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LOCALLY GRADED GROUPS WITH A CONDITION ON INFINITE SUBSETS

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Abstract. Let G be a group, we say that G satisfies the property $\mathcal{T}(\infty)$ provided that, every infinite set of elements of G contains elements $x \neq y, z$ such that $[x, y, z] = 1 = [y, z, x] = [z, x, y]$. We denote by $\mathcal{C}$ the class of all polycyclic groups, $\mathcal{S}$ the class of all soluble groups, $\mathcal{R}$ the class of all residually finite groups, $\mathcal{L}$ the class of all locally graded groups, $\mathcal{N}_2$ the class of all nilpotent group of class at most two, and $\mathcal{F}$ the class of all finite groups. In this paper, first we shall prove that if G is a finitely generated locally graded group, then G satisfies $\mathcal{T}(\infty)$ if and only if $G/Z_2(G)$ is finite, and then we shall conclude that if G is a finitely generated group in $\mathcal{T}(\infty)$, then

$$G \in \mathcal{L} \Leftrightarrow G \in \mathcal{R} \Leftrightarrow G \in \mathcal{S} \Leftrightarrow G \in \mathcal{C} \Leftrightarrow G \in \mathcal{N}_2\mathcal{F}.$$

1. Introduction

Nowadays, the study of the structure of groups with some combinatorial conditions on them, is one of the interesting topics that many researches are involved with. An important part of these researches comes from a famous question of Paul Erdös [12], where he posed:
Does there exist an upper bound for the order of (finite) subsets of a group G consisting of pairwise non-commuting elements, when every infinite set of elements of G contains a pair which commute?

Let G be a group and χ a class of groups. We say that G satisfies the condition (χ, ∞) if every infinite subset of G contains a pair of elements which generate a subgroup in the class χ. B. H. Neumann in [12] proved that a group G satisfies $(\mathcal{A}, \infty)$ if and only if G is center-by-finite, where $\mathcal{A}$ is

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the class of all abelian groups. Actually, it was an affirmative answer to Erdös's question. This result has initiated a great deal of research. Lennox and Wiegold proved in [10] that a finitely generated soluble group satisfies the property $(\mathcal{N}, \infty)$ if and only if it is $\mathcal{FN}$, i.e., finite-by-nilpotent, where $\mathcal{N}$ and $\mathcal{F}$ are the class of all nilpotent and the class of all finite groups, respectively. We denote by $\mathcal{N}_k$ the class of all nilpotent groups of class at most k, and by $\mathcal{E}_k$ the class of all k-Engel groups. A. Abdollahi and B. Taeri proved in [2] that if G is a finitely generated soluble group then G is in $(\mathcal{N}_k, \infty)$ if and only if G is finite-by-$\mathcal{N}(2, k)$, where $\mathcal{N}(2, k)$ is the class of all groups in which every 2-generator subgroup is nilpotent of class at most k. Extensions of problems of this type are studied in several articles (for example, [1]-[6], [8], [10], [11], [12]).

Our notation and terminology are standard. Let G be a group, a subset $X = \{x, y, z\}$ of G satisfies the property $\mathcal{T}$ (or $X \in \mathcal{T}$) whether $[x, y, z] = [y, z, x] = [z, x, y] = 1$. We say G satisfies the property $\mathcal{T}$ (or $G \in \mathcal{T}$) if $\{x, y, z\} \in \mathcal{T}$ for all elements x, y, z of G. Also we say that G satisfies the condition $\mathcal{T}(\infty)$ (or $G \in \mathcal{T}(\infty)$) provided that every infinite set of elements of G contains a subset $X = \{x, y, z\}$ such that $x \neq y$ and $X \in \mathcal{T}$. Let, as usual, $Z_n(G)$ denotes the $(n+1)$-th term of upper central series of G, and $\gamma_n(G)$ denotes the n-th term of lower central series of G. It is easy to see that:

$$(\mathcal{N}_2, \infty) \subseteq (\mathcal{E}_2, \infty) \subseteq \mathcal{T}(\infty). \qquad (\star)$$

The first author proved in [8] that for finitely generated soluble groups the inclusions which are labelled by $(\star)$ are equalities, furthermore those are equal to the class $\mathcal{N}_2\mathcal{F}$. The main result of [4] states that a finitely generated residually finite group G belongs to $(\mathcal{N}_2, \infty)$ if and only if $G/Z_2(G)$ is finite. C. Delizia, A. Rhemtulla and H. Smith, using deep results by Zelmanov and Lubotzky and Mann have shown a similar result for finitely generated locally graded groups (see [6]). They proved that if G is a finitely generated locally graded group in $(\mathcal{N}_k, \infty)$, then there is a positive integer c depending only on k such that $G/Z_c(G)$ is finite. In this paper we are going to use their techniques and generalize the main result of [8] as following.

Main Theorem. *Let G be a finitely generated locally graded group. Then G is in $\mathcal{T}(\infty)$ if and only if $G/Z_2(G)$ is finite.*

Let we denote by $\mathcal{C}$ the class of all polycyclic groups, $\mathcal{S}$ the class of all soluble groups, $\mathcal{R}$ the class of all residually finite groups, and $\mathcal{L}$ the class of all locally graded groups. Then as an interesting consequence of Main Theorem we shall mention the following result.

Corollary 1.1. *Let $\mathcal{U}$ denote the class of all finitely generated groups which satisfy the property $\mathcal{T}(\infty)$. Then*

$$\mathcal{N}_2\mathcal{F} \cap \mathcal{U} = \mathcal{C} \cap \mathcal{U} = \mathcal{S} \cap \mathcal{U} = \mathcal{R} \cap \mathcal{U} = \mathcal{L} \cap \mathcal{U}.$$

As another consequence of Main Theorem we shall include the following result, which asserts that the containments which are labelled by $(\star)$ will be equalities, provided that we consider finitely generated locally graded groups.

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Corollary 1.2. *Let $\mathcal{V}$ denote the class of all finitely generated locally graded groups. Then*

$$(\mathcal{N}_2, \infty) \cap \mathcal{V} = (\mathcal{E}_2, \infty) \cap \mathcal{V} = \mathcal{T}(\infty) \cap \mathcal{V}$$

Now the following question raises from previous results.

Quetion 1. *In general, does equality hold in $(\star)$? Does equality hold, when finitely generated groups are considered?*

2. Proof of Main Theorem

In this section, G is always a finitely generated group. Recall that a group H is called restrained group whether $\langle x\rangle^{\langle y\rangle}$, the normal closure of $\langle x\rangle$ in $\langle y\rangle$, is finitely generated for all $x, y \in H$. By Proposition 2.2 of [8], every group in $\mathcal{T}(\infty)$ is restrained. So we have the following key lemma for $\gamma_n(G)$ and $G^{(n)}$, the n-th term of the derived series of G.

Lemma 2.1. *Let G satisfy the property $\mathcal{T}(\infty)$. Then $G^{(n)}$ and $\gamma_n(G)$ are finitely generated, for all positive integers n.*

Proof. By Proposition 2.2 of [8], G is a restrained group. Then a repeated use of Corollary 4 of [9] shows that $G^{(n)}$ is finitely generated. It is well known that finitely generated nilpotent groups are polycyclic, so $G/\gamma_n(G)$ is polycyclic. Thus Lemma 3 of [9] implies that $\gamma_n(G)$ is finitely generated. □

A well-known theorem of Baer (see 14.5.1 of [13]) states that the finiteness of $G/Z_n(G)$ implies the finiteness of $\gamma_{n+1}(G)$. By this result we have the following lemma.

Lemma 2.2. *Let G satisfy the property $\mathcal{T}(\infty)$. Then $\gamma_3(G)/G''$ is finite. In particular $G^{(n)}/G^{(n+1)}$ is finite for all positive integers $n \geq 2$.*

Proof. It is evident that G/G'' is a finitely generated soluble group in $\mathcal{T}(\infty)$. It follows from the main theorem of [8] and a well-known result of Baer that $\gamma_3(G)/G''$ is finite. The last statement can be easily obtained by the first one. □

Lemma 2.3. *Let G satisfy the property $\mathcal{T}(\infty)$ and let H be a subgroup of finite index in G''. Then H/H' is finite.*

Proof. It is sufficient to prove the lemma for every normal subgroup H of G'' with finite index. On the contrary, suppose that H/H' is infinite and T/H' is the torsion subgroup of H/H'. One may easily see that $X = \{y, yh, yh^2, \ldots\}$ is an infinite subset of G for every element h of $H \setminus T$ and y of G''. There exist positive integers $i \neq j, k$ such that $\{yh^i, yh^j, yh^k\} \in \mathcal{T}$. Put $z = [yh^i, yh^j]$, we have $z = y^{-h^i} y^{h^j}$ which belongs to H. Now it follows from $[z, yh^k] = 1$ that $[z, y] \in T$. Let we denote by C_1 the set $\{h \in H : [y, h] \in T\}$, then $T < C_1 \leq H$ and $[C_1, y] \leq T$. If C_1 has infinite index in H, then we repeat the argument to obtain a subgroup C_2 of H such that $[C_2, y] \leq C_1$ and $T < C_1 < C_2$. Let m be a positive integer such that $y^m \in H$. Since H/T is torsion free and abelian, then we have $[C_2, y^m] = [C_2, y]^m \leq T$ which implies that $[C_2, y] \leq T$. So we have $C_2 \leq C_1$, which is a contradiction.

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Therefore C_1 has finite index in H, say n. We repeat the previous argument for every element h of H. Since $h^n \in C_1$, we have $[h,y]^n = [h^n, y] \in T$. So $[h,y] \in T$, which implies $H = C_1$. It follows that $[H, G''] \leq T$, thus H/T is in the center of G''/T. Since H/T has finite index in G''/T, we have $G^{(3)}T/T$ is finite, by a well-known theorem of Schur. On the other side, by Lemma 2.2, we have $G''/G^{(3)}$ is finite, so G''/T is finite, which yields a contradiction and completes the proof. □

Proposition 2.4. *Let G satisfy the property $\mathcal{T}(\infty)$, and let H be a subgroup of G'' with finite index such that $H \in \mathcal{T}$. Then $G/Z_2(G)$ is finite.*

Proof. Using Lemma 2.1, it follows that H is finitely generated nilpotent group of class at most two. On the other hand, H/H' is finite, by Lemma 2.3. Now a theorem of Robinson (see for example: 5.2.6 of [13]) implies that H and then G'' is finite. We may apply the main theorem of [8] to conclude that $|G/G'' : Z_2(G/G'')|$ and then $\gamma_3(G/G'')$ is finite. Now since G'' is finite, we have $\gamma_3(G)$ is finite. Therefore $G/Z_2(G)$ is finite. □

It immediately follows from Proposition 2.4 that if G has a subgroup H with finite index such that $H \in \mathcal{T}$, then $G/Z_2(G)$ is finite. Therefore we need to consider groups in which all subgroups with finite index does not satisfy $\mathcal{T}$. In this case we have the following lemma.

Lemma 2.5. *Let L be a finitely generated residually finite group which satisfies $\mathcal{T}(\infty)$ and suppose that $H \notin \mathcal{T}$ for each subgroup H of finite index in L. Then there exists a normal subgroup G of finite index in L such that $G = N\langle t\rangle$ for some normal subgroup N of G and element t satisfying $\{ta, tb, tb\} \in \mathcal{T}$ for all $a, b \in N$.*

Proof. Let we suppose that L has no such subgroup G. We may consider elements $x_0, y_0 \in L$, such that $\{x_0, y_0, y_0\} \notin \mathcal{T}$. Since L is residually finite, there exists a normal subgroup G_1 of finite index in L such that $\{x_0, y_0, y_0\} \notin \mathcal{T} \pmod{G_1}$. If $\{y_0x, y_0y, y_0y\} \in \mathcal{T}$ for all $x, y \in G_1$, then we obtain a contradiction by choosing $N = G_1$ and $t = y_0$. Therefore, there exist $x_1, y_1 \in G_1$ such that $\{w_0x_1, w_0y_1, w_0y_1\}$ is not in $\mathcal{T}$, where $w_0 = y_0$. In this case, since L is residually finite, there exists a normal subgroup G_2 of finite index in G_1 such that $\{w_0x_1, w_0y_1, w_0y_1\} \notin \mathcal{T} \pmod{G_2}$. Now we repeat the argument whit G_1 replaced by G_2 and w_0 by w_1, where $w_1 = w_0y_1$. By continuing this process, we obtain the chain $L > G_1 > G_2 > \cdots$ of subgroups of finite index in L, and elements $w_i = w_{i-1}y_i$ and $u_i = w_{i-1}x_i$ of L such that $\{u_i, w_i, w_i\} \notin \mathcal{T} \pmod{G_{i+1}}$, for all $i \geq 1$. Let $j > i > 1$ and $n = j - i$, then

$$u_j = w_{j-1}x_j = w_{j-2}y_{j-1}x_j = \cdots = w_{j-n}y_{j-(n-1)}\cdots y_{j-1}x_j = w_i \ (modG_{i+1}).$$

So $\{u_i, u_j, u_k\} \notin \mathcal{T}$ (modG_{i+1}) for all integers $i < j \leq k$. Therefore the infinite set $\{u_i\}_{i=1}^{\infty}$ of L contradicts $L \in \mathcal{T}(\infty)$, and this completes the proof. □

Now in view of the preceding results, we are going to consider G with the following properties, say (Γ).

- G is finitely generated residually finite;

- $G = N\langle t\rangle$ where N is a normal subgroup of G and t is an element satisfying $\{ta, tb, tb\} \in \mathcal{T}$ for all $a, b \in N$;
- H/H' is finite for every subgroup H of finite index in G.

Let p be a prime, a finite p-group H is called powerful whether $[H, H] \leq H^p$, where p is odd and $[H, H] \leq H^4$, where $p = 2$ (see [7], Chapter 2).

Lemma 2.6. *Let G satisfy the properties of (Γ). Then the following statements hold.*

(i) *Every finite image of G is nilpotent.*

(ii) *If G is a finite p-group, where p is a prime, then $(N^d)'$ is a powerful subgroup of N, for some integer d depending on p.*

Proof. Since $[ta, tb, tb] = 1$ for all $a, b \in N$, it is obvious that $\langle ta, tb\rangle$ is nilpotent of class at most two. Therefore our statements follow arguing as in the proofs of Lemmas 4 and 5 of [6]. □

Remark 2.7. *Let G satisfy (Γ). G/G' is abelian and finite. Consider π, the finite set consisting of all the factors of $|G/G'|$, hence G/G' is a finite π-group. For every non trivial element g of G there exists normal subgroup N in G such that G/N is finite and nilpotent. On the other hand, $G/G'N$ is finite, which implies that $(G/N)_{ab}$ is a finite π-group, and consequently G/N is a π-group, by 5.2.6 of [13]. For each $p \in \pi$, let R_p denote the finite p-residual of G, so $\bigcap_{p\in\pi} R_p = 1$. Now we need to consider the case where G is residually finite p-group. For, suppose that the result have been established for all G/R_p, where $p \in \pi$. Accordingly, $\gamma_3(G/R_p)$ and hence $\frac{\gamma_3(G)}{\gamma_3(G)\cap R_p}$ is finite. So $\gamma_3(G)$ is finite mod $\bigcap_{p\in\pi} R_p$. Therefore $G/Z_2(G)$ is finite.*

In the proof of the following result we need some properties of powerful p-groups, that the reader is referred to [7].

Proposition 2.8. *Let G be a residually finite group in the class $\mathcal{T}(\infty)$. Then $G/Z_2(G)$ is finite.*

Proof. Tanks to Proposition 2.4, we shall suppose that G satisfies (Γ), and by remark 2.7 that G is residually a finite p-group for some prime p. So for all $g \in G$, there exists a normal subgroup N_g of G such that G/N_g is a finite p-group. Since G/N_g satisfies (Γ), there exists an integer d, such that $\frac{(N^d)'N_g}{N_g}$ is powerful in $\frac{NN_g}{N_g}$, by applying lemma 2.6 to G/N_g. Now we argue as in the proof of Proposition 1 of [6] and conclude that G is a linear group. Since non-abelian free groups can not satisfy the property $\mathcal{T}(\infty)$, G has no such a subgroup, and so G is soluble-by-finite, by Corollary 1 of [14]. Now by lemma 2.6, every finite quotient of G is nilpotent, so G is soluble-by-nilpotent, thus G is soluble. Therefore G is finitely generated soluble and $G/Z_2(G)$ is finite, by [8] . □

Proof of Main Theorem. If $G/Z_2(G)$ is finite, then it is easy to see that G satisfies $\mathcal{T}(\infty)$, as Lemma 2.1 of [8] shows. Conversely, let G be a locally graded group in $\mathcal{T}(\infty)$, then by Lemma 2.2, $\Gamma_i(G)$ is finitely generated, where i is a positive integer. Now let R denote the finite residual of G. Since $G/\Gamma_i(G)$ is finitely generated and nilpotent, it is residually finite for all positive integers i. It follows that $R \leq \Gamma_i(G)$ for all $i \geq 1$. Moreover, $\Gamma_3(G)/R$ is finite by Proposition 2.8, thus R is finitely

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generated. If $R \neq 1$ then there exists a G-invariant subgroup $S < R$ of finite index in R. Hence G/S is finite, a contradiction. Therefore $R = 1$ and G is residually finite. Now Proposition 2.8 completes the proof. □

Proof of Corollary 1.1. Let $\mathcal{U}$ denote the class of finitely generated groups which satisfy $\mathcal{T}(\infty)$. It is well known (see for example, [13]) that finitely generated nilpotent groups are polycyclic, also polycyclic groups are residually finite, by 5.4.17 of [13], and in turn residually finite groups are locally graded. In fact we have the following diagram,

$$\begin{array}{ccccccc} \mathcal{F} & \subset & f.g.\,\mathcal{NF} & \subset & \mathcal{R} & \subset & \mathcal{L} \\ & & \cup & & \cup & & \cup \\ & & f.g.\,\mathcal{N} & \subset & \mathcal{C} & \subset & \mathcal{S} \end{array}$$

where f.g. is abbreviation of finitely generated. Now, if G belongs to $\mathcal{L} \cap \mathcal{T}(\infty)$, then the proof of Main Theorem implies that G belongs to the classes $\mathcal{R}$ and $\mathcal{S}$, and that $G/Z_2(G)$ is finite, that is, G is a group in $\mathcal{N}_2\mathcal{F}$. On the other hand, by the proof of Main Theorem of [8], we have $G \in \mathcal{C}$ whether $G \in \mathcal{S} \cap \mathcal{T}(\infty)$. Consequently, these give raise to the following equalities:

$$\mathcal{N}_2\mathcal{F} \cap \mathcal{U} = \mathcal{C} \cap \mathcal{U} = \mathcal{S} \cap \mathcal{U} = \mathcal{R} \cap \mathcal{U} = \mathcal{L} \cap \mathcal{U}.$$

□

Proof of Corollary 1.2. Theorem 3.4 of [5] states that $G \in (\mathcal{E}_2, \infty) \cap \mathcal{L}$ if and only if $G \in \mathcal{N}_2\mathcal{F}$. Also it follows from Theorem 2.8 of [5] that $G \in (\mathcal{N}_2, \infty) \cap \mathcal{R}$ if and only if $G \in \mathcal{N}_2\mathcal{F}$. Now Corollary 1.1 draws the following equalities:

$$(\mathcal{N}_2, \infty) \cap \mathcal{V} = (\mathcal{E}_2, \infty) \cap \mathcal{V} = \mathcal{T}(\infty) \cap \mathcal{V}.$$

□

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