www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 6 No. 4 (2017), pp. 35-40.

www.ui.ac.ir

FINITE NON-NILPOTENT GROUPS WITH FEW NON-NORMAL NON-CYCLIC SUBGROUPS

HAMID MOUSAVI* AND ZAHRA REZAZADEH

Communicated by Attila Maroti

Abstract. For a finite group G, let $\nu_{nc}(G)$ denote the number of conjugacy classes of non-normal non-cyclic subgroups of G. We characterize the finite non-nilpotent groups whose all non-normal non-cyclic subgroups are conjugate.

1. Introduction

Let G be a finite group. We denote by $\nu(G)$, $\nu_c(G)$ and $\nu_{nc}(G)$ the number of conjugacy classes of non-normal subgroups, non-normal cyclic subgroups and non-normal non-cyclic subgroups of G, respectively. Obviously, $\nu(G) = \nu_c(G)+\nu_{nc}(G)$. Many works have been done on $\nu(G)$; e.g. Dedekind [3] in (1978) classified all the finite groups with $\nu(G) = 0$. Groups with $\nu(G) = 1, 2, 3$ and 4 are classified in [1, 6, 9, 10, 11] and finite p-groups with $\nu(G) = p, p+1$ are classified in [2, 4]. In [12, 13], the authors classified the finite groups with $\nu_c(G) = 1, 2$. In [12], the authors prove that $\nu_c(G) = 1$ if and only if $\nu(G) = 1$, hence, if $\nu_{nc}(G) \neq 0$ then $\nu(G) \geq 3$. But the only research on $\nu_{nc}(G)$ is a relationship between $\nu_{nc}(G)$ and solvability: J. Lu and W. Meng [8] proved that if $\nu_{nc}(G) \leq |\pi(G)|$ then G is solvable.

The purpose of this paper is to consider the influence of the value of $\nu_{nc}(G)$ on the structure of G and classify the finite non-nilpotent groups G for which $\nu_{nc}(G) = 1$ holds.

Our notation is standard and can be found in e.g. [5]. Throughout this paper, $\Phi(G), \pi(G)$ and $Z(G)$ are the Frattini subgroup of G, the set of primes dividing the $|G|$ and the center of G, respectively; also Q_8, A_4 and

$$M_{p^{n+1}} = \langle x, y \,|\, x^{p^n}, y^p, x^y = x^{1+p^{n-1}} \rangle,$$

are the quaternion group of order 8, the alternating group of degree 4 and modular p-group of order p^{n+1}, respectively.

MSC(2010): Primary: 20D99; Secondary: 20E45.

Keywords: Non-normal subgroups; Conjugacy classes of non-normal subgroups; Non-nilpotent groups.

Received: 02 November 2016, Accepted: 12 January 2017.

*Corresponding author.

2. The Main Results

Let G be a finite non-nilpotent group. Thus there is some $P \in \mathcal{S}y\ell_p(G)$ such that $P \ntrianglelefteq G$. Hence $\mathcal{N}_G(P) < G$, and so $\mathcal{N}_G(P) \leq M$, where M is a maximal subgroup of G. Clearly, $M \ntrianglelefteq G$.

We consider the above notations fixed in the following Theorems.

Since $\nu_{nc}(G) = 1 \leq |\pi(G)|$, by [8, Theorem 1.2], any group G with $\nu_{nc}(G) = 1$ is solvable. Let us present the following alternative proof which is simpler.

Proposition 2.1. *Any finite non-nilpotent group G with $\nu_{nc}(G) = 1$ is solvable.*

Proof. Assume, for the sake of a contradiction, that G is not solvable. Then the Sylow 2-subgroups of G are non-normal and non-cyclic, and so they are maximal in G because $\nu_{nc}(G) = 1$. It follows that the Sylow subgroups of G of odd order are non-normal in G. Thus all of their normalizers are cyclic. By Burnside's theorem, G has a normal q-complement for each odd prime $q \in \pi(G)$. The intersection of all of these normal q-complements is a Sylow 2-subgroup of G, a contradiction. □

Theorem 2.2. *Let G be a finite non-nilpotent group with $\nu_{nc}(G) = 1$. If M is abelian, then G is isomorphic to one of the following groups:*

(1) $(\mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_p$, *where $p = q^2 + q + 1$, is prime;*
(2) $(\mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_{2^{\ell+1}}$, *where $2^\ell = (q+1)|Z(G)|$ and q is prime;*
(3) $\mathbb{Z}_p \times (\mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_p$, *where $p \mid q+1$ and $p \nmid q-1$;*
(4) $\mathbb{Z}_p \times \mathbb{Z}_q \rtimes \mathbb{Z}_{p^l}$ *and $Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_{p^{l-1}}$, where p, q are distinct primes with $p \mid q-1$;*
(5) $\mathbb{Z}_r \times \mathbb{Z}_r \times \mathbb{Z}_q \rtimes \mathbb{Z}_{p^l}$, *where p, q are distinct primes with $p \mid q-1$;*
(6) $(\mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_\ell$, *where $q \equiv -1 \pmod 3$ is a prime and $\ell = q^2 + q + 1$, is not a prime.*

Proof. Since M is abelian, $\mathcal{C}_G(P) = M$, and so $\mathcal{C}_G(P) = \mathcal{N}_G(P)$. By Burnside's Theorem, G has a normal p-complement; let N be one of them. We consider two cases:

Case 1. $P = M$.

Thus N is an elementary abelian q-group, for some prime $q \neq p$. Also we have $L := \mathcal{C}_P(N) = Z(G)$ and $G' = N$. Clearly, every proper subgroup of N is non-normal in G. So if $|N| \geq q^4$, then there are two non-conjugate proper subgroups of N, which are non-normal and non-cyclic in G; a contradiction with $\nu_{nc}(G) = 1$. Therefore, $|N| \leq q^3$ and so we have three cases as follows.

(i) $N \cong \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q$: We see that P must be cyclic. Since $\nu_{nc}(G) = 1$, N has no proper subgroup which is invariant under a subgroup of P, so $\mathcal{C}_P(N) = 1$ and $Z(G) = 1$. Thus since N has $q^2 + q + 1$ subgroups of order q^2 all of them must be conjugate, and so $|P| = q^2 + q + 1$. Now from [7, Theorem A] we have $|P| = p$. It follows that $G \cong (\mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_p$, where $q^2 + q + 1 = p$ is prime.

(ii) $N \cong \mathbb{Z}_q \times \mathbb{Z}_q$: We set $L = \mathcal{C}_P(N)$. First suppose that P is cyclic, then $L = Z(G)$. Let S be a non-normal non-cyclic subgroup of G. Thus, $S = H \rtimes K$ for some $K < P$ and $H < Q$ with $|H| = q$. Since S is non-cyclic, the action of K on H is non-trivial, and so $|K/L| \mid q-1$. Therefore, $q \neq 2$. We conclude from $\nu_{nc}(G) = 1$ that P/K acts transitively on the set of proper subgroups of N by conjugation, hence $|P/K| = q+1$, and finally that $p \mid (q-1, q+1)$. Thus, $p = 2$ and $|P| = (q+1)|K|$. Now we can choose K such that $|K/L| = 2$ and since HK is non-abelian, $HK/L \cong D_{2q}$ (the dihedral group of order $2q$). Thus, $|P| = 2(q+1)|L|$ and $G \cong (\mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_{2^{\ell+1}}$ where $2^\ell = (q+1)|Z(G)|$.

Now suppose that P is non-cyclic. If L contains a subgroup T of type (p,p), then for any non-trivial proper subgroup H of N, HT is non-normal and non-cyclic; which contradicts $\nu_{nc}(G)=1$. Therefore, L is cyclic. Assume that $K \leqslant P$ is of type (p,p). Since $K \ntrianglelefteq G$ then $P=K$ and $|L|=p$. Now since P/L acts on the set of proper subgroups of N by conjugation, so $p \mid q+1$. If $p \mid q-1$, then $p=2$ and G contains a normal subgroup of order q, a contradiction. Therefore, $G \cong \mathbb{Z}_p \times (\mathbb{Z}_q \times \mathbb{Z}_q) \rtimes \mathbb{Z}_p$, where $p \mid q+1$ and $p \nmid q-1$.

(iii) $N \cong \mathbb{Z}_q$: Then clearly P is non-cyclic and P/L is cyclic. Since $P \ntrianglelefteq G$ there is at most one maximal subgroup which is normal in G. If all maximal subgroups of P are non-normal in G, then they have to be cyclic. Therefore, $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$, which is impossible since $(1 \neq)L \trianglelefteq G$. Thus, there exists a maximal subgroup, say K, of P, which is normal in G. Hence, $[N,K]=1$ and so $K=L=Z(G)$. Let $a \in P\backslash L$. Since P is non-cyclic, there exists an element $b \in P\backslash\langle a\rangle$ of order p. It follows that $\langle a,b\rangle \ntrianglelefteq G$, and so $P=\langle a,b\rangle$. As L is maximal in P we may assume that $b \in L$. Thus, $L=\langle a^p,b\rangle$ and $G \cong \mathbb{Z}_p \times \mathbb{Z}_q \rtimes \mathbb{Z}_{p^\ell}$, where $p^\ell=|a|$, $p \mid q-1$ and $Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_{p^{\ell-1}}$.

Case 2. $P<M$.

If P is non-cyclic then M must be non-cyclic, which contradicts $\nu_{nc}(G)=1$. So, P is cyclic. Since G is solvable, $|G:M|$ is a power of q for some $(p \neq)q \in \pi(G)$. Let $Q \in \mathcal{S}y\ell_q(G)$. We show that $Q \trianglelefteq G$. Assume, on the contrary, that $Q \ntrianglelefteq G$. If M is non-cyclic, then $\mathcal{N}_G(Q)$ is cyclic, and so G is q-nilpotent, a contradiction. Therefore, M is cyclic. If Q is cyclic, then G' is cyclic and $Q \leq G'$. Thus, $Q \trianglelefteq G$; a contradiction. Hence, Q is non-cyclic, and so $Q=\mathcal{N}_G(Q) \leqslant N$, where N is normal complement of P, which is impossible. Therefore, $Q \trianglelefteq G$.

First suppose that M is non-cyclic. Then for some prime number $(p \neq)r$, M contains a non-cyclic Sylow r-subgroup R. Since $\nu_{nc}(G)=1$ then $R \trianglelefteq G$. Assume that K is an abelian subgroup of type (r,r) of R. Hence, PK is a non-normal and non-cyclic subgroup of G, and so $M=PK$ (i.e. $R=K \cong \mathbb{Z}_r \times \mathbb{Z}_r$). Also we get $R \leq Z(G)$, thus $G'=[Q,M]=[Q,P]$. If $|[Q,P]| \geq q^2$, then assume that $H<[Q,P]$ is of order q. Therefore, HR is a non-normal non-cyclic subgroup of G, a contradiction with $\nu_{nc}(G)=1$. Hence, $|[Q,P]|=q$, and so $G \cong R \times [Q,P]P$. On the other hand $R\Phi(P)$ is normal in G. Thus, $\Phi(P) \leq Z(G)$ as $\Phi(P) \trianglelefteq G$. If $r=q$ then by the assumptions $P=\langle x\rangle$, $[Q,P]=\langle y\rangle$ and $R=\langle z,t\rangle$, we have $\langle yt\rangle \ntrianglelefteq G$ and $\langle z,yt\rangle$ is non-normal non-cyclic in G; a contradiction. So, $r \neq q$ and we have $G \cong \mathbb{Z}_r \times \mathbb{Z}_r \times \mathbb{Z}_q \rtimes \mathbb{Z}_{p^n}$, where $|P|=p^n$ and $p \mid q-1$.

Now assume that M is cyclic. We claim that $Q \cap M=1$ and so Q is an elementary abelian q-group. Assume, contrary to our claim, that $Q \cap M \neq 1$. If Q is cyclic, then $|Q/(Q \cap M)|=q$, by $[Q,M]<Q$ we have $[Q,M] \leq Q \cap M<M$ and $M \trianglelefteq G$; a contradiction. Thus, Q is non-cyclic.

If $Q \cap M$ is a maximal subgroup of Q, then $q \neq 2$ and Q contains a maximal cyclic subgroup H of order q. Since $G=\Omega_1(Q)M$, by Maschke's Theorem, $G'=H$. Now let S be a non-normal non-cyclic subgroup of G. Since every non-cyclic subgroups of Q is normal in G, so $S \nleqslant Q$ is non-abelian, then $G'=S'$, and $S \trianglelefteq G$; a contradiction. Hence $Q \cap M$ is not maximal in Q. If Q has a generalized quaternion structure, then $Q \cap M=\Phi(Q)$ and all the maximal subgroups of Q are non-normal in G. Therefore $Q \cong Q_8$ (otherwise the maximal cyclic subgroup of Q must be normal in G) and hence $|M:\mathcal{C}_M(Q)|=3$. Assume that $S=HK \ntrianglelefteq G$ such that H is maximal in Q and $K \leqslant M$; thus $K \leqslant \mathcal{C}_M(Q)$ since H is K-invariant. Hence S is cyclic and $\nu_{nc}(G)=0$; a contradiction. So Q has an abelian subgroup H of type (q,q) such that $|H \cap M|=q$ (because the only subgroup $Q \cap M$ of order q is central); hence $H \ntrianglelefteq G$. Since all the maximal subgroups of Q which contain $Q \cap M$ are non-normal in G and Q dose not contain any cyclic maximal subgroup, then H is a maximal subgroup of Q and contains $Q \cap M$. Thus Q is of order q^3 and exponent q. Hence $Q \cap M = C_Q(P) \leqslant Z(G)$ is a subgroup of order q.

If Q is non-abelian, then q is odd also $Q_{q'}((\mathcal{C}_M(Q)) = 1$ (otherwise $H \gneqq HO_{q'}(\mathcal{C}_M(Q)) \ntrianglelefteq G$). Hence $Q \cap M = \mathcal{C}_M(Q)$ and $|M/(Q \cap M)| = q+1$. Therefore, M has a subgroup K of order 2 under which a maximal subgroup, say H_1, of H is an invariant. Since $H_1K \trianglelefteq G$ we have the contradiction $H_1 \trianglelefteq G$. Therefore Q is abelian.

Since $[Q, M]$ is the only maximal subgroup of Q that is normal in G (otherwise $[Q, M]$ contains a normal subgroup in G of order q), so Q has q^2+q maximal normal subgroups in G such that only $q+1$ of them contains $\mathcal{C}_M(Q)$. Hence we have the contradiction $\nu_{nc}(G) \geq 2$. Therefore $Q \cap M = \mathcal{C}_M(Q) = 1$ and $G' = [Q, M] = Q$ is a minimal normal subgroup of G.

Assume that HK is a non-normal non-cyclic subgroup of G, where $H < Q$ and $(1 \neq)K$ is the largest subgroup of M in which H is K-invariant. Let L be a complement of H in Q. Since L is K-invariant, LK and HK are conjugate in G, hence $|Q| = q^2$. If $HK < \mathcal{N}_G(HK)$, then $\mathcal{N}_G(HK) \trianglelefteq G$, and so the Sylow q-subgroup of $\mathcal{N}_G(HK)$ must be normal in G. It follows that $Q \leqslant \mathcal{N}_G(HK)$ and $\mathcal{N}_G(HK) = QK$. Since $\mathcal{N}_{QK}(K) = K$, by Frattini argument we get the contradiction $QK = (HK)\mathcal{N}_{QK}(K) = HK$. Thus $HK = \mathcal{N}_G(HK)$. We deduce that HK and LK are non-normal non-conjugate subgroups of QK and both of them have q conjugates in QK. Therefore q is odd and QK contains $q(q+1)$ subgroups isomorphic to HK all of which are conjugate in G. Since HK has $q|M:K|$ conjugate in G, then $|M:K| = q+1$ and Q dose not contain any M/K-invariant maximal subgroup. On the other hand $q+1$ is even, Hence M/K contains a subgroup T/K of order 2 which is invariant in at least one maximal subgroup of Q; so Q has a T-invariant maximal subgroup. This is contrary to the choice of K. It follows that $K = 1$, $|H| = q^2$ and $|Q| = q^3$. Also $\mathcal{C}_M(Q) = 1$. Therefore $|M| = q^2+q+1$ is not a prime number.

If $q \equiv 1 \pmod 3$, then $3 \mid |M|$. Assume that $y \in M$ is of order 3 and $x \in Q$ is of order q. Since y acts fixed-point free on Q, $xx^yx^{y^2} = 1$. Therefore $H = \langle x, x^y, x^{y^2}\rangle = \langle x, x^y\rangle$ is a $\langle y\rangle$-invariant subgroup of G. Since $H\langle y\rangle \ntrianglelefteq G$, this is contrary to $\nu_{nc}(G) = 1$. So $q \equiv -1 \pmod 3$. □

We remark that the smallest order of groups (6) presented in pervious Theorem, is 177023. This order occurs for $q = 11$ and $\ell = 11^2+11+1 = 133$ with the following presentation,

$$G = \langle a, b, c, x \mid R, C, W\rangle,$$

where;

$$R := \{a^{11}, b^{11}, c^{11}, x^{133}\},\ C := \{[a, b], [a, c], [b, c]\} \ \&\ W := \{a^x = a^2bc^7, b^x = a^7b^3c^{-1}, c^x = a^7c^{-1}\}.$$

Theorem 2.3. *Let G be a finite non-nilpotent group with $\nu_{nc}(G) = 1$. If M is non-abelian, then G is isomorphic to one of the following groups:*

(1) $\mathbb{Z}_{q^2} \rtimes \mathbb{Z}_{p^n}$, *where p, q are distinct primes with $p \mid q-1$ and $Z(G) = \Phi(P)$.*

(2) $\mathbb{Z}_q \rtimes Q_8$, *where q is prime, with following presentation*

$$G = \langle x, y, z \mid x^q = y^4 = z^4 = [x, y] = 1, y^z = y^{-1}, y^2 = z^2, x^z = x^{-1}\rangle.$$

(3) $\mathbb{Z}_q \rtimes (\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_p)$, *with following presentation*

$$\langle x, y, z \mid x^q = y^{p^n} = z^p = [x, z] = 1, y^z = y^{1+p^{n+1}}, x^y = x^i\rangle,$$

where p, q are distinct primes, $i \equiv 1$ (mod q), $n \geq 2$ if p is odd and $n \geq 3$ if $p = 2$.

(4) $Q_8 \times \mathbb{Z}_q \rtimes \mathbb{Z}_{p^l}$ *and* $Z(G) \cong \Phi(P)\Phi(Q_8)$*, where p, q are distinct primes.*

Proof. Suppose that $|G : M|$ is a power of q for some $(p \neq)q \in \pi(G)$. Let $Q \in \mathcal{S}y\ell_q(G)$. If $Q \ntrianglelefteq G$, then $\mathcal{N}_G(Q)$ must be cyclic, and so G is q-nilpotent; a contradiction. Thus $Q \trianglelefteq G$. We conclude from $G = QM$ that $Q \cap M \trianglelefteq G$, hence $|Q/(Q \cap M)| \leq q^2$. If Q is non-cyclic and $|Q/(Q \cap M)| = q^2$, then Q is isomorphic either to $\mathbb{Z}_q \times \mathbb{Z}_q$ with $Q \cap M = 1$ or Q_8 with central subgroup $Q \cap M$ of order 2. We consider two cases as follows.

Case 1. $P = \mathcal{N}_G(P)$.

First suppose that $Q \cap M \neq 1$. Since $Q \cap M$ is the only p-invariant subgroup of Q, then $\Phi(Q) \neq 1$ and so $M = P \cdot \Phi(Q)$. If $|Q/\Phi(Q)| = q^2$, then $Q \cong Q_8$. Hence, $\Phi(Q)$ is central, and since P is cyclic, M is abelian; a contradiction. Thus Q must be cyclic. It follows that $|Q| = q^2$, because any subgroup of Q is p-invariant. If $\Phi(P) \ntrianglelefteq M$, then $P = \mathcal{N}_G(\Phi(P))$. Since $\Phi(Q)\Phi(P)$ is non-abelian, it is a normal subgroup of G. So, $G = P\Phi(Q)$; a contradiction. Hence, $\Phi(P) \trianglelefteq M$ and $[\Phi(Q), \Phi(P)] = 1$. We deduce that $[Q, \Phi(P)] = 1$ because Q is cyclic. Thus, $Z(G) = \Phi(P)$ and $G \cong \mathbb{Z}_{q^2} \rtimes \mathbb{Z}_{p^n}$ where $p \mid q - 1$ and $n \in \mathbb{N}$.

Now assume that $Q \cap M = 1$. Then Q is an elementary abelian q-group of order at most q^2. If $P < M$, then PQ is a non-normal non-cyclic subgroup of G, which can not be conjugate to M; a contradiction. Thus, $P = \mathcal{N}_G(P) = M$.

We show that $|Q| = q$. Assume on the contrary that $|Q| = q^2$. Hence, for each proper subgroup K of P, KQ is normal in G, because KQ is non-cyclic, and So $K = KQ \cap P \trianglelefteq P$. It follows that P is a Dedekind group. Since P is non-abelian, P is a 2-group. Let x be an element of P of order 2. Thus there exists a proper subgroup, say H, of Q, which is invariant under $\langle x \rangle$. Therefore, $H\langle x \rangle$ is cyclic, for it is non-normal in G (i.e. $\langle x \rangle$ act trivially on H). Then by Maschke's Theorem, $\langle x \rangle$ act trivially on Q. So, $Z(P) \trianglelefteq G$. Now let y be an element of P of order 4. Thus, $\langle y^2 \rangle$ act trivially on Q. We conclude similarly that $[Q, y] = 1$. Hence, $[Q, P] = 1$; a contradiction. So, $|Q| = q$.

If $|\Omega_1(P)| = p$, then $P \cong Q_8$. So, $G \cong \mathbb{Z}_q \rtimes Q_8$ which has the presentation (2) of the theorem. Assume that $|\Omega_1(P)| > p$. Since $P \ntrianglelefteq G$, it has at least one non-normal maximal subgroup which must be cyclic. Clearly, $P \ncong Q_8$. Hence P has exactly one non-cyclic subgroup of index p. It follows that $P \cong M_{p^{n+1}}$. Note that $|\Omega_1(P)| = p^2$, therefore it is central, as $\Omega_1(P) \trianglelefteq G$. On the other hand the maximal subgroup of P consisting of $\Omega_1(P)$ is also central. So, $G \cong \mathbb{Z}_q \rtimes (\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_p)$ has the presentation (3) of Theorem.

Case 2. $P < \mathcal{N}_G(P)$.

Since PQ is non-cyclic, it must be normal in G. So, $G = Q\mathcal{N}_G(P)$ and $M = \mathcal{N}_G(P)(Q \cap M)$. We show that $M = \mathcal{N}_G(P)$. Assume on the contrary that $\mathcal{N}_G(P) < M$. Hence, $Q \cap M \neq 1$ and $M = P(Q \cap M)$, for $P(Q \cap M)$ is a non-normal non-cyclic subgroup of G. We conclude from $P < \mathcal{N}_G(P)$ that $|Q \cap M| \geq q^2$, therefore that $\mathcal{C}_Q(P) \neq 1$, and finally that Q is non-cyclic. Let H be a subgroup of type (q, q) of Q. Thus, $H \trianglelefteq G$, and so $M = PH$. Since $[P, H]P$ is a proper subgroup of M, which is non-normal in G, it is cyclic. Therefore, $[P, H] = 1$; a contradiction. Hence, $M = \mathcal{N}_G(P)$. Let N be the complement of P in M. If $N \ntrianglelefteq M$, then P is not central. Take a $R \in \mathcal{S}y\ell_r(M)$ such that $R \nleq \mathcal{C}_G(P)$, where $r \neq p$. Then $M = PR$ and since $R \ntrianglelefteq G$, R is cyclic. So, $Q \cap M = 1$. Now let $P = \langle x \rangle$ and $R = \langle y \rangle$, for some $x, y \in G$. Clearly, $[Q, R] = 1$. Therefore, $\langle xy \rangle \ntrianglelefteq G$. We infer that $Q\langle xy \rangle$ is a non-normal non-cyclic subgroup of G; a contradiction. Thus, $N \trianglelefteq M$, and so N is non-abelian. Therefore, $N \trianglelefteq G$. If K is a maximal subgroup of N, since $PK \ntrianglelefteq G$, then PK and so K is cyclic. It follows that $N \cong Q_8$, because the maximal subgroups of N are all cyclic. Also we have $N\Phi(P) \trianglelefteq G$. So, $\Phi(P) \trianglelefteq G$.

We show that $Q \cap M = 1$. Assume on contrary that $Q \cap M \neq 1$; then $q = 2$ and $N = Q \cap M$. Since $Q \ncong Q_8$, then $|Q/(Q \cap M)| = 2$ and so $|G : M| = 2$; a contradiction. Therefore, $Q \cap M = 1$. Now let H be a subgroup of order q of Q. Thus, $HN \trianglelefteq G$, since it is non-abelian. Hence, $H = HN \cap Q \trianglelefteq G$, and so $Q = H$. Then we have $G \cong Q_8 \times \mathbb{Z}_q \rtimes \mathbb{Z}_{p^\ell}$ and $Z(G) = \Phi(Q_8)\Phi(P)$, where $|P| = p^\ell$. □

Remark 2.4. *It can be easily verified that* $\nu_{nc}(G) = 1$*, for all groups presented in Theorems 2.2 and 2.3.*

References

[1] R. Brandl, Groups with few non-normal subgroups, *Comm. Algebra*, **23** (1995) 2091–2098.

[2] R. Brandl, Conjugacy classes of non-normal subgroups of finite p-groups, *Israel J. of Math.*, **195** (2013) 473–479.

[3] R. Dedekind, Über Gruppen deren samtliche teiler normalteiler sind, *Math. Ann.*, **48** (1897) 548–561.

[4] G. A. Fernandez-Alcober and L. Legarreta, The finite p-groups with p conjugacy classes of non-normal subgroups, *Israel J. of Math.*, **180** (2010) 189–192.

[5] D. Gorenstein, *Fnite Groups*, Harper and Row, New York, 1980.

[6] L. Gong, H. Cao and G. Chen, Finite nilpotent groups having exactly four conjugacy classes of non-normal subgroups, *Algebra Colloq.*, **20** (2013) 579–592.

[7] C. Yin Ho, Projective planes with a regular collineation group and a question about powers of a prime, *J. Algrbra*, **154** (1993) 141–151.

[8] J. Lu and W. Meng, On solvability of finite groups with few non-normal subgroups, *Comm. Algebra*, **1** (2015) 1752–1756.

[9] H. Mousavi, On finite groups with few non-normal subgroups, *Comm. Algebra*, **27** (1999) 3143–3151.

[10] H. Mousavi, Nilpotent groups with three conjugacy classes of non-normal subgroups, *Bulletin Iranian Math. Soc.*, **40** (2014) 1291–1300.

[11] H. Mousavi, Non-nilpotent groups with three conjugacy classes of non-normal subgroups, *Int. J. Group Theory*, **3** (2014) 1–7.

[12] D. Oggionni, G. Ponzoni and V. Zambelli, Groups with few non-normal cyclic subgroups, *J. Note Math.*, **30** (2010) 121–133.

[13] Shirong Li, The number of conjugacy classes of non-normal cyclic subgroups in nilpotent groups of odd order, *J. Group Theory*, **vv** (1998) 165–171.

Hamid Mousavi

Department of Mathematical Sciences, University of Tabriz, P.O.Box 51666-16471, Tabriz, Iran

Email: `hmousavi@tabrizu.ac.ir`

Zahra Rezazadeh

Department of Mathematical Sciences, Isfahan University of Technology, P.O.Box 84156-83111, Isfahan, Iran

Email: `zahra.rezazadeh@math.iut.ac.ir`