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ON ALMOST RECOGNIZABILITY BY SPECTRUM OF SIMPLE CLASSICAL GROUPS

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Abstract. The set of element orders of a finite group G is called the *spectrum*. Groups with coinciding spectra are said to be *isospectral*. It is known that if G has a nontrivial normal soluble subgroup then there exist infinitely many pairwise non-isomorphic groups isospectral to G. The situation is quite different if G is a nonabelain simple group. Recently it was proved that if L is a simple classical group of dimension at least 62 and G is a finite group isospectral to L, then up to isomorphism $L \leq G \leq \operatorname{Aut} L$. We show that the assertion remains true if 62 is replaced by 38.

1. **Introduction**

Given a finite group G, the *spectrum* of G is the set of its element orders and is denoted by $\omega(G)$. Finite groups are said to be isospectral if their spectra coincide. It is known that if a finite group G has a nontrivial normal soluble subgroup then there exist infinitely many pairwise non-isomorphic groups isospectral to G (see [9, 11]). In contrast, there are a lot of finite nonabelian simple groups that are uniquely determined by their spectra in the class of finite groups. Such groups are said to be *recognizable* (by spectrum). Moreover, recently it was proved that every "sufficiently large" nonabelain simple group L is *almost recognizable* (by spectrum), that is there are finitely many pairwise non-isomorphic finite groups isospectral to L. Namely, the following holds (see [6, 12, 16]).

Theorem A. *Let L be one of the following nonabelian simple groups:*

(1) *a sporadic group other than J_2;*
(2) *an alternating group A_n, where $n \neq 6, 10$;*
(3) *an exceptional group of Lie type other than ${}^3D_4(2)$;*
(4) *$L_n(q)$, where $n \geq 45$ or q is even;*
(5) *$U_n(q)$, where $n \geq 45$, or q is even and $(n, q) \neq (4, 2), (5, 2)$;*

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(6) $S_{2n}(q)$, $O_{2n+1}(q)$, *where* $n \geq 29$, *or* q *is even,* $n \neq 2, 4$, *and* $(n, q) \neq (3, 2)$*;*

(7) $O^{+}_{2n}(q)$, *where* $n \geq 31$, *or* q *is even and* $(n, q) \neq (4, 2)$*;*

(8) $O^{-}_{2n}(q)$, *where* $n \geq 30$ *or* q *is even.*

Then every finite group isospectral to L *is isomorphic to some group* G *with* $L \leq G \leq \operatorname{Aut} L$*. In particular, there are only finitely many pairwise non-isomorphic finite groups isospectral to* L*.*

Remark 1.1. *In the statements of Theorem* 2 *and Theorem* 3 *in* [12]*, the condition for* $S_{2n}(q)$ *and* $O_{2n+1}(q)$ *is* $n \geq 28$*, but in fact these theorems are proved for* $n \geq 29$*.*

Besides the theorem, A. Vasil′ev and M. Grechkoseeva in [6] formulated the following.

Conjecture B. *Let* L *be one of the following groups:*

(1) $L_n(q)$, *where* $n \geq 5$*;*

(2) $U_n(q)$, *where* $n \geq 5$ *and* $(n, q) \neq (5, 2)$*;*

(3) $S_{2n}(q)$, *where* $n \geq 3$, $n \neq 4$ *and* $(n, q) \neq (3, 2)$*;*

(4) $O_{2n+1}(q)$, *where* q *is odd,* $n \geq 3$, $n \neq 4$ *and* $(n, q) \neq (3, 3)$*;*

(5) $O^{\varepsilon}_{2n}(q)$, *where* $n \geq 4$ *and* $(n, q, \varepsilon) \neq (4, 2, +), (4, 3, +)$.

Then every finite group isospectral to L *is isomorphic to some group* G *with* $L \leq G \leq \operatorname{Aut} L$*.*

In Theorem A, Conjecture B, and throughout the paper, we use single-letter names for simple classical groups, following [3]. Also we use the standard abbreviation $L^{\varepsilon}_n(q)$, where $\varepsilon \in \{+, -\}$, $L^{+}_n(q) = L_n(q)$, and $L^{-}_n(q) = U_n(q)$. We continue investigation of the recognition problem for simple classical group and weaken restrictions on classical groups in Theorem A.

Theorem 1.2. *Let* L *be one of the simple groups* $L^{\varepsilon}_n(q)$ *with* $n \geq 27$, $S_{2n}(q)$, $O_{2n+1}(q)$ *with* $n \geq 16$, $L = O^{+}_{2n}(q)$ *with* $n \geq 19$*, and* $L = O^{-}_{2n}(q)$ *with* $n \geq 18$*. Then every finite group isospectral to* L *is isomorphic to some group* G *with* $L \leq G \leq \operatorname{Aut} L$*. In particular, there are only finitely many pairwise non-isomorphic finite groups isospectral to* L*.*

In fact, as we will show in the last section, Theorem 1.2 is a straightforward consequence of a series of previous results and the following theorem whose proof is the main goal of this paper.

Theorem 1.3. *Let* q *be a power of an odd prime* p*, and let* L *be one of the groups* $L^{\varepsilon}_n(q)$ *with* $27 \leq n \leq 44$, $S_{2n}(q)$, $O_{2n+1}(q)$ *with* $17 \leq n \leq 28$, $O^{+}_{2n}(q)$ *with* $19 \leq n \leq 30$*, and* $O^{-}_{2n}(q)$ *with* $18 \leq n \leq 29$*. If* $\omega(G) = \omega(L)$ *for a finite group* G*, then* G *has a unique nonabelian composition factor* S*, and* S *is not isomorphic to a group of Lie type over a field of characteristic distinct from* p*.*

Observe that Theorems 1.2 and 1.3 resemble Theorem 2 and Theorem 3 in [12] but their hypotheses are weaker. No wonder the structure of the present paper is similar to that of [12]. Moreover, we use a series of lemmas from [12] and either give a new proof to enhance their assertions or explain how to change the original proof to obtain required conclusions under weaker assumptions.

2. Preliminaries: arithmetic of Zsigmondy primes

Given a set of nonzero integers $n_1, \dots, n_k$, we denote by $(n_1, \dots, n_k)$ and $[n_1, \dots, n_k]$ their greatest common divisor and least common multiple, respectively. Given a nonzero integer n, we put $\varphi(n)$ for the Euler totient function of n, $\pi(n)$ for the set of prime divisors of n, and if G is a finite group then, as usual, $\pi(G)$ stands for $\pi(|G|)$. If π is a set of primes, then n_π denotes the π-part of n, that is, the largest divisor k of n with

$\pi(k) \subseteq \pi$; and $n_{\pi'}$ denotes the π'-part of n, that is, the ratio $|n|/n_\pi$. If n is a nonzero integer and r is an odd prime with $(r,n)=1$, then $e(r,n)$ denotes the multiplicative order of n modulo r. Given an odd integer n, we put $e(2,n)=1$ if $n \equiv 1 \pmod 4$, and $e(2,n)=2$ otherwise.

Fix an integer a with $|a|>1$. A prime r is said to be a *primitive prime divisor* of a^i-1 if $e(r,a)=i$. We write $r_i(a)$ to denote some primitive prime divisor of a^i-1, if such a prime exists, and $R_i(a)$ to denote the set of all such divisors. Zsigmondy [24] proved that primitive prime divisors exist for almost all pairs (a,i).

Lemma 2.1. (Zsigmondy [24]) *Let a be an integer and $|a|>1$. For every natural number i the set $R_i(a)$ is nonempty, except for the pairs $(a,i) \in \{(2,1),(2,6),(-2,2),(-2,3),(3,1),(-3,2)\}$.*

Let $i \neq 2$ be a positive integer. Then the product of all primitive prime divisors of a^i-1 taken with multiplicities is denoted by $k_i(a)$. Put $k_2(a)=k_1(-a)$. The number $k_i(a)$ is said to be the *greatest primitive divisor* of a^i-1. It follows from the definition that $(k_i(a),k_j(a))=1$ if $i \neq j$. It is easy to check that $k_1(a)=|a-1|$ if $a \not\equiv 3 \pmod 4$, and $k_1(a)=|a-1|/2$ if $a \equiv 3 \pmod 4$, as well as $k_2(a)=|a+1|$ if $a \not\equiv 1 \pmod 4$, and $k_2(a)=|a+1|/2$ if $a \equiv 1 \pmod 4$. It follows from [10] that for $i>2$,

$$k_i(a) = \frac{|\Phi_i(a)|}{(r, \Phi_{i_{\{r\}'}}(a))}, \tag{2.1}$$

where $\Phi_i(x)$ is the ith cyclotomic polynomial and r is the largest prime dividing i; moreover, if $i_{\{r\}'}$ does not divide $r-1$ then $(r,\Phi_{i_{\{r\}'}}(a))=1$.

Lemma 2.2. ([12, Lemma 1.3]) *Let a and i be integers with $|a|>1$ and $i>0$. If i is odd then $k_i(-a)=k_{2i}(a)$, and if i is a multiple of 4 then $k_i(-a)=k_i(a)$.*

Lemma 2.3. *Let $n<105$. Then all coefficients of $\Phi_n(x)$ belong to the set $\{-1,0,1\}$. In particular, for all $x>0$ it is true that $x^{\varphi(n)}-x^{\varphi(n)-1}-...-x-1 \leq \Phi_n(x) \leq x^{\varphi(n)}+x^{\varphi(n)-1}+...+x+1$. As a consequence we have that if $x \geq 2$ then $\Phi_n(x) \leq 2x^{\varphi(n)}$ and if $x \geq 3$ then $x^{\varphi(n)}/2 \leq \Phi_n(x)$.*

Proof. The fact that all coefficients of $\Phi_n(x)$ belong to the set $\{-1,0,1\}$ if $n<105$ is well-known and mentioned for example in [4]. Also it is known that $\deg(\Phi_n(x))=\varphi(n)$, whence $x^{\varphi(n)}-x^{\varphi(n)-1}-...-x-1 \leq \Phi_n(x) \leq x^{\varphi(n)}+x^{\varphi(n)-1}+...+x+1$ for all $x>0$. If $x \geq 2$ then $x^{\varphi(n)} \geq x^{\varphi(n)-1}+x^{\varphi(n)-2}+...+x+1$, so $\Phi_n(x) \leq 2x^{\varphi(n)}$. If $x \geq 3$ then $x^{\varphi(n)} \geq 3x^{\varphi(n)-1} \geq 2x^{\varphi(n)-1}+x^{\varphi(n)-1} \geq 2x^{\varphi(n)-1}+2x^{\varphi(n)-2}+x^{\varphi(n)-2} \geq ... \geq 2x^{\varphi(n)-1}+2x^{\varphi(n)-2}+...+2x+2$, so $x^{\varphi(n)}/2 \leq \Phi_n(x)$.

□

Lemma 2.4. ([12, Lemma 1.5]) *Let a and i be integers, and $\varepsilon \in \{+,-\}$. If $a \geq 2$, $i \geq 3$, and $(a,i) \notin \{(2,3),(2,6)\}$, then $k_i(\varepsilon a) > a^{\varphi(i)/2}$.*

Lemma 2.5. *Let q and u be prime powers, where $q \neq u$ and q is odd. Then the following hold.*

(i) If $k_{2i}(q)$ divides $k_{2i}(u)$ and $i \in \{8,12\}$ then $iq^i < u^i$.

(ii) If $k_9(q)k_{18}(q)$ divides $k_9(u)k_{18}(u)$ then $3q^{12} < u^{12}$.

(iii) If $k_9(\varepsilon q)$ divides $k_9(\tau u)$, where ε, $\tau \in \{+,-\}$ then $3q^6 < u^6$.

(iv) If $k_7(\varepsilon q)$ divides $k_7(\tau u)$, where ε, $\tau \in \{+,-\}$, and $(q-\varepsilon 1, 7)=1$ then $3q^6 < u^6$.

Proof. Let $k_{16}(q)$ divides $k_{16}(u)$. First we prove that $k_{16}(q) \neq k_{16}(u)$. Assume the contrary. According to (2.1), we obtain that $(q^8+1)/2 = (u^8+1)/(u-1,2)$. If $(u-1,2)=2$ then $u=q$; a contradiction. So $(u-1,2)=1$, and hence $q^8 = 2u^8+1$. Since u is even, we have that $(u,5)=1$. Then $u^4 \equiv 1 \pmod 5$, and hence $2u^8+1 \equiv 3 \pmod 5$. It is clearly that $q^8 \equiv 0,1 \pmod 5$, so we get a contradiction. Thus $k_{16}(u)/k_{16}(q) > 1$. By Fermat's little theorem, we have that $r_{16}(u) \equiv 1 \pmod{16}$, therefore $17k_{16}(q) \leq k_{16}(u)$, and hence $(17/2)(q^8+1) \leq u^8+1$. So $8q^8 < u^8 - 7 < u^8$.

Let $k_{24}(q)$ divides $k_{24}(u)$. Since for an integer x it is true that $k_{24}(x) = \Phi_{24}(x) = x^8 - x^4 + 1$, the equality $k_{24}(u) = k_{24}(q)$ is equivalent to $u = q$. Note that $r_{24}(u) \equiv 1 \pmod{24}$, so $25k_{24}(q) \leq k_{24}(u)$. It is clear that $k_{24}(u) < u^8$ and $q^8/2 < k_{24}(q)$, whence $12q^8 < u^8$, as required.

Now we prove (ii). Assume that $k_9(q)k_{18}(q)$ divides $k_9(u)k_{18}(u)$. If x is an integer then $k_9(x)k_{18}(x) = \frac{x^6+x^3+1}{(x-1,3)}\frac{x^6-x^3+1}{(x+1,3)} = \frac{x^{12}+x^6+1}{(x-1,3)(x+1,3)}$ and clearly $(x-1,3)(x+1,3) \in \{1,3\}$. Suppose that $k_9(q)k_{18}(q) = k_9(u)k_{18}(u)$. If $(q-1,3)(q+1,3) = (u-1,3)(u+1,3)$ then $q^{12}+q^6+1 = u^{12}+u^6+1$, and hence $u = q$; a contradiction. So we may assume that $(q-1,3)(q+1,3) = 1$, $(u-1,3)(u+1,3) = 3$, the other case is considered similarly. Then $3(q^{12}+q^6+1) = u^{12}+u^6+1$, and hence $3q^6(q^6+1) = (u^6+2)(u^6-1)$. Since $(u^6+2, u^6-1) \in \{1,3\}$, we have that either u^6+2 or u^6-1 is divisible by q^6. It is clear that $u^6+2 \neq q^6$ and $u^6-1 \neq q^6$, therefore $(u^6+2)(u^6-1) \geq 2q^6(2q^6-3)$. So $3(q^{12}+q^6+1) \geq 4q^{12} - 6q^6$, and hence $q^{12} \leq 9q^6+3$; a contradiction since this inequality is not true if $q \geq 2$. Thus $k_9(q)k_{18}(q)$ is a proper divisor of $k_9(u)k_{18}(u)$. Since $r_9(u) \equiv 1 \pmod{18}$ and $r_{18}(u) \equiv 1 \pmod{18}$, we have that $k_9(u)k_{18}(u) \geq 19k_9(q)k_{18}(q)$, and hence $2u^{12} > u^{12}+u^6+1 \geq (19/3)(q^{12}+q^6+1) > 6(q^{12}+q^6+1)$. Therefore $u^{12} > 3q^{12}$, as claimed.

Now we prove (iii). Assume that $k_9(\varepsilon q)$ divides $k_9(\tau u)$, where ε, $\tau \in \{+,-\}$. Note that $k_9(\varepsilon q) = (q^6 + \varepsilon q^3+1)/(q-\varepsilon 1,3)$, $k_9(\tau u) = (u^6+\tau u^3+1)/(u-\tau 1,3)$. Assume at first that $(q-\varepsilon 1,3) = (u-\tau 1,3)$. Then $q^6+\varepsilon q^3+1$ divides $u^6+\tau u^3+1$. Note that $q^6+\varepsilon q^3+1 \neq u^6+\tau u^3+1$, since otherwise either $u = q$ or $u^3 \pm q^3 = \pm 1$, which is impossible. Since $r_9(\tau u) \geq 19$, we have that $19(q^6+\varepsilon q^3+1) \leq u^6+\tau u^3+1$. Now $(19/2)q^6 \leq 19(q^6+\varepsilon q^3+1) \leq u^6+\tau u^3+1 \leq 2u^6$, and hence $4q^6 < u^6$, in particular, $3q^6 < u^6$, as required. So we may assume that $(q-\varepsilon 1,3) \neq (u-\tau 1,3)$. We have that $\frac{(u-\tau 1,3)}{(q-\varepsilon 1,3)}(q^6+\varepsilon q^3+1)$ divides $(u^6+\tau u^3+1)$. First we prove that these numbers are not equal. Assume the contrary and consider the case $(q-\varepsilon 1,3) = 1$, $(u-\tau 1,3) = 3$, since the other case is symmetric to this one. So $3q^3(q^3+\varepsilon 1) = (u^3-\tau 1)(u^3+\tau 2)$. Now $(u^3-\tau 1, u^3+\tau 2) = 3$, so only one of them is divisible by q^3. Since $u^3-\tau 1 \neq q^3$ and $u^3+\tau 2 \neq q^3$, we have that either $u^3-\tau 1 \geq 2q^3$ or $u^3+\tau 2 \geq 2q^3$. Therefore $(u^3-\tau 1)(u^3+\tau 2) \geq 2q^3(2q^3-3)$. So $3q^3(q^3+\varepsilon 1) \geq 4q^6 - 6q^3$, and hence $(6+\varepsilon 3)q^3 \geq q^6$; a contradiction, since $q \geq 3$. In the symmetric case $(q^3-\varepsilon 1)(q^3+\varepsilon 2) = 3u^3(u^3+\tau 1)$ and we obtain that $(6+\tau 3)u^3 \geq u^6$. Whence $u = 2$ and $\tau = +$, and hence $(q^3-\varepsilon 1)(q^3+\varepsilon 2) = 216$; a contradiction, since $q \geq 3$ and $(q^3-\varepsilon 1)(q^3+\varepsilon 2) > 216$. So we obtain that $\frac{(u-\tau 1,3)}{(q-\varepsilon 1,3)}(q^6+\varepsilon q^3+1)$ is a proper divisor of $(u^6+\tau u^3+1)$. Since $r_9(\tau u) \geq 19$, we have that $\frac{19(u-\tau 1,3)}{(q-\varepsilon 1,3)}(q^6+\varepsilon q^3+1) \leq (u^6+\tau u^3+1)$. So $(19/3)(q^6+\varepsilon q^3+1) \leq (u^6+\tau u^3+1)$. Observe that $18q^6 < 19(q^6+\varepsilon q^3+1)$, since $q \geq 3$. Thus $6q^6 \leq (u^6+\tau u^3+1) \leq 2u^6$, and hence $3q^6 < u^6$, as required.

Now we prove (iv). Assume that $k_7(\varepsilon q)$ divides $k_7(\tau u)$, where ε, $\tau \in \{+,-\}$, and $(q-\varepsilon 1,7) = 1$. Then $k_7(\varepsilon q) = q^6+\varepsilon q^5+q^4+\varepsilon q^3+q^2+\varepsilon q+1$ and $k_7(\tau u) = \frac{u^6+\tau u^5+u^4+\tau u^3+u^2+\tau u+1}{(u-\tau 1,7)}$. Suppose that $k_7(\varepsilon q) = k_7(\tau u)$. Since $u \neq q$, either $q \geq u+1$ or $u \geq q+1$ and hence if $(u-\tau 1,7) = 1$ then we obtain that either $k_7(\varepsilon q) > k_7(\tau u)$ or $k_7(\tau u) > k_7(\varepsilon q)$ respectively. So $(u-\tau 1,7) = 7$, and hence $u \geq 8$. Since $q \geq 3$, we have that $k_7(\varepsilon q) \geq q^6-q^5+q^4-q^3+q^2-q+1 > (2/3)q^6$, and since $u \geq 8$, we obtain that $k_7(\tau u) \leq (u^6+u^5+u^4+u^3+u^2+u+1)/7 \leq (3/14)u^6$. Therefore $(28/9)q^6 \leq u^6$, and hence $3q^6 < u^6$, as required. Thus $k_7(\varepsilon q)$ is a proper divisor of $k_7(\tau u)$. By Fermat's little theorem, for any prime r dividing $k_7(\tau u)$, we have that $r-1$ is divisible by 7. Therefore $r \geq 29$, and hence $29k_7(\varepsilon q) \leq k_7(\tau u)$. By Lemma 2.3, we have that $k_7(\varepsilon q) \geq q^6/2$ and $k_7(\tau u) \leq 2u^6$. So $(29/4)q^6 \leq u^6$, in particular $3q^6 < u^6$. The lemma is proved. □

Define the following function on positive integers:

$$\eta(k) = \begin{cases} k, & \text{if } k \text{ is odd,} \\ k/2, & \text{if } k \text{ is even.} \end{cases} \tag{2.2}$$

Lemma 2.6. *Let n be a positive integer. Then $\varphi(n) = 6$ is equivalent to $n \in \{7, 9, 14, 18\}$ and $\varphi(n) = 8$ is equivalent to $n \in \{15, 16, 20, 24, 30\}$.*

Proof. Let n be a positive integer with $\varphi(n) = 6$ and r be a prime divisor of n. Then $r-1$ divides 6, and hence $r = 2, 3$ or 7. If $r = 7$ then r^2 does not divide n, and hence $\varphi(n/7) = 1$, so $n = 7$ or $n = 14$. Therefore $r = 2$ or $r = 3$. If $n = r^\alpha$ for an integer $\alpha \geq 1$ then $n = 9$. It remains to consider the case $n = 2^\alpha 3^\beta$ for $\alpha, \beta \geq 1$. In this situation $\varphi(n) = 2^\alpha 3^{\beta-1}$. So $\alpha = 1$, $\beta = 2$, and hence $n = 18$. Since $6 = \varphi(7) = \varphi(14) = \varphi(9) = \varphi(18)$, the case $\varphi(n) = 6$ is done.

Let $\varphi(n) = 8$ and r be a prime divisor of n. Then $r - 1$ divides 8, and hence $r = 2, 3$ or 5. Let $n = r^\alpha$, where $\alpha \geq 1$ and $r \in \{2, 3, 5\}$. Then $r = 2$, and hence $n = 16$. If n is divisible by both 3 and 5 then $n_{\{3\}} = 3$ and $n_{\{5\}} = 5$. So $\varphi(n/15) = 1$, and hence $n \in \{15, 30\}$. Therefore either $n = 2^\alpha 3^\beta$ or $n = 2^\alpha 5^\beta$ for some integers $\alpha, \beta \geq 1$. Then $\beta = 1$ and either $\alpha = 3$ or $\alpha = 2$ respectively. So $n = 24$ or $n = 20$. Moreover, it is clear that if $n \in \{15, 16, 20, 24, 30\}$ then $\varphi(n) = 8$. The lemma is proved. □

Lemma 2.7. *Let j and u be integers, $u \geq 3$, and either $j \geq 15$ or $\eta(j) \geq 9$. Then either $\eta(j) = 9$ or $k_j(u) \geq u^8/6$.*

Proof. Suppose at first that $j < 105$. By the lemma assumption, we have that $\varphi(j) \geq 6$. Let $\varphi(j) = 6$. Then $j \in \{7, 9, 14, 18\}$ due to Lemma 2.6. Since either $j \geq 15$ or $\eta(j) \geq 9$, we obtain that $\eta(j) = 9$. So we may assume that $\varphi(j) > 6$, and hence $\varphi(j) \geq 8$. Let r be the greatest prime divisor of j. Suppose that $\varphi(j) = 8$. By Lemma 2.6, we obtain that $j \in \{15, 16, 20, 24, 30\}$. If $j \neq 16, 20$ then $j_{\{r\}'}$ does not divide $r - 1$, and hence $k_j(u) = \Phi_j(u)$ due to (2.1). Lemma 2.3 implies that $\Phi_j(u) \geq u^8/2 > u^8/6$, as required. If $j = 20$ then $k_j(u) = \frac{u^8-u^6+u^4-u^2+1}{(5,u^2+1)} \geq (u^8 - u^6 + u^4 - u^2 + 1)/5 > u^8/6$. If $j = 16$ then $k_j(u) = (u^8 + 1)/(u-1, 2) \geq u^8/2$. So we may assume that $\varphi(j) \geq 10$. Observe that $k_j(u) \geq \Phi_j(u)/r \geq u^{\varphi(j)}/(2r)$ due to Lemma 2.3. If $r \leq 17$ then $u^{\varphi(j)} \geq u^{10} \geq 9u^8$, and hence $k_j(u) \geq u^8/4$. It remains to treat the case $r > 17$. Since $\varphi(j)$ is divisible by $r - 1$, we have the inequality $\varphi(j) \geq r - 1$. Therefore $k_j(u) \geq u^{r-1}/(2r) \geq 3^{r-9}u^8/(2r)$. So it is sufficient to show that $3^{r-9}/(2r) \geq 1/6$ or equivalently $3^{r-8} \geq r$. This inequality is clearly true for $r = 19$, and hence it holds for all $r \geq 19$ because the next prime after r is less than $2r$.

Let now $j \geq 105$. We prove that $\varphi(j) \geq 16$. If j is divisible by rs, where r, s are prime greater than or equal to 5, then either $r \neq s$, and hence $(r - 1)(s - 1)$ divides $\varphi(j)$ or $r = s$, and hence $r(r - 1)$ divides $\varphi(j)$. It is clear that in the both cases $\varphi(j) \geq 16$. Let j has exactly one prime divisor r such that $r \geq 5$. If $r \geq 17$ then obviously $\varphi(j) \geq 16$. If $r \leq 13$ then $j/r > 8$, and hence $\varphi(j) = \varphi(r)\varphi(j/r) \geq 4 \cdot 4 = 16$. Thus we may assume that $j = 2^\alpha 3^\beta$, where α, β are integers. Observe that $\alpha, \beta \geq 1$, since otherwise the inequality $\varphi(j) \geq 16$ is clear. Now $\varphi(j) = 2^\alpha 3^{\beta-1}$. So if $\varphi(j) < 16$ then $\alpha < 4$ and $\beta < 3$. Whence $j \leq 8 \cdot 9 < 105$; a contradiction. Therefore we have that $\varphi(j) \geq 16$, and hence by Lemma 2.4 we obtain that $k_j(u) > u^{\varphi(j)/2} \geq u^8 > u^8/6$, as required. The lemma is proved. □

3. Preliminaries: the prime graph and the spectrum of a finite group

Let G be a finite group. Observe that the spectrum of G is completely determined by the set $\mu(G)$ consisting of all maximal elements of $\omega(G)$ with respect to divisibility. The *prime graph* $GK(G)$ of G is defined as follows: its vertices are elements of $\pi(G)$, and two distinct vertices r and s are adjacent if and only if $rs \in \omega(G)$. Recall that a subset of vertices of a graph is called a *coclique*, if every two vertices of this subset are nonadjacent. Denote by $t(G)$ the greatest size of a coclique in $GK(G)$. We refer to a coclique containing r as an $\{r\}$-coclique. If $r \in \pi(G)$ then $t(r, G)$ denotes the greatest size of $\{r\}$-cocliques and $\rho(r, G)$ is a set of vertices in some $\{r\}$-coclique of size $t(r, G)$.

Lemma 3.1. ([13, Proposition 2], [14, Theorem 2]) *Let L be a finite nonabelain simple group with $t(L) \geq 3$ and $t(2, L) \geq 2$, and let G be a finite group isospectral to L. Then the following hold.*
(i) There exists a nonabelain simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}\, S$ for the soluble radical K of G.
(ii) For every coclique ρ of $GK(G)$ containing at least three elements, at most one prime from ρ divides the product $|K| \cdot |\overline{G}/S|$. In particular, $t(S) \geq t(L) - 1$.
(iii) Every prime $r \in \pi(G)$ nonadjacent to 2 in $GK(G)$ does not divide $|K| \cdot |\overline{G}/S|$. In particular, $t(2, S) \geq t(2, L)$.

Alongside with the function $\eta(k)$ (see (2.2)), we define several functions of natural argument that were used in [20] for formulation an adjacency criterion in the prime graph of a simple classical group.

$$\nu(k) = \begin{cases} k, \text{ if } k \equiv 0 \pmod 4, \\ k/2, \text{ if } k \equiv 2 \pmod 4, \\ 2k, \text{ if } k \text{ is odd.} \end{cases} \tag{3.1}$$

For $\varepsilon \in \{+, -\}$, put

$$\nu_\varepsilon(k) = \begin{cases} k, \text{ if } \varepsilon = +, \\ \nu(k) \text{ if } \varepsilon = -. \end{cases} \tag{3.2}$$

For linear and unitary groups, we exploit also a reformulation of an adjacency criterion (see [19, Lemmas 2.1–2.3]), if it is more convenient for our goals than an initial formulation from [20] which used the function ν_ε. This reformulation is based on the equality $k_{\nu_\varepsilon(i)}(q) = k_i(\varepsilon q)$, which follows from Lemma 2.2 and the definition of ν_ε.

Now we introduce a function φ which was defined in [12] in order to unify further arguments. Namely, given a simple classical group L over a field of order q and a prime r coprime to q, we put

$$\varphi(r, L) = \begin{cases} e(r, \varepsilon q), \text{ if } L = L_n^\varepsilon(q), \\ \eta(e(r, q)), \text{ if } L \text{ is symplectic or orthogonal.} \end{cases} \tag{3.3}$$

It follows that

$$e(r,q)=\begin{cases}2\varphi(r,L), & \text{if either } e(r,q) \text{ is even and } L \text{ is symplectic or orthogonal,}\\ & \text{or } e(r,q)\equiv 2 \,(\mathrm{mod}\, 4) \text{ and } L \text{ is unitary;}\\ \varphi(r,L)/2, & \text{if } e(r,q)\equiv 1 \,(\mathrm{mod}\, 2) \text{ and } L \text{ is unitary;}\\ \varphi(r,L) & \text{otherwise.}\end{cases} \tag{3.4}$$

Observe that $e(r,-q)=\varphi(r,L)$ in the case of $e(r,q)=\varphi(r,L)/2$.

Following [12], for a classical group L, we write $\mathrm{prk}(L)$ to denote its dimension if L is a linear or unitary group, and its Lie rank if L is a symplectic or orthogonal group. Observe that $n=\mathrm{prk}(L)$ in Theorems 1.2, 1.3 in Introduction.

Lemma 3.2. *Let L be a finite simple group of Lie type over a field of characteristic p and $n=\mathrm{prk}(L)$. Let r, s be odd primes with $r, s\in\pi(G)\setminus\{p\}$. Put $k=e(r,-q)$ and $l=e(s,-q)$ if $L\simeq L_n^-(q)$ and $k=e(r,q)$ and $l=e(s,q)$ otherwise. Suppose that $2\le\varphi(r,L)\le\varphi(s,L)$. Then the following hold.*

(i) If $L=L_n^\varepsilon(q)$ then r and s are nonadjacent in $GK(L)$ if and only if $\varphi(r,L)+\varphi(s,L)>n$, and $\frac{l}{k}$ is not a natural number.

(ii) If $L\in\{O_{2n+1}(q),S_{2n}(q)\}$ then r and s are nonadjacent in $GK(L)$ if and only if $\varphi(r,L)+\varphi(s,L)>n$, and $\frac{l}{k}$ is not an odd natural number;

(iii) If $L=O_{2n}^\varepsilon(q)$ then r and s are nonadjacent in $GK(L)$ if and only if $2\varphi(r,L)+2\varphi(s,L)>2n-(1-\varepsilon(-1)^{k+l})$, $\frac{l}{k}$ is not an odd natural number, and, if $\varepsilon=+$, then the chain of equalities: $n=l=2\varphi(s,L)=2\varphi(r,L)=2k$ is not true.

Proof. An adjacency criterion for two odd vertices in $GK(L)$ was obtained in [20, 21]. In particular, the statement (i) follows from [20, Propositions 2.1,2.2] and the statements (ii), (iii) follow from [21, Propositions 2.4,2.5]. □

Lemma 3.3. ([12, Lemma 2.3]) *Let L be a simple classical group over a field of order q and characteristic p. If r is an odd prime from $\pi(L)\setminus\{p\}$ then $\varphi(r,L)$ divides $r-1$, and if L is a symplectic or orthogonal group then $2\varphi(r,L)$ divides $r-1$.*

Lemma 3.4. ([12, Lemma 2.4]) *Let L be a simple classical group over a field of order q and characteristic p, and let $\mathrm{prk}(L)=n\ge 4$.*

(i) *If $r\in\pi(L)\setminus\{p\}$, then $\varphi(r,L)\le n$.*

(ii) *If r and s are distinct primes from $\pi(L)\setminus\{p\}$ with $\varphi(r,L)\le n/2$ and $\varphi(s,L)\le n/2$, then r and s are adjacent in $GK(L)$.*

(iii) *If r and s are distinct primes from $\pi(L)\setminus\{p\}$ with $n/2<\varphi(r,L)\le n$ and $n/2<\varphi(s,L)\le n$, then r and s are adjacent in $GK(L)$ if and only if $e(r,q)=e(s,q)$.*

(iv) *If r and s are distinct primes from $\pi(L)\setminus\{p\}$ and $e(r,q)=e(s,q)$, then r and s are adjacent in $GK(L)$.*

Let L be a simple classical group over a field of order q and characteristic p. For $\sigma\subseteq\pi(L)\setminus\{p\}$, put $E(\sigma,L)=\{e(r,q)\mid r\in\sigma\}$. If $\mathrm{prk}(L)=n\ge 13$ then, by [21], every coclique ρ of greatest size in $GK(L)$ does not contain p, so the set $E(\rho,L)$ is well-defined for ρ. Define $J(L)$ as the union of sets $E(\rho,L)$, and $E(L)$ as the intersection of these sets, where ρ runs over all cocliques of greatest size in $GK(L)$.

Lemma 3.5. ([12, Lemma 2.5, Table 1]) *Let L be a simple classical group over a field of order q and characteristic p, and $\mathrm{prk}(L)=n\ge 13$. Let ρ be a coclique of greatest size in $GK(L)$. If $J(L)=E(L)$ then*

$E(\rho, L) = E(L)$. If $J(L) \neq E(L)$ then $E(\rho, L) = E(L) \cup \{j\}$ for some $j \in J(L) \setminus E(L)$. In particular, $|E(L)| \leq t(L) \leq |E(L)| + 1$. The sets $E(L)$, $J(L) \setminus E(L)$ and numbers $t(L)$ are listed in Table 1.

TABLE 1. Cocliques of greatest size ($n \geq 13$)

L	Conditions	$t(L)$	$E(L)$	$J(L) \setminus E(L)$
$L_n^\varepsilon(q)$	n odd	$\frac{n+1}{2}$	$\{i \mid \frac{n}{2} < \nu_\varepsilon(i) \leq n\}$	$\varnothing$
	n even	$\frac{n}{2}$	$\{i \mid \frac{n}{2} < \nu_\varepsilon(i) < n\}$	$\{\frac{n}{2}, n\}$
$S_{2n}(q)$ or $O_{2n+1}(q)$	$n \equiv 0 \pmod 4$	$\frac{3n+4}{4}$	$\{i \mid \frac{n}{2} \leq \eta(i) \leq n\}$	$\varnothing$
	$n \equiv 1 \pmod 4$	$\frac{3n+5}{4}$	$\{i \mid \frac{n}{2} < \eta(i) \leq n\}$	$\varnothing$
	$n \equiv 2 \pmod 4$	$\frac{3n+2}{4}$	$\{i \mid \frac{n}{2} < \eta(i) \leq n\}$	$\{\frac{n}{2}, n\}$
	$n \equiv 3 \pmod 4$	$\frac{3n+3}{4}$	$\{i \mid \frac{n+1}{2} < \eta(i) \leq n\}$	$\{\frac{n-1}{2}, n-1, n+1\}$
$O_{2n}^+(q)$	$n \equiv 0 \pmod 4$	$\frac{3n}{4}$	$\{i \mid \frac{n}{2} \leq \eta(i) \leq n, i \neq 2n\}$	$\varnothing$
	$n \equiv 1 \pmod 4$	$\frac{3n+1}{4}$	$\{i \mid \frac{n}{2} < \eta(i) \leq n, i \neq 2n, n+1\}$	$\{n-1, n+1\}$
	$n \equiv 2 \pmod 4$	$\frac{3n-2}{4}$	$\{i \mid \frac{n}{2} < \eta(i) \leq n, i \neq 2n\}$	$\{\frac{n}{2}, n\}$
	$n \equiv 3 \pmod 4$	$\frac{3n+3}{4}$	$\{i \mid \frac{n-1}{2} \leq \eta(i) \leq n, i \neq 2n, n-1\}$	$\varnothing$
$O_{2n}^-(q)$	$n \equiv 0 \pmod 4$	$\frac{3n+4}{4}$	$\{i \mid \frac{n}{2} \leq \eta(i) \leq n\}$	$\varnothing$
	$n \equiv 1 \pmod 4$	$\frac{3n+1}{4}$	$\{i \mid \frac{n}{2} < \eta(i) \leq n, i \neq n, \frac{n+1}{2}\}$	$\{\frac{n+1}{2}, n-1\}$
	$n \equiv 2 \pmod 4$	$\frac{3n+2}{4}$	$\{i \mid \frac{n}{2} < \eta(i) \leq n\}$	$\{\frac{n}{2}, n-2, n\}$
	$n \equiv 3 \pmod 4$	$\frac{3n+3}{4}$	$\{i \mid \frac{n-1}{2} \leq \eta(i) \leq n, i \neq n, \frac{n-1}{2}\}$	$\varnothing$

Following [12], we call a prime $r \in \pi(L)$ *large* (with respect to L), if r lies in some coclique of greatest size in the prime graph $GK(L)$, and *small* (with respect to L) otherwise.

Lemma 3.6. ([12, Lemma 2.7]) *Let L be a simple classical group and $n = \mathrm{prk}(L) \geq 13$.*

(i) *If $\varphi(r, L) \geq n/2$, then r is large with respect to L.*

(ii) *If r is large with respect to L, then $\varphi(r, L) \geq n/2 - 1$.*

(iii) *If ρ is a coclique in $GK(L)$ and $n/2 < \varphi(r, L)$ for every $r \in \rho$, then $GK(L)$ has a coclique σ of size $t(L)$ with $\rho \subseteq \sigma$.*

Lemma 3.7. ([12, Lemma 2.8]) *Let L be a simple classical group over a field of order q and $n = \mathrm{prk}(L) \geq 13$. Suppose that $r \in \pi(L)$ and ρ is an $\{r\}$-coclique of greatest size in $GK(L)$. Then every $s \in \rho' = \rho \setminus \{r\}$ is large with respect to L. Further, if r is small with respect to L, then $\varphi(s, L) > n/2$ for every $s \in \rho'$, and $E(\rho', L)$ is uniquely determined by r. If, in addition, $i = e(r, q) > 2$, then $E(\rho', L)$ is uniquely determined by i.*

Let L be a simple classical group and $n = \mathrm{prk}(L) \geq 13$. Let r be small with respect to L and ρ be an $\{r\}$-coclique of greatest size. By Lemma 3.7, the set $E(\rho \setminus \{r\}, L)$ is contained in $J(L)$ and does not depend on a choice of ρ, so we denote it by $J(r, L)$.

It follows from Lemma 3.1 that primes nonadjacent with 2 in $GK(L)$ are important for L. Let

$$m(L) = \{n \in \mu(L) \mid \text{there exists } r \in \pi(n) \text{ such that 2 and } r \text{ are nonadjacent in } GK(L)\}.$$

Lemma 3.8. *Let L be a simple classical group over a field of order q and $n = \mathrm{prk}(L) \geq 13$.*

(i) *The set $m(L)$ is as in Table 2, in particular, this set is nonempty.*

(ii) *If $r \in \pi(L)$ is nonadjacent with 2 in $GK(L)$ then there exists a unique number $x \in m(L)$ such that r divides x.*

(iii) *If $L = L_n^\varepsilon(q)$ and $x \in m(L)$ then $\frac{q^{n-1}-\varepsilon 1}{2(n,q-\varepsilon 1)} \leq x \leq \frac{2q^{n-1}}{(n,q-\varepsilon 1)}$.*

(iv) *If $L \in \{S_{2n}(q), O_{2n+1}(q)\}$ and $x \in m(L)$ then $\frac{q^n-1}{(2,q-1)} \leq x \leq \frac{q^n+1}{(2,q-1)}$.*

(v) *If $L = O_{2n}^+(q)$, n is even, and $x \in m(L)$, then $\frac{q^{n-1}}{2(2,q-1)} \leq x \leq \frac{2q^{n-1}}{(2,q-1)}$.*

(vi) *If $L = O_{2n}^-(q)$, n is even, and $x \in m(L)$, then $\frac{q^n}{2(2,q-1)} \leq x \leq \frac{2q^n}{(2,q-1)}$.*

(vii) *If $L = O_{2n}^\varepsilon(q)$, n is odd, and $x \in m(L)$, then $\frac{q^n}{2(4,q-\varepsilon 1)} \leq x \leq \frac{2q^n}{(4,q-\varepsilon 1)}$.*

Proof. It follows from [20, Propositions 6.3, 6.6], that if $r \in \pi(L)$ is nonadjacent with 2 in $GK(L)$ then there exists a maximal $\{2\}$-coclique $\rho(2, L)$ containing r. A description of sets $\rho(2, L)$ for a finite simple classical group L is contained in [20, Tables 4, 6]. Moreover, information from these tables implies that if r is nonadjacent with 2 then r is nonadjacent with the characteristic of the underlying field of L. So orders in $\mu(L)$ not divisible by r are orders of semisimple elements of L. Spectrum descriptions of simple classical groups are contained in [1, 2], in particular, for every r nonadjacent with 2 in $GK(L)$, there exists exactly one element in $\mu(L)$ dividing on r. We combine mentioned descriptions of $\rho(2, L)$ and $m(L)$ into Table 2. Further in this proof we use the description of $m(L)$ from Table 2. Denote by x an element from $m(L)$.

Let $L = L_n^\varepsilon(q)$. If $\varepsilon = +$ then $\frac{q^{n-1}-1}{(n,q-1)} \leq x \leq \frac{q^n-1}{(q-1)(n,q-1)}$. So $x > \frac{q^{n-1}-1}{2(n,q-1)}$. Moreover, $\frac{q^n-1}{(q-1)(n,q-1)} = \frac{q^{n-1}+q^{n-2}+\ldots+1}{(n,q-1)} \leq \frac{2q^{n-1}}{(n,q-1)}$, and hence the required inequalities hold for x in this case. If $\varepsilon = -$ then $\frac{q^n-(-1)^n}{(q+1)(n,q+1)} \leq x \leq \frac{q^{n-1}-(-1)^{n-1}}{(n,q+1)}$. Obviously, we have that $\frac{q^{n-1}-(-1)^{n-1}}{(n,q+1)} \leq \frac{2q^{n-1}}{(n,q+1)}$. It remains to check that $\frac{q^n-(-1)^n}{(q+1)(n,q+1)} \geq \frac{q^{n-1}+1}{2(n,q+1)}$. This is equivalent to $2(q^n - (-1)^n) \geq (q^{n-1} + 1)(q + 1)$ which is obviously true due to $n \geq 13$.

Now we prove statements (iv) and (v). If $L \in \{S_{2n}(q), O_{2n+1}(q)\}$ then $x = \frac{q^n \pm 1}{(2,q-1)}$. If $L = O_{2n}^+(q)$ with even n then $x = \frac{q^{n-1} \pm 1}{(2,q-1)}$. Obviously required inequalities hold in these cases.

Let $L = O_{2n}^-(q)$, where n is even. If q is odd then $x = (q^n + 1)/2$ and $q^n/4 < x < q^n$, as required. If q is even then $(q^{n-1} + 1)(q - 1) \leq x \leq (q^{n-1} - 1)(q + 1)$. Since $(q^{n-1} + 1)(q - 1) = q^n - q^{n-1} + q - 1 > q^n/2$ and $(q^{n-1} - 1)(q + 1) = q^n + q^{n-1} - q + 1 < 2q^n$, we obtain required inequalities for x.

Let finally $L = O_{2n}^\varepsilon(q)$ and n is odd. If $\varepsilon = +$ then we have that $\frac{q^n-1}{(4,q-1)} \leq x \leq \frac{(q^{n-1}+1)(q+1)}{(4,q-1)}$ and the inequalities for x hold due to $(q^{n-1} + 1)(q + 1) \leq 2q^n$ and $q^n - 1 > q^n/2$. If $\varepsilon = -$ then $\frac{(q^{n-1}+1)(q-1)}{(4,q+1)} \leq x \leq \frac{q^n+1}{(4,q+1)}$ and the inequalities for x hold due to $(q^{n-1} + 1)(q - 1) > q^n/2$ and $q^n + 1 < 2q^n$. The lemma is proved. □

TABLE 2. The set $m(L)$, $\mathrm{prk}(L) \geq 13$

S	Conditions	$\rho(2,S) \setminus \{2\}$	The elements of $m(L)$
$L_n^{\varepsilon}(q)$	$n_{\{2\}} < (q-\varepsilon 1)_{\{2\}}$	$\{r_n(\varepsilon q)\}$	$\frac{q^n-(\varepsilon 1)^n}{(q-\varepsilon 1)(q-\varepsilon 1,n)}$
	either $n_{\{2\}} > (q-\varepsilon 1)_{\{2\}} > 1$ or $n_{\{2\}} = (q-\varepsilon 1)_{\{2\}} = 2$	$\{r_{n-1}(\varepsilon q)\}$	$\frac{q^{n-1}-(\varepsilon 1)^{n-1}}{(q-\varepsilon 1,n)}$
	either $q_{\{2\}} > 1$ or $2 < n_{\{2\}} = (q-\varepsilon 1)_{\{2\}}$	$\{r_{n-1}(\varepsilon q), r_n(\varepsilon q)\}$	$\frac{q^{n-1}-(\varepsilon 1)^{n-1}}{(q-\varepsilon 1,n)}$, $\frac{q^n-(\varepsilon 1)^n}{(q-\varepsilon 1)(q-\varepsilon 1,n)}$
$O_{2n+1}(q)$ or $S_{2n}(q)$	n is odd and $(q-1)_{\{2\}} = 2$	$\{r_n(q)\}$	$\frac{q^n-1}{2}$
	n is even or $(q-1)_{\{2\}} > 2$	$\{r_{2n}(q)\}$	$\frac{q^n+1}{(2,q-1)}$
	n is odd and q is even	$\{r_n(q), r_{2n}(q)\}$	q^n-1, q^n+1
$O_{2n}^{\varepsilon}(q)$ n is odd	$q \equiv -\varepsilon 1 \pmod 4$	$\{r_n(\varepsilon q)\}$	$\frac{q^n-\varepsilon 1}{2}$
	$q-\varepsilon 1 \equiv 0 \pmod 8$	$\{r_{2n-2}(q)\}$	$\frac{(q^{n-1}+1)(q+\varepsilon 1)}{4}$
	$q-\varepsilon 1 \equiv 4 \pmod 8$	$\{r_n(\varepsilon q), r_{2n-2}(q)\}$	$\frac{q^n-\varepsilon 1}{4}$, $\frac{(q^{n-1}+1)(q+\varepsilon 1)}{4}$
	q is even	$\{r_n(\varepsilon q), r_{2n-2}(q)\}$	$q^n-\varepsilon 1$, $(q^{n-1}+1)(q+\varepsilon 1)$
$O_{2n}^{+}(q)$ n is even	$q \equiv \tau 1 \pmod 4$, $\tau \in \{+,-\}$	$\{r_{n-1}(-\tau q)\}$	$\frac{q^{n-1}+\tau 1}{2}$
	q is even	$\{r_{n-1}(q), r_{2n-2}(q)\}$	$q^{n-1}-1$, $q^{n-1}+1$
$O_{2n}^{-}(q)$ n is even	q is odd	$\{r_{2n}(q)\}$	$\frac{q^n+1}{2}$
	q is even	$\{r_{n-1}(q), r_{2n-2}(q), r_{2n}(q)\}$	$(q^{n-1}-1)(q+1)$, $(q^{n-1}+1)(q-1)$, q^n+1

Lemma 3.9. *Let L, S be finite simple classical groups such that $\mathrm{prk}(L) \geq 13$, $\mathrm{prk}(S) \geq 13$. Assume that G is a finite group isospectral to L and $S \leq G/K \leq \mathrm{Aut}\, S$, where K is the soluble radical of G. Then for every $m_1 \in m(L)$ there exists $m_2 \in m(S)$ such that m_2 divides m_1.*

Proof. Let $m_1 \in m(L)$. Then there exists r such that r is nonadjacent with 2 in $GK(L)$ and r divides m_1. Lemma 3.1 implies that $r \in \pi(S)$, and r is nonadjacent with 2 in $GK(S)$. Therefore there exists $m_2 \in m(S)$ dividing by r. Applying Lemma 3.8 (ii), we obtain that m_2 divides m_1, as required. □

Lemma 3.10. *Let L be a simple classical group over a field of order q and characteristic p.*

(i) *If $L = L_n^{\varepsilon}(q)$ and $n \geq 23$, then $\omega(L)$ contains a number k with $k \geq q^{4t(L)/3}$ and all prime divisors of k are large with respect to L.*

(ii) *If $L \in \{S_{2n}(q), O_{2n+1}(q)\}$ and $n \geq 17$, or $L = O_{2n}^{\varepsilon}(q)$ and $n \geq 18$, then $\omega(L)$ contains a number k with $k \geq q^{t(L)}/2$ and all prime divisors of k are large with respect to L.*

(iii) *If $\mathrm{prk}(L) \geq 14$ then the numbers from $\omega(L)$ do not exceed $q^{2t(L)}/(q-1)$.*

Proof. The statement (i) is exactly [12, Lemma 2.14 (i)]. It is also proved in [12, Lemma 2.14 (ii)] that if $L \in \{S_{2n}(q), O_{2n+1}(q)\}$ and $n \geq 29$, or $L = O_{2n}^{\varepsilon}(q)$ and $n \geq 30$, then $\omega(L)$ contains a number k with $k \geq q^{10t(L)/9}$ and all prime divisors of k are large with respect to L. So we may assume that if $L \in \{S_{2n}(q), O_{2n+1}(q)\}$ then $n \leq 28$, and if $L = O_{2n}^{\varepsilon}(q)$ then $n \leq 29$. It is easy to see that under these assumptions there exists j such that $r_j(q)$ is large with respect to L, j is prime, $t(L) < j \leq \mathrm{prk}(L)$, and $j \neq \mathrm{prk}(L)$ if $L = O_{2n}^{\varepsilon}(q)$. Then $r_{2j}(q)$ is also large with respect to L. Remind that $r_{2j}(q) = r_j(-q)$ and choose $\varepsilon \in \{+,-\}$ such that $(q-\varepsilon 1, j) = 1$. Now we prove that $k = k_j(\varepsilon q)$ is required. It is clear that all prime divisor of k are large with respect to L, so it is sufficient to show that $k \geq q^{t(L)}/2$. Since $(q-\varepsilon 1, j) = 1$, we have that $k = (q^j - \varepsilon 1)/(q-\varepsilon 1) = (\varepsilon q)^{j-1} + (\varepsilon q)^{j-2} + \ldots + \varepsilon q + 1$. Since $q \geq 2$, we have that $q^{j-1} \geq 2q^{j-2}$, and hence $k > q^{j-1}/2$. Since we choose j so that $j > t(L)$, we obtain the required inequality $k > q^{t(L)}/2$.

Now we prove (iii). It follows from [17, Lemma 1.3] that elements of $\omega(L)$ do not exceed $q^{m+1}/(q-1)$, where m is the Lie rank of L. If L is linear or unitary then by Table 1, we obtain that $t(L) \geq \operatorname{prk}(L)/2 = (m+1)/2$, and hence elements of $\omega(L)$ do not exceed $q^{2t(L)}/(q-1)$, as claimed. If L is symplectic or orthogonal then by Table 1 we have that $t(L) \geq (3m-2)/4$. So it is sufficient to prove that $(3m-2)/2 \geq m+1$. This is equivalent to $m \geq 4$. The lemma is proved. □

Lemma 3.11. ([12, Lemma 2.12]) *Let L be a simple classical group over a field of order q and characteristic p, and let $\operatorname{prk}(L) = n \geqslant 4$. If $r \in \pi(L) \setminus \{p\}$, $i = e(r,q)$, and $n/2 < \varphi(r,L) \leqslant n$, then L includes a cyclic Hall subgroup of order $k_i(q)$.*

4. Proof: restrictions on K and $\overline{G}/S$

Let $\mathcal{L}$ be the class of simple classical groups whose dimension are as in the hypothesis of Theorem 1.3. It is not hard to check using information in Table 1 that $L \in \mathcal{L}$ if and only if $14 \leq t(L) \leq 22$.

Let L be a simple classical group over a field of order q and characteristic p and $L \in \mathcal{L}$. Suppose that G is a finite group isospectral to L. By [20, Theorem 7.1], we have that $t(2,L) \geq 2$. Lemma 3.1 implies that there exists a nonabelain simple group S such that $S \leq G/K \leq \operatorname{Aut} S$, where K is the soluble radical of G. Further in this section we fix groups G, K, and S. Moreover, we suppose that S is a simple classical group over a field of order u and characteristic v.

The following three propositions are the analogies of the [12, Propositions 3, 4, 5]. There are only two differences. First, we require $L \in \mathcal{L}$ instead of $\dim L \geq 40$ and, second, there are no exceptions in Proposition 4.2 in contrast to [12, Proposition 4], where there are some additional restrictions in cases of linear and unitary groups.

Proposition 4.1. *Suppose that $L \in \mathcal{L}$ and q is odd. Then the soluble radical K of G is nilpotent. If $r \in \pi(K) \setminus \{v\}$, then $t(r,L) = 2$, and $(s, |K| \cdot |\overline{G}/S| \cdot |P|) = 1$ for every $s \in \pi(L)$ nonadjacent to r in $GK(L)$ and every proper parabolic subgroup P of S.*

Proposition 4.2. *Suppose that $L \in \mathcal{L}$ and q is odd. If a prime r not equal to p divides the order of $\overline{G}/S$, then $\varphi(r,L) \leq n/3$. In particular, r is small with respect to L.*

Proposition 4.3. *Suppose that $L \in \mathcal{L}$ and q is odd. If a prime r is large with respect to L, then $(r, pv|K| \cdot |\overline{G}/S|) = 1$ and $k_{e(r,q)}(q) \in \omega(S)$. In particular, $t(S) \geq t(L)$.*

Proposition 4.2 will follow from Lemma 4.5 and Lemma 3.6. Proposition 4.1 is proved as [12, Proposition 3] with Lemma 4.5 in place of [12, Lemma 4.1]. The proof of [12, Propositon 5] uses [12, Proposition 4], and the condition $\dim L \geq 40$ is used to conclude that p is not large with respect to L. Actually it is sufficient to have that $t(L) \geq 6$, which is true since $L \in \mathcal{L}$. So the proof of Proposition 4.3 goes as the proof of [12, Propostion 5] with the changes pointed above.

Lemma 4.4. *Let $L \in \mathcal{L}$, q is odd, $r \in \pi(L)$, and r divides $|\overline{G}/S|$. Suppose that there exist i, j, $i \neq j$ such that $\{r, r_i(q), r_j(q)\}$ is a coclique in $GK(L)$. Then r divides either $k_i(q) - 1$ or $k_j(q) - 1$.*

Proof. Put $m = \operatorname{prk}(S)$. Since r divides $|\overline{G}/S|$, Lemma 3.1(ii) implies that $(r_i \cdot r_j, |\overline{G}/S| \cdot |K|) = 1$ and r_i, $r_j \in \pi(S)$. Assume that $\varphi(r_i, S) \leq m/2$. By Lemma 3.4(ii), for all $r_j \in R_j(q)$ we obtain that $\varphi(r_j, S) > m/2$. So $e(r_j, S)$ is the same for all $r_j \in R_j(q)$ due to Lemma 3.4(iii). Put $t = e(r_j, u)$ and observe that S has a cyclic Hall subgroup of order $k_t(u)$ due to Lemma 3.11. Let $s \in \pi(k_j(q))$ and $P \in Syl_s(S)$. By Frattini argument, r divides $|N_{\overline{G}}(P)|$. Let $x \in N_{\overline{G}}(P)$ and $|x| = r$. Then $S\langle x\rangle$ is a Frobenius group with kernel S and complement $\langle x\rangle$, and hence $|S| \equiv 1 \pmod r$. Note that Sylow s-subgroup of G and S are isomorphic and since s is an arbitrary element of $\pi(k_j(q))$ we obtain that $k_j(q) - 1$ is divisible by r. The case $\varphi(r_i, S) > m/2$ is exactly the same.

□

Lemma 4.5. *Let $L \in \mathcal{L}$ and q is odd. If $r \in \pi(L) \setminus \{p\}$ and $\varphi(r, L) > n/3$, then r does not divide $|\overline{G}/S|$.*

Proof. Assume the contrary and let there exists r such that r divides $|\overline{G}/S|$ and $\varphi(r, L) > n/3$. First, we consider the case $L \simeq L_n^{\varepsilon}(q)$, $n \geq 27$. Put $i = e(r, \varepsilon q) = \varphi(r, L)$ and so $i > n/3$. Let M be the set from Table 3 which depends on n and i. The set consists of two elements that we denote by i_1, i_2. It is ease to see that $n/2 < i_1, i_2 \leq n$, so $r_{i_1}(\varepsilon q)$, $r_{i_2}(\varepsilon q)$ are large with respect to L by Lemma 3.6(i). Moreover, we have that $i_1 + i > n$, $i_2 + i > n$ and i_1, i_2 are not divisible by i. Lemma 3.2 implies that $r_{i_1}(\varepsilon q)$ and $r_{i_2}(\varepsilon q)$ are nonadjacent with r in $GK(L)$, so $\{r, r_{i_1}(\varepsilon q), r_{i_2}(\varepsilon q)\}$ is a coclique of size 3 in this graph. By Lemma 4.4, there exists $j \in M$ such that r divides $k_j(\varepsilon q) - 1$. Now we prove that it is impossible.

TABLE 3. Set M for $L_n^{\varepsilon}(q)$

$n = 27$		$28 \leq n \leq 31$	
i	M	i	M
$18, 27$	$\{16, 24\}$	$i \geq 14$	see M for $n = 27$
$12, 24,\ \varepsilon q \equiv -1 \pmod 3$	$\{16, 27\}$	$i = 10, 11, 13$	$\{24, 28\}$
$12, 24,\ \varepsilon q \not\equiv -1 \pmod 3$	$\{16, 18\}$	$i = 12,\ \varepsilon q \equiv 1 \pmod 5$	$\{20, 27\}$
$i \neq 12, 18, 24, 27,\ \varepsilon q \equiv -1 \pmod 3$	$\{24, 27\}$	$i = 12,\ \varepsilon q \not\equiv 1 \pmod 5$	$\{25, 27\}$
$i \neq 12, 18, 24, 27,\ \varepsilon q \not\equiv -1 \pmod 3$	$\{18, 24\}$		
$32 \leq n \leq 39$		$40 \leq n \leq 44$	
i	M	i	M
$i \neq 11, 12, 14, 16, 28, 32$	$\{28, 32\}$	$i \neq 16, 18, 20, 36, 40$	$\{36, 40\}$
$i = 16, 32$	$\{24, 28\}$	$20, 40$	$\{32, 36\}$
$i = 28$	$\{24, 32\}$	$18, 36$	$\{32, 40\}$
$i = 12$	$\{30, 32\}$	$i = 16,\ n < 44$	$\{28, 36\}$
$i = 14,\ n \leq 37$	$\{24, 32\}$	$i = 16,\ n = 44$	$\{36, 44\}$
$i = 11,\ n = 32$	$\{28, 32\}$		
$i = 14,\ n \geq 38$	$\{32, 36\}$		

Assume that $n = 27$. Then $i \geq 10$. By the definition of M, we have that $j \in \{16, 18, 24, 27\}$. If $j = 16$ then $k_j(\varepsilon q) - 1 = (q^8 - 1)/2$, and hence $i \leq 8$; a contradiction. If $j = 24$ then $k_j(\varepsilon q) - 1 = q^4(q^4 - 1)$, and hence $i \leq 4$; a contradiction. If $j = 18$ then by Table 3, we have that $\varepsilon q \not\equiv -1 \pmod 3$, and hence $k_j(\varepsilon q) - 1 = \varepsilon q^3(\varepsilon q^3 - 1)$, $i \leq 3$; a contradiction. Finally, if $j = 27$ then $\varepsilon q \equiv -1 \pmod 3$, $k_j(\varepsilon q) - 1 = \varepsilon q^9(\varepsilon q^9 + 1)$, and hence $i = 18$; a contradiction with the choice of M.

Assume that $28 \leq n \leq 31$, and hence $i \geq 10$. If $i \geq 14$ then M is the same as in the previous case, so this case is considered above. Let now $i < 14$ and $i \neq 12$. By the definition of M, we have that $j \in \{24, 28\}$. The case $j = 24$ is regarded as for $n = 27$. If $j = 28$ then r divides $k_{28}(\varepsilon q) - 1 = q^{12} - q^{10} + q^8 - q^6 + q^4 - q^2$, and hence r divides $q^{10} - q^8 + q^6 - q^4 + q^2 - 1 = (q^2 - 1)(q^8 + q^4 + 1)$. Therefore r divides $q^8 + q^4 + 1$, and hence $q^{12} - 1$ is divisible by r; a contradiction. So $i = 12$ and r divides $k_j(\varepsilon q) - 1$ where $j = 20, 25$ or 27. If $j = 20$ then $\varepsilon q \equiv 1 \pmod 5$, and hence $k_{20}(\varepsilon q) - 1 = q^8 - q^6 + q^4 - q^2 = q^2(q^2 - 1)(q^4 + 1)$. So r divides either $q^2 - 1$ or $q^4 + 1$; a contradiction since $e(r, \varepsilon q) = 12$. Let $j = 25$ then $\varepsilon q \not\equiv 1 \pmod 5$, and hence $k_{25}(\varepsilon q) - 1 = \varepsilon q^5(\varepsilon q^5 + 1)(q^{10} + 1)$. So r divides either $\varepsilon q^5 + 1$ or $q^{10} + 1$; a contradiction. Thus $j = 27$. In this case r divides $k_{27}(\varepsilon q) - 1 = \frac{q^{18} + \varepsilon q^9 + 1}{(q - \varepsilon 1, 3)} - 1$. If $(q - \varepsilon 1, 3) = 1$ then r divides $q^9(\varepsilon q^9 + 1)$; a contradiction. Let $(q - \varepsilon 1, 3) = 3$ then $k_{27}(\varepsilon q) - 1 = (q^9 - \varepsilon 1)(q^9 + \varepsilon 2)/3$. Since $e(r, \varepsilon q) = 12$, we have that $q^9 \equiv -\varepsilon 2 \pmod r$. Then $1 \equiv q^{12} \equiv -\varepsilon 2q^3 \pmod r$, and hence $1 \equiv -\varepsilon 8q^9 \equiv 16 \pmod r$. So r divides 15; a contradiction since $e(3, \varepsilon q), e(5, \varepsilon q) \leq 4$.

Assume that $32 \leq n \leq 39$, and hence $i \geq 11$. Observe that $i = 11$ only when $n = 32$. By the definition of M, we have that $j \in \{24, 28, 30, 32, 36\}$. If $j = 24$ then r divides $k_j(\varepsilon q) - 1 = q^4(q^4 - 1)$; a contradiction. If $j = 28$ then $k_j(\varepsilon q) - 1 = q^2(q^3 - 1)(q^3 + 1)(q^4 - q^2 + 1)$. Since $(q^4 - q^2 + 1) \mid (q^{12} - 1)$ and $i \neq 12$, we obtain a contradiction. If $j = 32$ then by definition of M we have $i \neq 16$ and r divides $k_j(\varepsilon q) - 1 = (q^{16} - 1)/2$; a contradiction. if $j = 36$ then $r_i(\varepsilon q)$ divides $k_j(\varepsilon q) - 1 = q^6(q^3 - 1)(q^3 + 1)$; a contradiction. Thus $j = 30$, and hence $i = 12$ by Table 3. Since $k_{30}(\varepsilon q) - 1 = \varepsilon q(q^4 - 1)(\varepsilon q^3 + q^2 - 1)$, we have that r divides $\varepsilon q^3 + q^2 - 1$. However, r divides $k_{12}(\varepsilon q) = q^4 - q^2 + 1$. So r divides $\varepsilon q(\varepsilon q^3 + q^2 - 1) - (q^4 - q^2 + 1) = \varepsilon q^3 + q^2 - 1 - \varepsilon q$. Therefore r divides εq; a contradiction.

Finally we assume that $40 \leq n \leq 44$, and hence $i \geq 14$. By the definition of M, we have that $j \in \{28, 32, 36, 40, 44\}$. If $j = 28$ then $k_j(\varepsilon q) - 1 = q^2(q^3 - 1)(q^3 + 1)(q^4 - q^2 + 1)$. We get a contradiction, since $i \geq 14$. If $j = 32$ then by Table 3, we have that $i \neq 16$. Then r divides $k_j(\varepsilon q) - 1 = (q^{16} - 1)/2$; a contradiction. If $j = 36$ then $r_i(\varepsilon q)$ divides $k_j(\varepsilon q) - 1 = q^6(q^3 - 1)(q^3 + 1)$; a contradiction. If $j = 40$ then by Table 3, $j \neq 16$ and r divides $k_{40}(\varepsilon q) - 1 = q^4(q^4 - 1)(q^8 + 1)$; a contradiction. Thus $j = 44$, and hence $i = 16$. Then $k_j(\varepsilon q) - 1 = q^2(q^5 + 1)(q^5 - 1)(q^8 - q^6 + q^4 - q^2 + 1)$; a contradiction, since $(q^8 - q^6 + q^4 - q^2 + 1) \mid (q^{10} + 1)$. Thus L is not a linear or unitary group.

Assume that L is a symplectic or orthogonal group. We have the similar proof as in the case of linear and unitary groups. Put $i = e(r, q)$. Then $\varphi(r, L) = \eta(i)$ (see (3.3)). Let M be the set from Table 4 which depends on n, i. We have the similar properties of M as early: M consists of two numbers i_1, i_2 such that $\varphi(r_{i_1}(q), L) = \eta(i_1) > n/2$, $\varphi(r_{i_2}(q), L) = \eta(i_2) > n/2$, $\varphi(r_i(q), L) + \varphi(r_{i_1}(q), L) > n$, $\varphi(r_i(q), L) + \varphi(r_{i_2}(q), L) > n$ and i_1/i, i_2/i are not odd integers. So $\{r_i, r_{i_1}, r_{i_2}\}$ is a coclique in $GK(L)$. Lemma 4.4 implies that there exists $j \in M$ such that r divides $k_j(\varepsilon q) - 1$. Now we prove that it is impossible sorting out values of n. By the assumption of Theorem 1.3, we have that $17 \leq n \leq 30$. Assume at first that $n = 17$ or $n = 18$. Then $\eta(i) \geq 6$ or $\eta(i) \geq 7$ respectively.

Table 4. Set M for orthogonal and symplectic groups

$n = 17, 18$		$19 \le n \le 20$	
i	M	i	M
$i \ne 16, 24, 32$	$\{24, 32\}$	$i \ne 16, 32, 36$	$\{32, 36\}$
16	$\{24, 28\}$	$16, 32$	$\{28, 36\}$
24	$\{28, 32\}$	36	$\{28, 32\}$
32	$\{24, 28\}$		
$21 \le n \le 25$		$26 \le n \le 30$	
i	M	i	M
$i \ne 16, 36, 40$	$\{36, 40\}$	$i \ne 20, 44, 48$	$\{44, 48\}$
36	$\{28, 40\}$	$i = 48$	$\{40, 44\}$
40	$\{28, 36\}$	$i = 44$	$\{40, 48\}$
$16,\ q \equiv -1 \,(\mathrm{mod}\,19)$	$\{36, 19\}$	$i = 20,\ q \equiv -1 \,(\mathrm{mod}\,23)$	$\{23, 48\}$
$16,\ q \not\equiv -1 \,(\mathrm{mod}\,19)$	$\{36, 38\}$	$i = 20,\ q \not\equiv -1 \,(\mathrm{mod}\,23)$	$\{46, 48\}$

By Table 4, we obtain that $j \in \{24, 28, 32\}$. Assume that $j = 28$. Then $k_{28}(q) - 1 = q^{12} - q^{10} + q^8 - q^6 + q^4 - q^2 = (q^4 - q^2)(q^8 + q^4 + 1)$. Hence r divides $q^8 + q^4 + 1$, whence $i = 12$; a contradiction with the choice of M. Let $j = 24$. Then r divides $q^8 - q^4 = q^4(q^4 - 1)$; a contradiction. If $j = 32$, then r divides $q^{16} - 1$. So $i = 16$; a contradiction with the choice of M.

Let now $19 \le n \le 20$. In this case $\eta(i) \ge 7$. The cases $j = 28, 32$ are treated in the same way as for $n = 18$. So $j = 36$, and hence $k_j(q) - 1 = q^{12} - q^6 = q^6(q^6 - 1)$; a contradiction with $\eta(i) \ge 7$.

Assume that $21 \le n \le 25$. So $\eta(i) \ge 8$. The cases $j = 28, 36$ are considered above. Let $j = 40$. Since $k_{40}(q) - 1 = q^4(q^4 - 1)(q^8 + 1)$, we have that $r \mid (q^8 + 1)$, and hence $i = 16$; a contradiction with the choice of M.

Let $j = 19$. Then $(q - 1, 19) = 1$, and hence $k_{19}(q) = \frac{q^{19} - 1}{q - 1}$. Since $i = 16$, we have r divides $q^{16} - 1$. Since r divides $k_{19}(q) - 1$, we have that $q^{19} \equiv q \,(\mathrm{mod}\, r)$, and hence $q^3 \equiv q \,(\mathrm{mod}\, r)$. So r divides $q(q^2 - 1)$; a contradiction. Thus $j = 38$ and $(q + 1, 19) = 1$. Therefore $k_{38}(q) - 1 = \frac{q^{19} + 1}{q + 1} - 1 = \frac{q^{19} - q}{q + 1}$. Since $i = 16$, we have that $q^{16} \equiv 1 \,(\mathrm{mod}\, r)$, and hence $q^{19} - q \equiv q^3 - q \,(\mathrm{mod}\, r)$. So $q^3 - q$ is divisible by r; a contradiction.

Assume that $26 \le n \le 30$, and hence $\eta(i) \ge 9$. Then $j \in \{23, 40, 44, 46, 48\}$. If $j = 40$ then similarly to the previous case we have that $i = 16$, however $\eta(16) = 8 < 9$; a contradiction. Let $j = 44$, and hence $i \ne 20$. Then $k_j(q) - 1 = q^2(q^{10} - 1)(q^8 - q^6 + q^4 - q^2 + 1)$. So $r \mid q^8 - q^6 + q^4 - q^2 + 1$, and hence $r \mid q^{10} + 1$; a contradiction; If $j = 48$ then $k_j(q) - 1 = q^8(q^8 - 1)$, so r divides $q^8 - 1$; a contradiction. Thus $j = 23$ or $j = 46$. In this case $i = 20$. Note that $k_j(q) = k_{23}(\tau q)$ for $\tau \in \{+, -\}$ and $q \not\equiv \tau 1 \,(\mathrm{mod}\, 23)$. Then $k_{23}(\tau q) - 1 = \frac{q^{23} - q}{q - \tau 1}$. So $q^{23} \equiv q \,(\mathrm{mod}\, r)$ and since $q^{20} \equiv 1 \,(\mathrm{mod}\, r)$, we obtain that $q^3 \equiv q \,(\mathrm{mod}\, r)$. Therefore $r \mid (q^2 - 1)q$; a contradiction. The lemma is proved. □

5. Proof: restrictions on S

Let L be as in the hypothesis of Theorem 1.3, that is, $L \in \mathcal{L}$ and the characteristic of underlying field of L is odd. As in the previous section, we assume further that G is a finite group isospectral to L, and S is a unique nonabelian composition factor of G, so $S \le G/K \le \mathrm{Aut}\, S$, where K is the soluble radical of G.

Lemma 5.1. *S is a finite simple classical group.*

Proof. Since $L \in \mathcal{L}$, we have that $t(L) \geq 14$. By Lemma 3.1, we obtain that $t(S) \geq t(L) - 1 \geq 13$. By [19, Theorems 1, 2] and [18, Theorems 1, 2], S is neither an alternating group nor sporadic simple group nor the Tits group. By [21, Table 4], for any exceptional simple group of Lie type U, we have that $t(U) \leq 12$. So S is a simple classical group. □

Further we assume that S is a simple classical group and numbers p, q, u, v are as in the previous section. Put $m = \mathrm{prk}(S)$ and $t = t(L)$. Following to the hypothesis of Theorem 1.3, we assume that $2 \neq p \neq v$, in particular, $q \neq u$ and $q \geq 3$.

Proposition 5.2. *Suppose that $L \in \mathcal{L}$ and q is odd, and for some positive integer a the number $k_a(u)$ has a prime divisor large with respect to S. Then $k_a(u)$ has a prime divisor large with respect to L. In particular, $t(L) = t(S)$ and every prime r large with respect to L is large with respect to S.*

Proof. The assertion is an analogue of Proposition 7 in [12], where the conclusion is proved for L with $t(L) \geq 23$. We will proof Proposition 5.2 in the similar manner. Suppose to the contrary that non of prime divisors of $k_a(u)$ is large with respect to L.

Lemma 5.3. *For every coclique σ of greatest size in $GK(S)$ containing $r_a(u)$ there exists a set $J \subseteq E(\sigma, S)$ of size $d = \max\{1, t(S) - t(L)\}$ satisfying the following:*

(i) *for any $j \in J$, every $r \in R_j(u)$ is large with respect to S and divides the number $|\overline{G}/S| \cdot |K|$;*

(ii) *if $t(S) > t(L)$ and every coclique ρ of the greatest size in $GK(L)$ contains a prime s with $\varphi(s, S) \leq m/2$, then $\varphi(r, S) > m/2$ for any $j \in J$ and every $r \in R_j(u)$.*

Proof. The assertion is close to [12, Lemma 6.1]. One can check that [12, Lemma 6.1] states that there exists a set J of positive integers satisfying (i) and (ii). However, the proof starts from arbitrary coclique, say σ, of greatest size in $GK(S)$ containing $r_a(u)$ and the set J is proved to be a subset of $E(\sigma, S)$. Moreover, the proof is not used the condition $t(L) \geq 23$ from the mentioned above Proposition 7 from [12]. So the proof of the lemma goes as the proof of Lemma 6.1 in [12]. □

Lemma 5.4. *Let a set J be as in Lemma 5.3. Then for each $j \in J$ and every prime r from $R_j(u)$ the number $(k_j(u))_{\{r\}}$ divides $p(q^2 - 1) \log_v u$. Moreover, the inequality*

$$\frac{\prod_{j \in J} k_j(u)}{\log_v u} \leq p(q^2 - 1)$$

holds true.

Proof. The statement is the exactly same as [12, Lemma 6.4]. The proof of Lemma 6.4 [12] uses [12, Propositions 3, 5] and auxiliary [12, Lemmas 6.2, 6.3]. One can check that Lemmas 6.2, 6.3 hold true for $L \in \mathcal{L}$. As was mentioned above, in place of Propositions 3, 5 from [12] we can use Propositions 4.1, 4.3. So the proof goes as the proof of [12, Lemma 6.4] with the pointed changes. □

We are ready to prove Proposition 5.2. By Proposition 4.3, we conclude that $t(S) \geq t(L)$. Let J be a corresponding set from Lemma 5.3 and $d = |J|$. Then $d \geq t(S) - t$. By Lemma 5.3, we have that $J \subseteq E(\sigma, S)$, where σ is a coclique of greatest size in $GK(S)$.

Assume at first that $d \geq 2$ and $u \neq 2$. In this case $d = t(S) - t$, in particular $t(S) \geq 15$. By Table 1, we have that for any $j \in J$ either $j \geq 15$ or $\eta(j) \geq 9$. Lemma 2.7 implies that if $\eta(j) > 9$ and $u \geq 3$

then $k_j(u) \geq u^8/6$. If $\eta(j) = 9$ then $j \in \{9, 18\}$. Since $t(S) \geq 15$ and $J \subseteq E(\sigma, S)$, information from Table 1 implies that the both numbers 9 and 18 cannot be elements of J simultaneously. If $j \in \{9, 18\}$ then $k_j(u) \geq (u^6 - u^3 + 1)/3 \geq u^6/6$. So we obtain that $\prod_{j \in J} k_j(u) \geq u^6(u^8)^{d-1}/6^d = u^{8d-2}/6^d$. On the other hand, $\prod_{j \in J} k_j(u) \leq p(q^2-1)\log_v u < q^3 u$. Therefore $u^{8d-2}/6^d < q^3 u$, and hence $u^{8d-3}/6^d < q^3$. Since $u^2 \geq 9$, we have that $u^{8d-3} = u^{2d}u^{6d-3} \geq 9^d u^{6d-3} > 6^d u^{6d-3}$. So $u^{6d-3} < q^3$, and hence $u^{2d-1} < q$. By Lemma 3.10, the spectrum of L contains a number b such that $b \geq q^t/2$ and all its prime divisors are large with respect to L. It follows from Proposition 4.3 that $b \in \omega(S)$, so $b \leq u^{2t(S)}/(u-1) \leq u^{2t+2d}/(u-1) \leq u^{2t+2d}/2$ by Lemma 3.10 (iii). Therefore $q^t \leq u^{2t+2d}$. Using the inequality $u^{2d-1} < q$, we obtain that $u^{(2d-1)t} < q^t \leq u^{2t+2d}$. To get a contradiction, it is sufficient to show that $(2d-1)t > 2t + 2d$ or equivalently $2dt > 3t + 2d$. Since $d \geq 2$, we have that $d \geq 3/2 + d/4$, and hence $2dt \geq 3t + td/4$. Now $3t + td/4 > 3t + 2d$ due to $t \geq 14$. Therefore $(2d-1)t > 2t + 2d$, as claimed; a contradiction.

Let now $d = 1$ and $u > 2$. Then by Lemma 5.4, we have that $k_j(u)$ divides $p(q^2-1)\log_v u$, where $j \in J$. Observe that $p(q^2-1)$ is divisible by 24 and $k_j(u)$ is coprime to 6, so actually $24k_j(u)$ divides $p(q^2-1)\log_v u$. By Table 1, we have that either $j \geq 15$ or $\eta(j) \in \{7, 8, 9\}$. Assume at first that $j \geq 15$ and $\eta(j) \neq 7, 9$. Lemma 2.7 implies that $k_j(u) \geq u^8/6$. Since $8k_j(u)$ divides $p(q^2-1)\log_v u$ and $\log_v u \leq u$, we obtain that $8u^8/6 \leq q^3 u$, and hence $u^7 < q^3$. By Lemma 3.10, the spectrum of L contains a number b such that $b \geq q^t/2$ and all its prime divisors are large with respect to L. It follows from Proposition 4.3 that $b \in \omega(S)$, so $b \leq u^{2t(S)}/(u-1) \leq u^{2t+2}/(u-1)$ by Lemma 3.10 (iii). So $q^t < 2u^{2t+2}/(u-1) < 4u^{2t+1}$. Since $u^7 < q^3$, we have that $u^{7t/3} < q^t$, and hence $u^{7t/3} < 4u^{2t+1}$. The last inequality is equivalent to $u^{t/3-1} < 4$; a contradiction, since $u \geq 3$ and $t \geq 14$. Therefore we have that $\eta(j) \in \{7, 9\}$. By Table 1, we have that either $t(S) \leq 15$ or $m \leq 36$ and S is a linear or unitary group in the latter case.

Assume that $t(L) = t(S)$. Then every coclique of greatest size in $GK(L)$ forms a coclique of greatest size in $GK(S)$ due to Propostion 4.3. Since $24 \in E(S)$, we have that there exists $r \in R_{24}(u)$ such that r is large with respect to L. Also we may assume that $r \in \sigma$. Let $r \in R_i(q)$. Now we prove that $R_i(q) \subseteq R_{24}(u)$. Let $r' \in R_i(q) \setminus \{r\}$. Then $r' \in \pi(S)$ and $t(r', S) = t$ due to Proposition 4.3. However r' is adjacent with r, and hence $r' \in R_{24}(u)$. So $R_i(q) \subseteq R_{24}(u)$, and hence $k_i(q)$ divides $k_{24}(u) = u^8 - u^4 + 1$. By Table 1, we have that either $\eta(i) \in \{7, 8, 9\}$ or $\eta(i) > 9$. In the first case we have that $k_i(q) \geq q^6/14$ and in the other cases by Lemma 2.7 we obtain that $k_i(q) \geq q^8/6$. Therefore $q^6 \leq 14u^8$, and hence $q^3 < 4u^4$. On the other hand, we have that $24k_j(u)$ divides $p(q^2-1)\log_v u$, and hence $24k_j(u) < q^3 u$. Since $\eta(j) \in \{7, 9\}$, we have that $k_j(u) \geq u^6/14$. Therefore $(24/14)u^6 \leq q^3 u$, and hence $u^5 < (3/4)q^3$. Comparing with the inequality $q^3 < 4u^4$, we obtain that $u^5 < 3u^4$; a contradiction with $u > 2$. So $t(S) - t(L) = 1$. Since $\eta(j) \in \{7, 9\}$ and $r_j(u)$ is large with respect to S, we obtain that either $S \simeq L_m^{\pm}(u)$ with $29 \leq m \leq 36$ or $S \in \{O_{38}^{\pm}(u), O_{39}(u), S_{38}(u)\}$. By Table 1, we may choose σ such that if $r \in \sigma$ and $\varphi(r, S) \leq m/2$ then $\varphi(r, S) = j$. Assume that there exists $i \in E(\sigma, S)$ such that for every $r \in R_i(u)$ it is true that r divides $|\overline{G}/S| \cdot |K|$. Then the set $\{i, j\}$ satisfies conditions of Lemma 5.3 and we obtain a contradiction as in the case $d \geq 2$ considering this set instead of J. Thus we can choose for every $i \in E(\sigma, S) \setminus \{j\}$ an element $r_i(u) \in R_i(u)$ such that $(r_i(u), |\overline{G}/S| \cdot |K|) = 1$. These $t(S) - 1$ elements form a coclique of greatest size in $GK(L)$, in particular $r_{24}(u)$ is large with respect to L, since $24 \in E(S)$. Let $r_{24}(u) \in R_i(q)$ and $r' \in R_i(q) \setminus \{r_{24}(u)\}$. Then $r' \in \pi(S)$, $t(r', S) \geq t(S) - 1$, and r' is adjacent to $r_{24}(u)$ in $GK(S)$. By the adjaceny criterion, we obtain that $r' \in R_{24}(u)$. So $R_i(q) \subseteq R_{24}(u)$, and hence $k_i(q)$ divides $k_{24}(u)$. Arguing as above, we obtain a contradiction.

Let finally $u = 2$, in particular, $\log_v u = 1$. Then by Lemma 5.4, we have that $\prod_{j\in J} k_j(u)$ divides $p(q^2-1)$. Let r be a prime divisor of $\prod_{j\in J} k_j(u)$. Then r divides $k_j(u)$ for some $j \in J$, and hence $r-1$ is divisible by j. Since for any $j \in J$, we have that $\eta(j) \geq 7$, and r is odd, it is true that $r \geq 17$. Moreover, we have that r divides one of the numbers p, $q+1$, $q-1$ and therefore $q+1 \geq 17$. Since q is odd, we obtain that $q \geq 17$. By Lemma 3.10, the spectrum of L contains a number b such that $b \geq q^t/2$ and all its prime divisors are large with respect to L. It follows from Proposition 4.3 that $b \in \omega(S)$, so $b \leq 2^{2t+2d}$ by Lemma 3.10 (iii). Since $q \geq 17$, we have that $17^t/2 \leq b \leq 2^{2t+2d}$, and hence $2^{4t} < 2^{2t+2d+1}$. So $2t+2d \geq 4t$, and hence $d \geq t$. As above, we obtain that $r_j(u) \geq 17$ for any $j \in J$. Observe that at most one $r_j(u)$ is equal to p, and hence either $q+1$ or $q-1$ is divisible by at least $(d-1)/2$ different primes $r_j(u)$. So $q+1 \geq 17^{(d-1)/2} > 2^{2(d-1)}+1$, and hence $q > 2^{2(d-1)}$. Therefore $b \geq q^t/2 > 2^{2t(d-1)-1}$. Whence we have that $2^{2t(d-1)-1} < 2^{2(t+d)}$. Since $d \geq t$, it is true that $t(d-1) > t+d$, and hence $2^{2t(d-1)} > 2^{2(t+d)}$; a contradiction. The proposition is proved. □

Lemma 5.5. *Let $L \in \mathcal{L}$ and q is odd. If $s \in \pi(L)$ is chosen so that $\varphi(s, L) > n/3$, then $t(s, L) = t(s, S)$.*

Proof. This lemma is an analogue of [12, Lemma 7.1], where the assertion is proved for L with $t(L) \geq 23$. The proof is based on Propostions 5, 7, 8 from [12]. To adopt this proof for $L \in \mathcal{L}$, we can use Propostion 4.3, Propostion 5.2, and Lemma 4.5 respectively. □

Following [12], define

$$M(L) = \{i \mid \varphi(r_i(q), L) > n/3, t(r_i(q), L) < t(L)\},$$

$$M(S) = \{i \mid \varphi(r_i(u), S) > m/3, t(r_i(u), S) < t(S)\}.$$

Lemma 5.6. *Let $L = L_n^\varepsilon(q)$, where $n \geq 27$ and $i \in M(L)$. Then*

(i) $t(r_i(\varepsilon q), L) = i$;

(ii) if $s \in \pi(L)$ such that $\varphi(s, L) \leq n/3$, then $t(s, L) < t(r_i(\varepsilon q), L)$.

Proof. Let $i \in M(L)$. Arguing as in the proof of [12, Lemma 7.3], we obtain the equality $t(r_i(\varepsilon q), L) = i$. If $j \leq n/3$ then the similar reasoning shows that $t(r_j(\varepsilon q), L) \leq j$. Now we have that $t(r_j(\varepsilon q), L) \leq j \leq n/3 < i = t(r_i(\varepsilon q), L)$, and hence (ii) is proved. □

Lemma 5.7. *Let $L \in \{S_{2n}(q), O_{2n+1}(q)\}$, where $n \geq 17$ and $i \in M(L)$. Then*

(i) $t(r_i(q), L) = \begin{cases} [\frac{3\eta(i)+2}{2}] & \text{if } n \text{ is even,} \\ [\frac{3\eta(i)+3}{2}] & \text{if } n \text{ is odd.} \end{cases}$

In particular, if n is even then $t(r_i(q), L) \not\equiv 0 \pmod 3$ and if n is odd then $t(r_i(q), L) \not\equiv 2 \pmod 3$.

(ii) if $s \in \pi(L)$ such that $\varphi(s, L) \leq n/3$, then $t(s, L) < t(r_i(q), L)$.

Proof. Let $i \in M(L)$. Define a function $\tau(k)$ on positive integer k:

$$\tau(k) = \begin{cases} [\frac{3\eta(k)+2}{2}] & \text{if } n \text{ is even,} \\ [\frac{3\eta(k)+3}{2}] & \text{if } n \text{ is odd.} \end{cases}$$

Observe that if k_1, k_2 are integers and $\eta(k_1) > \eta(k_2)$, then $\tau(k_1) > \tau(k_2)$. Arguing as in the proof of [12, Lemma 7.4], we obtain the required equality $t(r_i(q), L) = \tau(i)$. If $j \leq n/3$ then the similar reasoning shows that $t(r_j(\varepsilon q, L) \leq \tau(j)$. Since $\varphi(r_j(q), l) = \eta(j) \leq n/3 < \varphi(r_i(q), L) = \eta(i)$, we have that $\tau(j) < \tau(i)$, and hence $t(r_i(q), L) < t(r_j(q), L)$, so (ii) is proved. □

Lemma 5.8. *Let $L \simeq O^+_{2n}(q)$, where $n \geq 19$ and $i \in M(L)$. Then*

$$(i)\ t(r_i(q), L) = \begin{cases} [\frac{3\eta(i)+1}{2}] & \text{if } n \text{ is even,} \\ [\frac{3\eta(i)+2}{2}] & \text{if } n \text{ is odd and } i \text{ is even,} \\ [\frac{3\eta(i)+3}{2}] & \text{if } n \text{ is odd and } i \text{ is odd.} \end{cases}$$

In particular, if n is even then $t(r_i(q), L) \not\equiv 1 \pmod 3$.

(ii) if $s \in \pi(L)$ such that $\varphi(s, L) \leq n/3$, then $t(s, L) < t(r_i(\varepsilon q), L)$.

Proof. Similar arguments as in the proof of Lemma 5.7 show that the assertions (i), (ii) follow from the proof of [12, Lemma 7.5]. □

Lemma 5.9. *Let $L \simeq O^-_{2n}(q)$, where $n \geq 18$ and $i \in M(L)$. Then*

$$(i)\ t(r_i(q), L) = \begin{cases} [\frac{3\eta(i)+4}{2}] & \text{if } n \text{ is even,} \\ [\frac{3\eta(i)+2}{2}] & \text{if } n \text{ is odd and } i \text{ is odd,} \\ [\frac{3\eta(i)+3}{2}] & \text{if } n \text{ is odd and } i \text{ is even.} \end{cases}$$

In particular, if n is even then $t(r_i(q), L) \not\equiv 1 \pmod 3$.

(ii) if $s \in \pi(L)$ such that $\varphi(s, L) \leq n/3$, then $t(s, L) < t(r_i(\varepsilon q), L)$.

Proof. Similar arguments as in the proof of Lemma 5.7 show that the assertions (i), (ii) follow from the proof of [12, Lemma 7.6]. □

Put

$$T(L) = \{t(r_i(q), L) \mid i \in M(L)\}.$$

Recall that L is a group from the assumption of Theorem 1.3 if and only if $L \in \mathcal{L}$ and q is odd, since this is a definiton of $\mathcal{L}$. Note that the equality $t(L) = t(S)$ implies that the nonabelian composition factor S of G also lies in $\mathcal{L}$. Define the following classes for groups from $\mathcal{L}$.

$$\mathcal{Y}_0 = \{L \in \mathcal{L} \mid \{t-1, t-2, t-3\} \subseteq T(L)\};$$
$$\mathcal{Y}_1 = \{L \in \mathcal{L} \mid \{t-1, t-2, t-3\} \cap T(L) = \{t-2, t-3\}\};$$
$$\mathcal{Y}_2 = \{L \in \mathcal{L} \mid \{t-1, t-2, t-3\} \cap T(L) = \{t-1, t-3\}\};$$
$$\mathcal{Y}_3 = \{L \in \mathcal{L} \mid \{t-1, t-2, t-3\} \cap T(L) = \{t-1, t-2\}\};$$
$$\mathcal{X} = \{O^-_{36}(q)\};$$

Lemma 5.10. *If $L \in \mathcal{L}$ then either $L \in \mathcal{Y}_i$ with $0 \leq i \leq 3$ or $L \in \mathcal{X}$ according to Table 5. Moreover, if $L \in \mathcal{Y}_3$ then $t - 4 \in T(L)$. If $L \in \mathcal{X}$, then $\{t-1, t-2\} \cap T(L) = \{t-2\}$.*

Proof. Let $L \simeq O^-_{2n}(q)$. Assume that $n = 4k$, where k is integer and $k \geq 5$. Then by Table 1, we have that $t(L) = 3k + 1$. Lemma 5.9 implies that $t - 3 \notin T(L)$. To prove that $L \in \mathcal{Y}_3$ it is sufficient to show that $t-1, t-2 \in T(L)$. Moreover, we need also show that $t - 4 \in T(L)$. Using Table 1, we obtain that $r_{2k-1}(q)$, $r_{4k-4}(q)$, and $r_{2k-3}(q)$ are small with respect to L. Since $k \geq 5$, we have that $2k-3 = \eta(2k-3) > 4k/3 = n/3$, so $2k-1, 4k-4, 2k-3 \in M(L)$. By Lemma 5.9, we have that $t(r_{2k-1}(q), L) = 3k = t - 1$, $t(r_{4k-4}(q), L) = 3k - 1 = t - 2$, and $t(r_{2k-3}(q), L) = 3k - 3 = t - 4$, as required. Assume that $n = 4k + 1$, then $k \geq 5$ and $t(L) = 3k + 1$ by Table 1. To prove that $L \in \mathcal{Y}_0$, we need to show that $t - 1$, $t - 2$, and $t - 3$ are in $T(L)$. Using Table 1, we obtain that $4k - 2$, $2k - 1$, and $4k - 4$ are small with respect to L. Since $k \geq 5$, we have that $\eta(4k-4) > (4k+1)/3 = n/3$, and hence $4k-2,\ 2k-1,\ 4k-4 \in M(L)$. Now Lemma 5.9 implies that $t(r_{4k-2}(q), L) = 3k = t-1$, $t(r_{2k-1}(q), L) = 3k-1 = t-2$, and $t(r_{4k-4}(q), L) = 3k-2 = t-3$, as required. Assume that $n = 4k+2$. Then $k \geq 4$ and $t(L) = 3k+2$ by Table 1. Note that $k = 4$ corresponds to $L \simeq O_{36}^-(q)$. Lemma 5.9 implies that $t-1 \notin M(L)$. So to prove that $L \in \mathcal{Y}_1$ we need to show that $t-2, t-3 \in T(L)$. Note that $r_{2k-1}(q)$ and $r_{4k-4}(q)$ are small with respect to L by Table 1. If $k > 4$, we have that $2k-2 = \eta(4k-4) > (4k+2)/3 = n/3$, so $4k-4,\ 2k-1 \in M(L)$. In the case $k = 4$ we have that $2k-1 = \eta(2k-1) > (4k+2)/3 = n/3$, so $2k-1 \in M(L)$. Lemma 5.9 implies that $t(r_{2k-1}(q), L) = 3k = t-2$, $t(r_{4k-4}(q), L) = 3k-1 = t-3$, as required. Assume that $n = 4k+3$. By Table 1, we have that $k \geq 4$, $t(L) = 3k+3$. To prove that $L \in \mathcal{Y}_0$, we need to show that $t-1, t-2, t-3 \in M(L)$. Using Table 1, we obtain that $r_{2k+1}(q)$, $r_{4k}(q)$, and $r_{4k-2}(q)$ are small with respect to L and since $\eta(4k-2) = 2k-1 > (4k+3)/3 = n/3$, we obtain that $4k,\ 4k-2,\ 2k+1 \in M(L)$. Lemma 5.9 implies that $t(r_{2k+1}(q), L) = 3k+2 = t-1$, $t(r_{4k}(q), L) = 3k+1 = t-2$, and $t(r_{4k-2}(q), L) = 3k = t-3$, as required.

All other cases for L are considered similarly. If $L \in \mathcal{Y}_i$ in Table 5 then we have that $t-i \notin T(L)$ by Lemmas 5.7, 5.8, 5.9. The last column of Table 5 contains elements of $\{t-1, t-2, t-3\} \cap T(L)$ if L is not in $\mathcal{Y}_3$ and $t-1$, $t-2$, $t-4$ otherwise. □

TABLE 5. Description of classes $\mathcal{X}$ and $\mathcal{Y}_i$ with $0 \leq i \leq 3$

L	class	$t(L)$	elements of $T(L)$
$L_n^\varepsilon(q)$, $n = 2k$, $k \geq 14$	$\mathcal{Y}_0$	k	$t(r_{k-1}, L) = k-1$, $t(r_{k-2}, L) = k-2$, $t(r_{k-3}, L) = k-3$
$L_n^\varepsilon(q)$, $n = 2k+1$, $k \geq 13$	$\mathcal{Y}_0$	$k+1$	$t(r_k, L) = k$, $t(r_{k-1}, L) = k-1$, $t(r_{k-2}, L) = k-2$
$S_{2n}(q), O_{2n+1}(q)$, $n = 4k$, $k \geq 5$	$\mathcal{Y}_1$	$3k+1$	$t(r_{2k-1}, L) = 3k-1$, $t(r_{4k-4}, L) = 3k-2$
$S_{2n}(q), O_{2n+1}(q)$, $n = 4k+1$, $k \geq 4$	$\mathcal{Y}_3$	$3k+2$	$t(r_{4k}, L) = 3k+1$, $t(r_{2k-1}, L) = 3k$, $t(r_{4k-4}, L) = 3k-2$
$S_{2n}(q), O_{2n+1}(q)$, $n = 4k+2$, $k \geq 4$	$\mathcal{Y}_2$	$3k+2$	$t(r_{4k}, L) = 3k+1$, $t(r_{2k-1}, L) = 3k-1$
$S_{2n}(q), O_{2n+1}(q)$, $n = 4k+3$, $k \geq 4$	$\mathcal{Y}_1$	$3k+3$	$t(r_{4k}, L) = 3k+1$, $t(r_{2k-1}, L) = 3k$
$O_{2n}^+(q)$, $n = 4k$, $k \geq 5$	$\mathcal{Y}_2$	$3k$	$t(r_{2k-1}, L) = 3k-1$, $t(r_{4k-4}, L) = 3k-3$
$O_{2n}^+(q)$, $n = 4k+1$, $k \geq 5$	$\mathcal{Y}_0$	$3k+1$	$t(r_{2k-1}, L) = 3k$, $t(r_{4k-2}, L) = 3k-1$, $t(r_{4k-4}, L) = 3k-2$
$O_{2n}^+(q)$, $n = 4k+2$, $k \geq 5$	$\mathcal{Y}_3$	$3k+1$	$t(r_{4k}, L) = 3k$, $t(r_{2k-1}, L) = 3k-1$, $t(r_{4k-4}, L) = 3k-3$
$O_{2n}^+(q)$, $n = 4k+3$, $k \geq 4$	$\mathcal{Y}_0$	$3k+3$	$t(r_{4k+2}, L) = 3k+2$, $t(r_{4k}, L) = 3k+1$, $t(r_{2k-1}, L) = 3k$
$O_{2n}^-(q)$, $n = 4k$, $k \geq 5$	$\mathcal{Y}_3$	$3k+1$	$t(r_{2k-1}, L) = 3k$, $t(r_{4k-4}, L) = 3k-1$, $t(r_{2k-3}, L) = 3k-3$
$O_{2n}^-(q)$, $n = 4k+1$, $k \geq 5$	$\mathcal{Y}_0$	$3k+1$	$t(r_{4k-2}, L) = 3k$, $t(r_{2k-1}, L) = 3k-1$, $t(r_{4k-4}, L) = 3k-2$
$O_{2n}^-(q)$, $n = 4k+2$, $k \geq 5$	$\mathcal{Y}_1$	$3k+2$	$t(r_{2k-1}, L) = 3k$, $t(r_{4k-4}, L) = 3k-1$
$O_{36}^-(q)$	$\mathcal{X}$	14	$t(r_7, L) = 12$
$O_{2n}^-(q)$, $n = 4k+3$, $k \geq 4$	$\mathcal{Y}_0$	$3k+3$	$t(r_{2k+1}, L) = 3k+2$, $t(r_{4k}, L) = 3k+1$, $t(r_{4k-2}, L) = 3k$

As a consequence we obtain that $\mathcal{L} = \mathcal{Y}_0 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3 \cup \mathcal{X}$.

Lemma 5.11. *If $S \in \mathcal{Y}_i$ with $1 \leq i \leq 3$, then $L \in \mathcal{Y}_i \cup \mathcal{X}$. If $L \in \mathcal{X}$, then $S \notin \mathcal{Y}_2$. In particular, if $L \in \mathcal{Y}_0$, then $S \in \mathcal{Y}_0 \cup \mathcal{X}$.*

Proof. Assume that $S \in \mathcal{Y}_i$, where $i > 0$. Then $t-i \notin T(S)$. Let $L \notin \mathcal{Y}_i$ and $L \not\simeq O_{36}^-(q)$. Then $t-i \in T(L)$ by Lemma 5.10. So there exists $r \in \pi(L)$ such that $e(r,q) \in M(L)$ and $t(r, L) = t-i$. Lemma 5.5 implies that $t(r, S) = t-i$. Since $S \in \mathcal{Y}_i$, we have that $e(r, S) \leq m/3$. By Lemmas 5.7, 5.8, 5.9, we have that

$t(r,S) < t(s,S)$ for any $t(s,S) \in T(S)$. Lemma 5.10 implies that the minimal element in $T(S)$ is less than $t-i$. So $t(r,S) < t-i$; a contradiction.

Let $L \in \mathcal{X}$, so $L \simeq O^-_{36}(q)$. Assume that $S \in \mathcal{Y}_2$ and so $t-2 \notin T(S)$. By Lemma 5.10, there exists $r \in \pi(L)$ such that $e(r,q) \in M(L)$ and $t(r,L) = t-2$. Lemma 5.5 implies that $t(r,S) = t-2$. Therefore we have that $e(r,S) \leq m/3$. As above we have that $t(r,S) < t(s,S)$, where $t(s,S) \in T(S)$. So $t(r,S) < t-2$; a contradiction. □

Lemma 5.12. *If $L \in \mathcal{L}$ and $L \simeq L^\varepsilon_n(q)$, where q is odd, then $S \simeq L$.*

Proof. Assume the contrary and let at first $S \simeq L^\tau_m(u)$, where $\tau \in \{+,-\}$ and $(n,\varepsilon,q) \neq (m,\tau,u)$. Proposition 5.2 yields the equality $t(L) = t(S)$ and so by Table 1, we have that $m \in \{n-1, n, n+1\}$. Suppose that $m = n-1$. If n is odd then $t(S) \neq t(L)$ by Table 1. Therefore n is even. Using Table 1, we have that $|J(L)| = t(L)+1$ and $|J(S)| = t(S) = t(L)$. By Lemma 5.5, we obtain that $t(r_n(\varepsilon q), S) = t(r_{n/2}(\varepsilon q), S) = t(S)$. Therefore primes $r_n(\varepsilon q)$ and $r_{n/2}(\varepsilon q)$ are large with respect to S and adjacent in $GK(S)$. Since $|J(S)| = t(S)$, we have that $e(r_n(\varepsilon q), \tau u) = e(r_{n/2}(\varepsilon q), \tau u)$. Observe that $r_{n/2-1}(\varepsilon q)$ is adjacent to $r_{n/2}(\varepsilon q)$ and nonadjacent to $r_n(\varepsilon q)$ in $GK(L)$. By Lemmas 5.5, 5.6, we have that $r_{n/2-1}(\varepsilon q) \in R_k(\tau u)$ for some k and let $r_{n/2}(\varepsilon q) \in R_l(\tau u)$. Since $r_{n/2-1}(\varepsilon q)$, $r_{n/2}(\varepsilon q)$, $r_n(\varepsilon q)$ are coprime to $|\overline{G}/S| \cdot |K|$, we obtain that $r_{n/2-1}(\varepsilon q)$ is adjacent to $r_{n/2}(\varepsilon q)$ and is nonadjacent to $r_n(\varepsilon q)$ in $GK(S)$. However, existence of an edge between representatives from $R_k(\tau u)$ and $R_l(\tau u)$ depends only on k and l but not on representatives; a contradiction. Thus we have that $m \geq n$.

Put $i = 12$ if $n \leq 32$ and $i = 16$ otherwise. Note that $n/3 < i < n/2$, and hence $t(r_i(\varepsilon q), L) < t(L)$. By Lemmas 5.5, 5.6, we obtain that $i = t(r_i(\varepsilon q), L) = t(r_i(\varepsilon q), S)$, and hence $R_i(\varepsilon q) \subseteq R_i(\tau u)$. So $k_i(\varepsilon q)$ divides $k_i(\tau u)$. If $i = 16$ then Lemma 2.5 yields that $8q^8 \leq u^8$. Let $i = 12$. Since $q \neq u$, we have that $k_{12}(q) \neq k_{12}(u)$. So $k_i(\varepsilon q)$ is a proper divisor of $k_i(\tau u)$. By Fermat's little theorem, we obtain that $13(q^4 - q^2 + 1) \leq u^4 - u^2 + 1$. Since $q^4 - q^2 + 1 > q^4/2$, we have that $6q^4 < u^4$, and hence $36q^8 < u^8$. Thus in the all cases it is true that $8q^8 < u^8$. Lemma 3.9 yields that some element x from $m(S)$ divides some element y from $m(L)$. By Lemma 3.8, we obtain that $(u^{m-1} - 1)/(2m) \leq x \leq y \leq 2q^{n-1}$. So $u^{m-1} - 1 \leq 4mq^{n-1}$. Since $n \leq m \leq n+1$ and $n \leq 44$, we have that $u^{n-1} \leq 4(n+1)q^{n-1} + 1 < 181q^{n-1}$. On the other hand, we have that $8q^8 < u^8$, and hence $8^{(n-1)/8}q^{n-1} < u^{n-1} < 181q^{n-1}$; a contradiction, since $8^{(n-1)/8} > 8^3 > 181$.

Assume that S is an orthogonal or symplectic group. Lemma 5.11 implies that either $S \in \mathcal{X}$ or $S \in \mathcal{Y}_0$. If $S \in \mathcal{X}$ then $T(S) \cap \{t-1, t-2\} = \{t-2\}$ due to Lemma 5.10. Considering an element $r \in \pi(L)$ such that $t(r,L) = t-1$ we obtain a contradiction similarly as in Lemma 5.11. So $S \in \mathcal{Y}_0$. By Table 5, we have that $S \simeq O^\tau_{2m}(u)$, where m is odd. Now we prove that n is odd. Assume the contrary and n is even. Then $|J(L)| = t(L)+1$. By Table 1, if $m \equiv 3 \pmod 4$ then $|J(S)| = t(S)$ and this case is considered similarly as the case $S = L^\varepsilon_m(u)$ and $m = n-1$ above. If $m \equiv 1 \pmod 4$ then $|J(S)| = t(S)+1 = |J(L)|$. The similar arguments with $r_{n/2-1}(\varepsilon q)$ as above show that $e(r_n(\varepsilon q), S) \neq e(r_{n/2}(\varepsilon q), S)$. Since $r_n(\varepsilon q)$, $r_{n/2}(\varepsilon q)$ are large with respect to S and adjacent in $GK(S)$, by Table 1, we obtain that either $R_n(q) \subseteq R_{m-1}(\tau u)$ and $R_{n/2}(q) \subseteq R_{m+1}(\tau u)$ or $R_n(q) \subseteq R_{m+1}(\tau u)$ and $R_{n/2}(q) \subseteq R_{m-1}(\tau u)$. Let $k = e(r_{n/2-1}(\varepsilon q), S)$. Then $r_k(u)$ is small with respect to S. Therefore we obtain that $\eta(k) \leq (m-3)/2$ by Table 1. Lemma 3.2 implies that $r_k(u)$ is adjacent to both $r_{m-1}(\tau u)$ and $r_{m+1}(\tau u)$; a contradiction, since $r_{n/2-1}(\varepsilon q)$ and $r_n(\varepsilon q)$ are nonadjacent in $GK(L)$. Thus n is odd. Observe that if n is a prime then by Lemma 3.2, a prime $r_n(\varepsilon q)$ is nonadjacent to $r_i(\varepsilon q)$, where $i \neq n$. So $GK(L)$ is disconnected and $R_n(\varepsilon q)$ is a connected component. Therefore $GK(S)$ is also disconnected. The classification of nonabelain finite simple groups a with disconnected prime graph was obtained in [7], [22]. The full list of nonabelian simple groups with a disconnected prime graph can be found, e.g., in [8, Tables 1-3]. In particular, we obtain that if $GK(O^{\tau}_{2m}(u))$ with odd m is disconnected, then either m is a prime or $m = 2^t + 1$ for some integer t. Sorting out odd values of n and using that $t(L) = t(S)$, we obtain that only the pairs $(n, m) = (29, 19)$, $(35, 23)$, and $(43, 29)$ are possible.

Let $(n, m) = (35, 23)$. By Lemmas 5.6, 5.8, 5.9, we have that $t(r_{17}(\varepsilon q), L) = 17 = R_{11}(-\tau u)$, and hence $R_{17}(\varepsilon q) \subseteq R_{11}(-\tau u)$. Observe that $r_{17}(\varepsilon q)$ and $r_{34}(\varepsilon q)$ are adjacent in $GK(L)$. Since $r_{34}(\varepsilon q)$ is large with respect to L and $|J(O^{\tau}_{46}(u))| = t(O^{\tau}_{46}(u))$, we have that $t(r_{34}(\varepsilon q), S) = t(S)$ and $R_{34}(\varepsilon q) \subseteq R_i(u)$ for some i such that $\eta(i) \geq 11$ due to Table 1. Then $r_i(u)$ is adjacent with $r_{11}(-\tau u)$. By the adjacent criterion, we obtain that $22 + 2\eta(i) \leq 46$, and hence $\eta(i) \leq 12$. Observe that $r_{18}(\varepsilon q)$ is large with respect to L, adjacent with $r_{17}(\varepsilon q)$ in $GK(L)$, and nonadjacent with $r_{34}(\varepsilon q)$. So $r_{34}(\varepsilon q) \notin R_{11}(-\tau u)$. Since $\eta(i) \geq 11$, we have that either $r_{34}(\varepsilon q) \in R_{11}(\tau u)$ or $r_{34}(\varepsilon q) \in R_{24}(u)$. Note that $r_{11}(\tau u)$ and $r_{24}(u)$ are adjacent with $r_9(u)$ in $GK(S)$, so $r_{34}(\varepsilon q)$ is adjacent with $r_9(u)$ in $GK(L)$. Observe that $9 \in M(S)$, therefore by Lemmas 5.8, 5.9, we have that $t(r_9(u), S) \in \{14, 15\}$. Since $14, 15 \in M(L)$ and $t(r_{14}(\varepsilon q)) = 14$, $t(r_{15}(\varepsilon q)) = 15$, we obtain that either $R_{14}(\varepsilon q) \subseteq R_9(u)$ or $R_{15}(\varepsilon q) \subseteq R_9(u)$. So we can choose $r_9(u) \in R_{14}(\varepsilon q) \cup R_{15}(\varepsilon q)$. We get a contradiction, since $r_{34}(\varepsilon q)$ is adjacent to neither $r_{14}(\varepsilon q)$ nor $r_{15}(\varepsilon q)$ in $GK(L)$.

Assume that $(n, m) = (29, 19)$ or $(43, 29)$. Recall that $GK(L)$, $GK(S)$ are disconnected. Moreover, it is showed in [8, Lemma 4] that $R_n(q)$ and $R_m(\tau u)$ are connected components that do not contain 2 in $GK(L)$ and $GK(S)$ respectively. So $k_n(\varepsilon q) = k_m(\tau u)$. Let $\tau = +$. Then by [8, Table 1], we have that $k_n(\varepsilon q) = \frac{q^n - \varepsilon 1}{(q-\varepsilon)(q-\varepsilon 1, n)}$, $k_m(u) = \frac{u^m - 1}{u-1}$, and $u \in \{2, 3, 5\}$. Let r be a prime divisor of $k_m(u)$. Then $r \in R_n(\varepsilon q)$, and hence $r \equiv 1 \pmod{n}$. So $k_m(u) \equiv 1 \pmod{n}$, and hence $u^m - u$ is divisible by n. It is not hard to check that n does not divide $u^m - u$ under our assumptions. Let $\tau = -$. Then by [8, Table 1], we obtain that $k_m(-u) = (3^m + 1)/4$. By the similar arguments as in the previous case, we have that $k_m(-u) - 1$ is divisible by n, what is not true under our assumptions; a contradiction. □

Lemma 5.13. *Let $L \in \mathcal{L}$ and q is odd.*

(i) If $L \simeq O^{\varepsilon}_{2n}(q)$, $S \simeq O^{\tau}_{2m}(u)$ then $n - m$ is even.

(ii) If $L \in \{O_{2n+1}(q), S_{2n}(q)\}$ then $S \not\simeq O^{\varepsilon}_{2m}(u)$.

(iii) If $L \simeq O^{\varepsilon}_{2n}(q)$ then $S \notin \{O_{2m+1}(u), S_{2m}(u)\}$.

Proof. If $L \simeq O^{\varepsilon}_{2n}(q)$, where n is odd, then $L \in \mathcal{Y}_0$. Moreover, by Table 1, we have that $t(L) \geq 15$, so $S \notin \mathcal{X}$. By Lemma 5.11, we obtain that $S \in \mathcal{Y}_0$. Table 5 yields that $S \not\simeq O^{\tau}_{2m}(u)$, where m is even, and $S \notin \{O_{2n+1}(u), S_{2n}(u)\}$. Let $L \in \{O_{2n+1}(q), S_{2n}(q)\}$ and $S \simeq O^{\varepsilon}_{2m}(u)$, where m is even. Since $t(L) = t(S)$, we get a contradiction with Table 5 and Lemma 5.11. By the same reasons, we get a contradiction if $L \simeq O^{\varepsilon}_{2n}(u)$, where n is even, $(n, \varepsilon) \neq (18, -)$, and $S \in \{O_{2m+1}(u), S_{2m}(u)\}$. Assume that $L = O^{-}_{36}(q)$ and $S \in \{O_{2m+1}(u), S_{2m}(u)\}$. Then Lemma 5.11 yields that $S \notin \mathcal{Y}_2$. Since $t(S) = 14$, we have that $r_9(q)$, $r_{16}(q)$, $r_{18}(q)$ are large with respect to L and pairwise adjacent in $GK(L)$. Lemma 5.5 yields that $r_9(q)$, $r_{16}(q)$, $r_{18}(q)$ are large with respect to S. By Table 1, we have that $J(S) = E(S)$. So there exists i such that $r_9(q)$, $r_{16}(q)$, $r_{18}(q) \in R_i(u)$. Therefore $r_9(q) \cdot r_{16}(q) \cdot r_{18}(q)$ divides $k_i(u)$, so $r_9(q) \cdot r_{16}(q) \cdot r_{18}(q) \in \omega(L)$. The spectrum description of $O^{-}_{36}(q)$ with odd q contains in [2, Corollaries 8-9], in particular, we have that $r_9(q) \cdot r_{16}(q) \cdot r_{18}(q) \notin \omega(O^{-}_{36}(q))$; a contradiction. Thus (iii) follows.

Assume that $S \simeq O^{\tau}_{2m}(u)$, where m is odd, $\tau \in \{+, -\}$, and either $L \in \{O_{2n+1}(q), S_{2n}(q)\}$ or n is even and $L \simeq O^{\pm}_{2n}(q)$. By Lemma 3.9, we have that there exist $x \in m(S)$ and $y \in m(L)$ such that x divides y. Moreover, Lemma 3.8 yields that $u^m/8 \leq x$ and since q is odd, $y \leq q^n$. Proposition 5.2 implies that $t(S) = t(L)$ and using Table 1 we obtain that $n - 1 \leq m \leq n + 1$. Whence we have the inequality $u^{n-1} \leq 8q^n$.

Now we prove that $q^2 < 2u$. Since $L \in \mathcal{L}$, we have that $n \leq 30$. Assume that $n \leq 20$. Then by Table 1, we have that $7, 14 \in M(L)$. Using Lemmas 5.7, 5.8, 5.9, we obtain that $t(r_7(q), L) = t(r_{14}(q), L)$ and either $t(r_7(q), L) = 11$ or $t(r_7(q), L) = 12$. Lemma 5.5 implies that $t(r_7(q), L) = t(r_7(q), S)$ and $t(r_{14}(q), L) = t(r_{14}(q), S)$. Observe that by Lemmas 5.8, 5.9, if s is a prime and $t(s, S) = 11$ then $e(s, u) = 14$ in the case $\tau = +$, and $e(s, u) = 7$, when $\tau = -$. Similarly, we have that $t(s, S) = 12$ is equivalently to $e(s, u) = 7$, when $\tau = +$, and $e(s, u) = 14$, when $\tau = -$. So either $R_7(q), R_{14}(q) \subseteq R_7(u)$ or $R_7(q), R_{14}(q) \subseteq R_{14}(u)$. Therefore $k_7(q)k_{14}(q)$ divides either $k_7(u)$ or $k_{14}(u)$. Observe that $k_7(q)k_{14}(q) = \frac{q^{14}-1}{(q^2-1)(q-1,7)(q+1,7)} \geq (q^{12}+q^{10}+...+q^2+1)/7 \geq q^{12}/7$. It is clear that both numbers $k_7(u)$ and $k_{14}(u)$ are not greater than $u^6+u^5+u^4+u^3+u^2+u+1$. So $q^{12}/7 < 2u^6$, and hence $q^{12} < 14u^6$. Since $2^6 > 14$, we obtain the required inequality $q^2 < 2u$. Assume that $20 < n \leq 26$. Then by Table 1, we have that $9, 18 \in M(L)$. Arguing as above, we obtain that $k_9(q)k_{18}(q)$ divides either $k_9(u)$ or $k_{18}(u)$. It is clear that $k_9(u) \leq u^6 + u^3 + 1 < 2u^6$ and $k_{18}(u) \leq u^6 - u^3 + 1 < 2u^6$. Moreover, $k_9(q)k_{18}(q) = \frac{(q^6+q^3+1)(q^6-q^3+1)}{(q-1,3)(q+1,3)} \geq (q^{12} + q^6 + 1)/3 \geq q^{12}/3$. So we have that $q^{12} < 6u^6$, and hence $q^2 < 2u$, as claimed. Let finally $27 \leq n \leq 30$. Then by Table 1, we have that $11, 22 \in M(L)$. Arguing as above, we obtain that $k_{11}(q)k_{22}(q)$ divides either $k_{11}(u)$ or $k_{22}(u)$. It is clear that $k_{11}(u) < 2u^{10}$ and $k_{22}(u) < 2u^{10}$. Moreover, $k_{11}(q)k_{22}(q) = \frac{q^{22}-1}{(q^2-1)(q-1,11)(q+1,11)} \geq q^{20}/11$. So we have that $q^{20} < 22u^{10}$, and hence $q^2 < 2u$, as claimed. Now we prove that obtained inequalities $u^{n-1} \leq 8q^n$ and $q^2 < 2u$ contradict to each other. Observe that $q^2 < 2u$ is equivalent to $q^{2n-2} < 2^{n-1}u^{n-1}$. Therefore we have that $q^{2n-2} < 2^{n+2}q^n$, and hence $q^{n-2} < 2^{n+2}$. Since $q \geq 3$, we have $q^{n-2} \geq 3^{n-9}3^7 > 2^{n-9}2^{11} > 2^{n+2}$; a contradiction. The lemma is proved. □

Lemma 5.14. *Let $L \in \mathcal{L}$ and $L \in \{O_{2n+1}(q), S_{2n}(q), O^{\pm}_{2n}(q)\}$ with odd q. Then $S \not\simeq L^{\tau}_m(u)$, where $\tau \in \{+,-\}$.*

Proof. Assume the contrary and let $S \simeq L^{\tau}_m(u)$, where $\tau \in \{+,-\}$. Since $t(S) \leq 22$, we have that $m \leq 44$. Let at first $L \in \{O_{2n+1}(q), S_{2n}(q)\}$ or n is even and $L \simeq O^{\pm}_{2n}(q)$. Lemma 3.9 implies that for some $x \in m(S)$, $y \in m(L)$ it is true that x divides y. By Lemma 3.8, we have that $x \geq (u^{m-1}-1)/(2m)$ and $y \leq 2q^n$. Therefore $u^{m-1} \leq 4mq^n + 1 \leq 180q^n$.

Now we prove that $q < u$. Observe that $17 \leq n \leq 30$ and by Table 1, we have that $\{7, 9, 11\} \cap M(L) \neq \varnothing$. Let i be the minimal element from this intersection and note that $2i \in M(L)$. Let $j = t(r_i(q), L)$. Then Lemmas 5.7, 5.8, 5.9 imply that $j \in \{11, 12\}$, $j \in \{14, 15\}$, or $j \in \{17, 18\}$, if $i = 7, 9$ or 11 respectively. Lemma 5.6 yields that $t(r, S) = j$ is equivalent to $r \in R_j(\tau u)$. So $R_i(q)$, $R_{2i}(q) \subseteq R_j(\tau u)$, and hence $k_i(q)k_{2i}(q)$ divides $k_j(\tau u)$. If $i = 7$ and $d = (q-1, 7)(q+1, 7)$, then $k_7(q)k_{14}(q) = \frac{q^{14}-1}{d(q^2-1)} \geq q^{12}/d$ and $k_j(\tau u) \leq 2u^{10}$. So $q^{12} \leq 2du^{10}$. If $d = 1$ then since $q^2 > 2$, we obtain that $q < u$. If $d > 1$ then $d = 7$, and hence $q > 7$. In this case $q^2 > 2d$, and hence again $q < u$. If $i = 9$ then $k_9(q)k_{18}(q) \geq (q^{12} + q^6 + 1)/3 \geq q^{12}/3$ and $k_j(\tau u) \leq u^8$. So $q^{12} \leq 3u^6$, and hence $q < u$. If $i = 11$ then then $k_{11}(q)k_{22}(q) \geq \frac{q^{22}-1}{11(q^2-1)} \geq q^{20}/11$ and $k_j(\tau u) \leq 2u^{16}$. So $q^{20} \leq 22u^{16}$, and hence $q < u$. Thus we have the inequality $q < u$ in all cases. We proved above that $u^{m-1} < 180q^n$. So $q^{m-1} < 180q^n$, and hence $q^{m-n-1} < 180$. Since $t(S) = t(L)$, it is not hard to check using Table 1 that $m - n > 5$, in particular $q^{m-n-1} \geq q^5 \geq 3^5 = 243$; a contradiction.

Consider now the case $L \simeq O^{\varepsilon}_{2n}(q)$, where n is odd and $\varepsilon \in \{+,-\}$. First we prove that $q \geq 7$. Assume that $q < 7$. Then since q is odd, we have that either $q = 3$ or $q = 5$. If $n < 24$ then by Table 1, we have that $16 \in M(L)$. Lemma 5.5 implies that $t(r_{16}(q), L) = t(r_{16}(q), S)$. By Lemmas 5.8, 5.9, we obtain that $t(r_{16}(q), L) = 13$ and so $r_{16}(q) \in R_{13}(\tau u)$ due to Lemma 5.6. Observe that $17 \in R_{16}(3)$ and $17 \in R_{16}(5)$. So $17 \in R_{13}(\tau u)$; a contradiction with Lemma 3.3. If $n \geq 24$ then by Table 1 we have that $11, 22 \in M(L)$. Lemma 5.5 implies that $t(r_{11}(q), L) = t(r_{11}(q), S)$ and $t(r_{22}(q), L) = t(r_{22}(q), S)$. By Lemmas 5.8, 5.9, we obtain that

$t(r_{11}(q),L), t(r_{22}(q),L) \in \{17,18\}$ and so $r_{11}(q), r_{22}(q) \in R_{17}(\tau u) \cup R_{18}(\tau u)$ due to Lemma 5.6. Observe that $23 \in R_{11}(3)$ and $23 \in R_{22}(5)$. So either $23 \in R_{17}(\tau u)$ or $23 \in R_{18}(\tau u)$; a contradiction with Lemma 3.3. Thus we have that $q \geq 7$.

Now Lemma 3.8 yields that there exist $x \in m(S)$, $y \in m(L)$ such that x divides y. Since q is odd, by Lemma 3.8 we obtain that $y \leq q^n$. Moreover, we have that $\frac{u^{m-1}-\tau 1}{2(m,u-\tau 1)} \leq x$. Observe that $(u+1)u^{m-3} < u^{m-1}-1$, and hence $u^{m-3}/2 < \frac{u^{m-1}-\tau 1}{2(m,u-\tau 1)}$. So $u^{m-3} < 2q^n$. Now we prove that $q^6 < 7u^6$. Let at first $n < 27$. Then by Table 1, we have that $18 \in M(L)$ if $\varepsilon = +$ and $9 \in M(L)$ if $\varepsilon = -$. Lemmas 5.8, 5.9 yield that $t(r_9(-\varepsilon q),L) = 14$. So $R_9(-\varepsilon q) \subseteq R_{14}(\tau u)$ due to Lemma 5.6, and hence $k_9(-\varepsilon q)$ divides $k_{14}(\tau u)$. Observe that $k_9(-\varepsilon q) = (q^6-\varepsilon q^3+1)/(q+\varepsilon 1,3)$ and $k_{14}(\tau u)$ divides $u^6-\tau u^5+u^4-\tau u^3+u^2-\tau u+1$. If $q \leq u$ then the inequality $q^6 < 7u^6$ obviously holds. Let $q \geq u+1$. Then $q^6-\varepsilon q^3+1 > (u+1)^6-\varepsilon(u+1)^3+1 > u^6-\tau u^5+u^4-\tau u^3+u^2-\tau u+1$. So $k_{14}(\tau u)/k_9(-\varepsilon q) < 3$. Since $k_9(-\varepsilon q)$ is odd, we obtain that $q^6-\varepsilon q^3+1 = 3(u^6-\tau u^5+u^4-\tau u^3+u^2-\tau u+1)$. Since $q \geq 7$, we have that $q^6-\varepsilon q^3+1 > 6/7q^6$. On the other hand, $u^6-\tau u^5+u^4-\tau u^3+u^2-\tau u+1 < 2u^6$. So $6/7q^6 < 6u^6$, and hence $q^6 < 7u^6$, as required. Let now $n \geq 27$. Then by Table 1, we have that $11 \in M(L)$ if $\varepsilon = +$ and $22 \in M(L)$ if $\varepsilon = -$. Lemmas 5.8, 5.9 yield that $t(r_{11}(\varepsilon q),L) = 18$. So $R_{11}(\varepsilon q) \subseteq R_{18}(\tau u)$, and hence $k_{11}(\varepsilon q)$ divides $k_{18}(\tau u)$. Observe that $k_{11}(\varepsilon q) \geq \frac{(q^{11}+1)}{11(q+1)} > q^{10}/22$ and $k_{18}(\tau u) < u^6+u^3+1 < 2u^6$. Therefore we have that $q^{10} < 44u^6$, and hence $q^6 < 7u^6$ due to $q \geq 7$. Thus we have that $q^6 < 7u^6$ in all cases.

Now we prove that the obtained inequalities $u^{m-3} \leq 2q^n$ and $q^6 < 7u^6$ contradict to each other. The inequality $q^6 < 7u^6$ is equivalent to $q^{m-3} < 7^{(m-3)/6}u^{m-3}$. So $q^{m-3} < 2\cdot 7^{(m-3)/6}q^n$ or equivalently $q^{m-n-3} < 2\cdot 7^{(m-3)/6}$. Therefore $q^{m-n-3} < 7^{(m+3)/6}$. We have that $q \geq 7$ and so to obtain a contradiction it is sufficient to show that $6(m-n-3)/(m+3) > 1$. This inequality is equivalent to $5(m-3) > 6n+6$. By Table 1, we have that $(m+1)/2 \geq t(S)$ and $t(L) \geq (3n+1)/4$. Therefore $(m+1)/2 \geq (3n+1)/4$, and hence $2m+1 \geq 3n$. So $5(m-3) = 2(2m+1)+m-17 > 6n+6$ due to $m \geq 27$ by Table 1. Thus $5(m-3) > 6n+6$ and as was pointed above this inequality leads to a contradiction. The lemma is proved. □

Lemma 5.15. *Assume that $L \in \mathcal{L}$ and $L \in \{O_{2n+1}(q), S_{2n}(q)\}$ with odd q. Then $S \in \{O_{2n+1}(q), S_{2n}(q)\}$.*

Proof. By Lemmas 5.13, 5.14, we have that $S \not\simeq O^{\pm}_{2m}(u)$ and $S \not\simeq L^{\pm}_m(u)$. So $S \in \{O_{2m+1}(u), S_{2m}(u)\}$. Proposition 4.3, Lemma 5.10, and Table 5 imply that $n = m$.

By Lemma 3.9, we obtain that for some $x \in m(S)$, $y \in m(L)$ it is true that x divides y. Lemma 3.8 implies that $y \leq (q^n+1)/2$ and $(u^n-1)/2 \leq x$. Therefore $u^n-2 \leq q^n$.

Assume that $17 \leq n \leq 23$. Then by Table 1, we obtain that $16 \in M(L)$. Lemmas 5.5, 5.7 yield that $t(r_{16}(q),L) = t(r_{16}(q),S)$ and $t(r_{16}(q),L) = 13$. Therefore $k_{16}(q)$ divides $k_{16}(u)$. Lemma 2.5 implies that $8q^8 < u^8$. We rewrite this as $8^{n/8}q^n \leq u^n$ and since $n \geq 17$, we obtain that $8q^n < u^n$. On the other hand, we have that $u^n \leq q^n+2$; a contradiction.

Assume that $24 \leq n \leq 26$. Then $9, 18 \in M(L)$. Lemma 5.7 implies that $t(r_9(q),L) = t(r_{18}(q),L)$. Since $t(r_9(q),L) = t(r_9(q),S)$ and $t(r_{18}(q),L) = t(r_{18}(q),S)$, we have that $R_9(q) \cup R_{18}(q) \subseteq R_9(u) \cup R_{18}(u)$. So $k_9(q)k_{18}(q)$ divides $k_9(u)k_{18}(u)$. Lemma 2.5 implies that $u^{12} > 3q^{12}$. So $u^n > 3^{n/12}q^n \geq 9q^n$. However, we obtained above that $u^n-2 \leq q^n$; a contradiction.

Assume finally that $n = 27, 28$. Note that $24 \in M(L)$. Lemma 5.7 implies that $t(r_{24}(q),L) = t(r_{24}(q),S)$. So $R_{24}(q) \subseteq R_{24}(u)$ and $k_{24}(q)$ divides $k_{24}(u)$. Lemma 2.5 implies that $u^8 > 12q^8$, and hence $u^n > 12^{n/8}q^n > 12^3q^n$; a contradiction with the inequality $u^n \leq q^n+2$. The lemma is proved. □

Lemma 5.16. *Let $L \in \mathcal{L}$ and q is odd. If $L = O^+_{2n}(q)$, where n is even, then $S \in \{O^+_{2n}(q), O^-_{2n-4}(q)\}$ and if $L = O^-_{2n}(q)$, where n is even, then $S \in \{O^-_{2n}(q), O^+_{2n+4}(q)\}$.*

Proof. Let $L = O^\varepsilon_{2n}(q)$, where n is even and $\varepsilon \in \{+,-\}$. Lemmas 5.13, 5.14 yield that $S \simeq O^\tau_{2m}(u)$, where m is even and $\tau \in \{+,-\}$. Since $t(L) = t(S)$, by Lemma 5.10 and Table 5.10, we obtain that if $L = O^+_{2n}(q)$ then $S \in \{O^+_{2n}(u), O^-_{2n-4}(u)\}$ and if $L = O^-_{2n}(q)$ then $S \in \{O^-_{2n}(u), O^+_{2n+4}(u)\}$.

Now we prove that $u = q$. Assume the contrary. Let at first $m = n$, and hence $\tau = \varepsilon$. By Lemma 3.9, we obtain that there exist $x \in m(S)$, $y \in m(L)$ such that x divides y. Lemma 3.8 yields that if $\varepsilon = +$ then $u^{n-1}/4 \leq x$, $y \leq q^{n-1}$, and if $\varepsilon = -$ then $u^n/4 \leq x$, $y \leq q^n$. So either $u^{n-1} < 4q^{n-1}$ or $u^n < 4q^n$. Let $n = 18$. Since $t(O^+_{36}(q)) = 13$ due to Table 1, we conclude that $\varepsilon = -$. Then $7, 14 \in M(L)$. Since $t(r_7(q), L) = t(r_7(q), S)$, we obtain that $R_7(q) \cup R_{14}(q) \subseteq R_7(u) \cup R_{14}(u)$. Using the adjacency criterion from Lemma 3.2, we obtain that if $r_i(q)$ is large with respect to L and adjacent with $r_7(q)$ in $GK(L)$ then $i \in \{9, 11, 16, 18, 20\}$. Similarly, if $r_i(q)$ is large with respect to L and adjacent with $r_{14}(q)$ in $GK(L)$ then $i \in \{9, 16, 18, 20, 22\}$. So we obtain that $R_{11}(q) \cup R_{22}(q) \subseteq R_{11}(u) \cup R_{22}(u)$. Assume that there exist $r, r' \in R_7(q)$ such that $r \in R_7(u)$ and $r' \in R_{14}(u)$. We have that $r_{11}(q) \in R_i(u)$, where $i \in \{11, 22\}$. By the adjacency criterion, we have that $r_{11}(q)$ is adjacent in $GK(S)$ with exactly one of r and r'; a contradiction, since $r_{11}(q)$ is adjacent with both of them in $GK(L)$. So either $R_7(q) \subseteq R_7(u)$ or $R_7(q) \subseteq R_{14}(u)$. The similar is true for $R_{14}(q)$. Let $\epsilon, \lambda \in \{+,-\}$ such that $(q - \epsilon 1, 7) = 1$ and $R_7(\epsilon q) \subseteq R_7(\lambda u)$. Then $k_7(\epsilon q)$ divides $k_7(\lambda u)$. By Lemma 2.5, we have that $3q^6 < u^6$, and hence $3^{3/2} q^{18} < u^{18}$. We obtained earlier that $u^{18} \leq 4q^{18}$; a contradiction, since $3^{3/2} > 4$. Assume that $20 \leq n \leq 22$. Then $16 \in M(L)$. Lemma 5.5 implies that $t(r_{16}(q), L) = t(r_{16}(q), S)$. So $R_{16}(q) \subseteq R_{16}(u)$, and hence $k_{16}(q)$ divides $k_{16}(u)$. Lemma 2.5 implies that $u^8 > 8q^8$. However, either $u^{n-1} > 8^{(n-1)/8} q^{n-1} > 64 q^{n-1}$ or $u^n > 8^{n/8} q^n > 64 q^n$; a contradiction with $u^{n-1} \leq 4q^{n-1}$ or $u^n \leq 4q^n$ respectively. If $24 \leq n \leq 26$ then $9, 18 \in M(L)$. Since $t(r_9(q), L) = t(r_{18}(q), L)$, we obtain that $R_9(q) \cup R_{18}(q) \subseteq R_9(u) \cup R_{18}(u)$. Therefore $k_9(q) k_{18}(q)$ divides $k_9(u) k_{18}(u)$. Lemma 2.5 implies that $3q^{12} < u^{12}$. So either $u^{n-1} > 3^{(n-1)/12} q^{n-1} > 5q^{n-1}$ or $u^n > 3^{n/12} q^n > 5q^n$; a contradiction. Let finally $28 \leq n \leq 30$. Then $24 \in M(L)$. So $t(r_{24}(q), L) = t(r_{24}(q), S)$, and hence $R_{24}(q) \subseteq R_{24}(u)$. Therefore $k_{24}(q)$ divides $k_{24}(u)$ and by Lemma 2.5, we obtain that $12q^{12} < u^{12}$. Thus either $u^{n-1} > 12^{(n-1)/12} q^{n-1} > 144 q^{n-1}$ or $u^n > 12^{n/12} q^n > 144 q^n$; a contradiction.

Assume that $L = O^+_{2n}(q)$, $S = O^-_{2n-4}(u)$. Since $t(S) = t(L)$, by Table 1 we obtain that $n \equiv 2 \pmod 4$. By Lemma 3.9, we obtain that for some $x \in m(S)$ and $y \in m(L)$ it is true that x divides y. Lemma 3.8 yields that $u^{n-2}/4 \leq x$ and $y \leq q^{n-1}$. So $u^{n-2} \leq 4q^{n-1}$.

Observe that $n \in \{22, 26, 30\}$. If $n = 22$ then $9, 18 \in M(L)$ and $t(r_9(q), L) = t(r_{18}(q)) = 14$. Since $t(r_{16}(u), S) = 14$, we conclude that $R_9(q) \cup R_{18}(q) \subseteq R_{16}(u)$. Then $k_9(q) k_{18}(q)$ divides $k_{16}(q)$. Since $k_9(q) k_{18}(q) = \frac{(q^6+q^3+1)(q^6-q^3+1)}{(q-1,3)(q+1,3)}$, we obtain that $q^{12}/3 \leq k_9(q) k_{18}(q)$. Since $k_{16}(u) = (u^8+1)/(2, u-1)$, we have that $q^{12} < 3(u^8+1) < 4u^8$. So $q^6 \leq 2u^4$. Therefore $q^{30} < 32 u^{20} \leq 128 q^{21}$; a contradiction with $q \geq 3$. If $n = 26, 30$ then we have that $11, 22 \in M(L)$. Since $t(r_{11}(q), L) = t(r_{22}(q), L) = 17$ and $t(r_{20}(u), S) = 17$, we have that $R_{11}(q) \cup R_{22}(q) \subseteq R_{20}(u)$. Therefore $k_{11}(q) k_{22}(q)$ divides $k_{20}(u)$. Observe that $k_{11}(q) k_{22}(q) = \frac{q^{22}-1}{(q^2-1)(q-1,11)(q+1,11)}$, $k_{20}(u) = (u^8 - u^6 + u^4 - u^2 + 1)/(u^2+1, 5)$. So $k_{11}(q) k_{22}(q) \geq (q^{20} + q^{18} + ... + q^2 + 1)/11 \geq q^{20}/11$ and $k_{20}(u) \leq u^8$. Therefore $q^{20} < 11u^8$. Then $q^{10} < 4u^4$, and hence if $n = 26$ then $q^{60} < 4^6 u^{24} < 4^7 q^{25}$, and if $n = 30$ then $q^{70} < 4^7 u^{28} < 4^8 q^{29}$. Again we get a contradiction with $q \geq 3$.

Assume finally that $L = O^-_{2n}(q)$, $S = O^+_{2n+4}(u)$. Since $t(S) = t(L)$, by Table 1 we obtain that $n \equiv 0 \pmod 4$. By Lemma 3.9, we have that for some $x \in m(S)$, $y \in m(L)$ it is true that x divides y. Lemma 3.8 yields that $u^{n+1}/4 \leq x$ and $y \leq q^n$. Therefore $u^{n+1} \leq 4q^n$. Observe that since $L \in \mathcal{L}$, we have that $n \in \{20, 24, 28\}$. Let $n = 20$ or $n = 24$. Then $9, 18 \in M(L)$. Since $t(r_9(q), L) = t(r_{18}(q), L) = 15$ and $t(r_{20}(u), S) = 15$ we have that $R_9(q) \cup R_{18}(q) \subseteq R_{20}(u)$. Therefore $k_9(q)k_{18}(q)$ divides $k_{20}(u)$. As in the previous case, we obtain that $q^{12}/3 \leq k_9(q)k_{18}(q)$. Since $k_{20}(u) = (u^8 - u^6 + u^4 - u^2 + 1)/(u^2 + 1, 5) < u^8$, we have that $q^{12} < 3u^8 < 4u^8$. So $q^6 < 2u^4$. If $n = 20$ then we conclude that $q^{30} < 2^5 u^{20}$. On the other hand, $u^{21} \leq 4q^{20}$, and hence $q^{30} < 128q^{20}$; a contradiction with $q \geq 3$. If $n = 24$ then we have that $q^{36} < 2^6 u^{24}$ and $u^{25} < 4q^{24}$. So $q^{36} < 256q^{24}$; a contradiction. Assume that $n = 28$. Then $11, 22 \in M(L)$, $t(r_{11}(q), L) = t(r_{22}(q), L) = 18$, and $t(r_{24}(u), S) = 18$. So $k_{11}(q)k_{22}(q)$ divides $k_{24}(u)$. Above we obtained that $k_{11}(q)k_{22}(q) \geq q^{20}/11$. Since $k_{24}(u) = u^8 - u^4 + 1$, we have that $q^{20} < 11u^8$. Therefore $q^5 < 2u^2$, and hence $q^{70} < 2^{14}u^{28} < 2^{14}u^{29} < 2^{16}q^{28}$; a contradiction with $q \geq 3$. The lemma is proved. □

Lemma 5.17. *Let $L \in \mathcal{L}$ and $L \in \{O^+_{2n}(q), O^-_{2n}(q)\}$, where n and q are odd. Then $S \in \{O^+_{2n}(q), O^-_{2n}(q)\}$.*

Proof. Let $\varepsilon \in \{+, -\}$ is chosen so that $L = O^\varepsilon_{2n}(q)$. By the hypothesis, we have that $19 \leq n \leq 29$. By Lemmas 5.13, 5.14, we obtain that $S \simeq O^\tau_{2m}(u)$, where m is odd and $\tau \in \{+, -\}$. By Proposition 5.2, we have $t(L) = t(S)$, and hence $m = n$ due to Table 1. Now Lemma 3.9 yields there exist $x \in m(S)$ and $y \in m(L)$ such that x divides y. Moreover, Lemma 3.8 implies that $u^n/8 \leq x$ and $y \leq q^n$. So $u^n \leq 8q^n$. Suppose that $n < 27$. By Table 1, we obtain that $9 \in M(O^-_{2n}(q))$, $18 \in M(O^+_{2n}(q))$. Observe that $t(r_9(-\varepsilon q), L) = 14$ due to Lemmas 5.8, 5.9. By Lemma 5.5, we have that $t(r_9(-\varepsilon q), L) = t(r_9(-\varepsilon q), S)$. So $R_9(-\varepsilon q) \subseteq R_9(-\tau u)$, and hence $k_9(-\varepsilon q)$ divides $k_9(-\tau u)$. Lemma 2.5 implies that $3q^6 < u^6$. So $3^{n/6}q^n < u^n$; a contradiction with the inequality $u^n \leq 8q^n$. Let $n = 27$ or $n = 29$. Then $24 \in M(L)$. By Lemmas 5.8, 5.9, we have that $t(r_{24}(q), L) = 19$. Therefore we obtain that $R_{24}(q) \subseteq R_{24}(u)$, and hence by Lemma 2.5, it is true that $12q^{12} < u^{12}$; a contradiction with the inequality $u^n \leq 8q^n$. The lemma is proved. □

6. Proof of Theorems 1.2 and 1.3

In this section we prove Theorems 1.2, 1.3. It is not hard to check using the information from Table 1 that if L is a group in odd characteristic from the hypothesis of Theorem A, then $t(L) \geq 23$. Similarly, if L is a classical group from the hypothesis of Theorem 1.2, then either $L \in \{S_{32}(q), O_{33}(q)\}$, where q is a prime power, or $t(L) \geq 14$. It was proved in [15, Theorem 2], that if G is a finite group such that $\omega(G) = \omega(L)$, where $L \in \{S_{32}(q), O_{33}(q)\}$ and q is odd, then G has a unique nonabelain composition factor, and this factor is isomorphic to L. So it remains to prove Theorem 1.2 for a classical group L either in even characteristic, or with $14 \leq t(L) \leq 22$ or equivalently for $L \in \mathcal{L}$.

Let $L \in \mathcal{L}$, p be the characteristic of the underlying field of L, G be a finite group with $\omega(G) = \omega(L)$ and S be a nonabelian composition factor of G. By the main theorem of [16], we may assume that $p \neq 2$. By Lemmas 5.12, 5.15, 5.16, 5.17, we have that if $L \in \mathcal{L}$ and p is odd then S is a classical group of Lie type over a field of characteristic p, and hence Theorem 1.3 is proved. The next two lemmas allow us to finish the proof of Theorem 1.2.

Lemma 6.1. ([6, Theorem 2], [19, Theorem 3]) *Let q be a power of a prime p, L be one of the groups $L^\varepsilon_n(q)$, where $n > 3$ and $(\varepsilon, n, q) \neq (-, 4, 2)$, $S_{2n}(q)$, where $n > 2$ and $(n, q) \notin \{(2, 2), (2, 3)\}$, $O_{2n+1}(q)$, where $n > 3$, or $O^\pm_{2n}(q)$, where $n > 4$, and let G be a finite group with $\omega(G) = \omega(L)$. Suppose that some nonabelian* *composition factor S of G is a group of Lie type over a field of characteristic p. Then either $S \simeq L$ or one of the following holds:*

(i) $L = S_4(q)$, where $q \neq 3^{2k+1}$, and $S = L_2(q^2)$;

(ii) $L \in \{O_9(q), S_8(q)\}$ and $S = O_8^-(q)$;

(iii) $\{L, G\} = \{O_8^+(2), S_6(2)\}$;

(iv) $\{L, G\} = \{O_8^+(3), O_7(3)\}$.

Lemma 6.2. ([5, Corollary 1.1]) *Let L be a finite nonabelian simple group other than $L_4(q)$, $U_3(q)$, $U_4(q)$, $U_5(2)$, and ${}^3D_4(2)$. Let G be a finite group, $1 \neq K \lhd G$ and $G/K \simeq L$. Then $\omega(G) \neq \omega(L)$.*

Theorem 1.3 and Lemma 6.1 provide $S \simeq L$, and hence $L \leq G/K \leq \mathrm{Aut}\, L$, where K is the soluble radical of G. Now Lemma 6.2 yields that $K = 1$, so Theorem 1.2 is proved.

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