subgroup of index $p$. It follows that $P \cong M_{p^{n+1}}$. Note that $|\Omega_1(P)| = p^2$, therefore it is central, as $\Omega_1(P) \leq G$. On the other hand the maximal subgroup of $P$ consisting of $\Omega_1(P)$ is also central. So, $G \cong \mathbb{Z}_q \rtimes (\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_p)$ has the presentation (3) of Theorem.

Case 2. $P < N_G(P)$.

Since $PQ$ is non-cyclic, it must be normal in $G$. So, $G = QN_G(P)$ and $M = N_G(P)(Q \cap M)$. We show that $M = N_G(P)$. Assume on the contrary that $N_G(P) < M$. Hence, $Q \cap M \neq 1$ and $M = P(Q \cap M)$, for $P(Q \cap M)$ is a non-normal non-cyclic subgroup of $G$. We conclude from $P < N_G(P)$ that $|Q \cap M| \geq q^2$, therefore that $\mathbb{C}(P) \neq 1$, and finally that $Q$ is non-cyclic. Let $H$ be a subgroup of type $(q,q)$ of $Q$. Thus, $H \leq G$, and so $M = PH$. Since $[P, H]$ is a proper subgroup of $M$, which is non-normal in $G$, it is cyclic. Therefore, $[P, H] = 1$; a contradiction. Hence, $M = N_G(P)$. Let $N$ be the complement of $P$ in $M$. If $N \not\in M$, then $P$ is not central. Take a $R \in \text{Sy}_r(M)$ such that $R \not\in \mathbb{C}(P)$, where $r \neq p$. Then $M = PR$ and since $R \not\in G$, $R$ is cyclic. So, $Q \cap M = 1$. Now let $P = \langle x \rangle$ and $R = \langle y \rangle$, for some $x, y \in G$. Clearly, $[Q, R] = 1$. Therefore, $\langle xy \rangle \neq G$. We infer that $Q\langle xy \rangle$ is a non-normal non-cyclic subgroup of $G$; a contradiction. Thus, $N \leq M$, and so $N$ is non-abelian. Therefore, $N \leq G$. If $K$ is a
maximal subgroup of \( N \), since \( PK \not\trianglelefteq G \), then \( PK \) and so \( K \) is cyclic. It follows that \( N \cong Q_8 \), because the maximal subgroups of \( N \) are all cyclic. Also we have \( N\Phi(P) \leq G \). So, \( \Phi(P) \leq G \).

We show that \( Q \cap M = 1 \). Assume on contrary that \( Q \cap M \neq 1 \); then \( q = 2 \) and \( N = Q \cap M \). Since \( Q \neq Q_8 \), then \( |Q/(Q \cap M)| = 2 \) and so \( |G : M| = 2 \); a contradiction. Therefore, \( Q \cap M = 1 \). Now let \( H \) be a subgroup of order \( q \) of \( Q \). Thus, \( HN \leq G \), since it is non-abelian. Hence, \( H = HN \cap Q \leq G \), and so \( Q = H \). Then we have \( G \cong Q_8 \times Z_q \rtimes Z_p^\ell \) and \( Z(G) = \Phi(Q_8)\Phi(P) \), where \( |P| = p^\ell \).

**Remark 2.4.** It can be easily verified that \( \nu_{nc}(G) = 1 \), for all groups presented in Theorems 2.2 and 2.3.

**References**


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