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COUNTABLY RECOGNIZABLE CLASSES OF GROUPS WITH RESTRICTED CONJUGACY CLASSES

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Abstract. A group class $\mathfrak{X}$ is said to be countably recognizable if a group belongs to $\mathfrak{X}$ whenever all its countable subgroups lie in $\mathfrak{X}$. It is proved here that most of the relevant classes of groups defined by restrictions on the conjugacy classes are countably recognizable.

1. Introduction

A group class $\mathfrak{X}$ is said to be *countably recognizable* if a group G is an $\mathfrak{X}$-group whenever all its countable subgroups belong to $\mathfrak{X}$. Countably recognizable classes of groups were introduced and studied by R. Baer [1] in 1962, but already in the fifties the property of being hyperabelian and that of being hypercentral were proved to be detectable from the behaviour of countable subgroups by Baer himself and S. N. Černikov, respectively (see for instance [19, Part 1, Theorem 2.15 and Theorem 2.19]).

Among the countably recognizable group classes there are of course the so-called local classes: a group class $\mathfrak{X}$ is said to be *local* if it contains all groups in which every finite subset lies in an $\mathfrak{X}$-subgroup. On the other hand, the classes $\mathfrak{S}$ of soluble groups and $\mathfrak{N}$ of nilpotent groups are examples of countably recognizable group classes which are not local. In his paper, Baer produced many interesting examples of countably recognizable group classes which are not local; it follows for instance from Baer's methods that if $\mathfrak{X}$ is a countably recognizable group class, closed with respect to subgroups and homomorphic

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images, then the class of all groups admitting an ascending normal series with $\mathfrak{X}$-factors is likewise countably recognizable, and this class is not local for many natural choices of $\mathfrak{X}$.

Later, many other relevant countably recognizable group classes were discovered. In fact, B. H. Neumann [15] proved that residually finite groups (i.e. groups in which the intersection of all normal subgroups of finite index is trivial) form a countably recognizable class. Moreover, R. E. Phillips ([16],[17]) proved that the class of groups in which every subgroup has all its maximal subgroups of finite index is countably recognizable, and that the same conclusion holds for the class of groups whose simple sections belong to a subgroup closed and countably recognizable group class.

Further interesting examples of countably recognizable group classes can be found in [4] and [22], where it is proved for instance that groups for which some term (with finite ordinal type) of the derived series has finite rank form a countably recognizable class, and that a corresponding result holds if the derived series is replaced by the lower central series. We point out also that several other classes of groups have been added to the list of countably recognizable classes in the forthcoming paper [9].

Examples of important group classes which are not countably recognizable are known in the literature. It was proved by G. Higman [10] that there exists a group of cardinality $\aleph_1$ which is not free but whose countable subgroups are free. Therefore the class of free groups is not countably recognizable. Note that also the property of being free abelian cannot be detected from the behaviour of countable subgroups; in fact, the cartesian product of any infinite collection of infinite cyclic groups cannot be decomposed into a direct product of infinite cyclic groups, but all its countable subgroups are free abelian (see for instance [6, Theorem 19.2]). Moreover, M. I. Kargapolov [11] constructed a locally nilpotent group with no abelian non-trivial ascendant subgroups, and this example shows that the class SN^* of all groups admitting an ascending series with abelian factors is not countably recognizable.

It is straightforward to show that the class of groups with finite conjugacy classes is countably recognizable, and the aim of this paper is to prove that also many other relevant group classes defined by restrictions on the conjugacy classes are countably recognizable. These restrictions are introduced and described in Section 2, while Section 3 contains a systematic study of their properties of countable character.

Most of our notation is standard and can be found in [19].

2. **Conjugacy classes**

If G is a group, the elements of G admitting only finitely many conjugates form a subgroup $FC(G)$, called the *FC-centre* of G, and G is an *FC-group* if it coincides with the FC-centre, i.e. if all conjugacy classes of elements of G are finite. Thus a group G has the FC-property if and only if the index $|G : C_G(x)|$ is finite for each element x of G. Clearly, all abelian groups and all finite groups have the FC-property, and the study of FC-groups was initially developed with the aim of finding properties shared by these two relevant group classes. We refer to the monographs [24] and [3] for a detailed

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description of results and properties concerning this important chapter of the theory of infinite groups. It is for instance obvious that finitely generated FC-groups are finite over the centre, while an arbitrary FC-group G is residually finite over $Z(G)$. Among the basic results, it should be also mentioned that if G is any FC-group, then the factor group $G/Z(G)$ is locally finite, so that it follows from the celebrated theorem of Schur that the commutator subgroup of any FC-group is locally finite. In particular, torsion-free FC-groups are abelian.

Moreover, it is easy to show that a periodic group has the FC-property if and only if it is covered by finite normal subgroups (this result is usually known as Dietzmann's Lemma). If $\mathfrak{X}$ is any group class, we shall denote by $\boldsymbol{M}\mathfrak{X}$ the class of all groups in which every finite subset lies in a normal $\mathfrak{X}$-subgroup. Thus Dietzmann's Lemma just says that $\boldsymbol{M}\mathfrak{F}$ is the class of all periodic FC-groups, where $\mathfrak{F}$ denotes the class of all finite groups.

Recall also that a group G is said to be a *BFC-group* if it has boundedly finite conjugacy classes, i.e. if there is a positive integer k such that $|G : C_G(x)| \leq k$ for all elements x of G. It was proved by Neumann [13] that a group has the BFC-property if and only if its commutator subgroup is finite.

If G is a group, the *upper FC-central series* of G is the ascending characteristic series $\{FC_\alpha(G)\}_\alpha$ defined by setting $FC_0(G) = \{1\}$,

$$FC_{\alpha+1}(G)/FC_\alpha(G) = FC\big(G/FC_\alpha(G)\big)$$

for each ordinal α and

$$FC_\lambda(G) = \bigcup_{\alpha<\lambda} FC_\alpha(G)$$

if λ is a limit ordinal. The last term of the upper FC-central series of G is called the *FC-hypercentre* of G, and G is said to be *FC-hypercentral* if it coincides with the FC-hypercentre. Moreover, G is called *FC-nilpotent* if $FC_k(G) = G$ for some non-negative integer k, and in this case the smallest such k is the *FC-nilpotency class* of G; then a group has the FC-property if and only if it is FC-nilpotent of class ≤ 1.

Clearly, all nilpotent-by-finite groups are FC-nilpotent. On the other hand, if p is any prime number and G is the semidirect product of a group P of type p^∞ by the cyclic group generated by an automorphism of P of infinite order, then G is FC-nilpotent, but it is not nilpotent-by-finite, since P is the Fitting subgroup of G. This situation cannot occur in the case of finitely generated groups. In fact, the following result, due to D. H. McLain [12], shows that for finitely generated groups the properties of being FC-hypercentral, FC-nilpotent and nilpotent-by-finite are equivalent. In particular, finitely generated FC-hypercentral groups satisfy the maximal condition on subgroups.

Lemma 2.1. *Let G be a finitely generated FC-hypercentral group. Then G is nilpotent-by-finite.*

Proof. Let μ be the smallest ordinal such that the factor group $G/FC_\mu(G)$ satisfies the maximal condition on subgroups, and assume that $\mu > 0$. Then $G/FC_\mu(G)$ is FC-nilpotent and all factors of its upper FC-central series are central-by-finite, so that in particular $G/FC_\mu(G)$ is polycyclic-by-finite. It follows that $FC_\mu(G)$ is the normal closure of a finite subset of G, and so μ cannot be a limit ordinal. Thus the FC-centre $FC_\mu(G)/FC_{\mu-1}(G)$ of $G/FC_{\mu-1}(G)$ is finitely generated, so that it satisfies the maximal condition and hence also $G/FC_{\mu-1}(G)$ satisfies the maximal condition. This contradiction shows that $\mu = 0$, and so G satisfies the maximal condition on subgroups; in particular, G is FC-nilpotent, and so $G = FC_k(G)$ for some non-negative integer k.

For each positive integer $i \leq k$, the FC-centre $FC_i(G)/FC_{i-1}(G)$ of $G/FC_{i-1}(G)$ is finitely generated, and so the centralizer $C_G\big(FC_i(G)/FC_{i-1}(G)\big)$ has finite index in G. Therefore

$$C = \bigcap_{i=1}^{k} C_G\big(FC_i(G)/FC_{i-1}(G)\big)$$

is a nilpotent subgroup of finite index of G, and so G is nilpotent-by-finite. □

Let FC^0 be the class of all finite groups, and for each non-negative integer n define by induction FC^{n+1} as the class consisting of all groups G such that $G/C_G(\langle x\rangle^G)$ belongs to FC^n for every element x of G. Notice that FC^1 is precisely the class of all FC-groups, and that the class FC^n is closed with respect to subgroups and homomorphic images for all n. It is also clear that FC^n contains all nilpotent groups of class at most n.

Groups with the FC^n-property have been introduced in [8], where it was proved in particular that if G is an FC^n-group for some positive integer n, then the subgroup $\gamma_n(G)$ is contained in the FC-centre of G, and so G is FC-nilpotent of class at most n. It follows easily that $\gamma_{n+1}(G)$ is periodic for every FC^n-group G, and in particular torsion-free groups with the FC^n-property are nilpotent of class at most n. The consideration of the infinite dihedral group shows that

$$FC^* = \bigcup_{n\in\mathbb{N}} FC^n$$

is properly contained in the class FC_* of FC-nilpotent groups.

In [8] it was also studied the class FC^∞ consisting of all groups G such that for each element x the factor group $G/C_G(\langle x\rangle^G)$ belongs to FC^n for some non-negative integer n depending on x. For each positive integer n, let G_n be a finitely generated nilpotent group of class n such that $G_n/Z_{n-1}(G_n)$ is infinite. Then G_n belongs to FC^n but not to FC^{n-1}, and hence the direct product

$$G = \operatorname*{Dr}_{n\in\mathbb{N}} G_n$$

is an FC^∞-group which does not have the FC^n-property for any n. Therefore FC^∞ properly contains the class FC^*. Note also that the class FC^∞ and the class FC_* are incomparable.

In a celebrated paper of 1955, B. H. Neumann [14] started the investigation of groups in which all subgroups are normal up to the obstruction of a finite section, and proved that such groups are close to be abelian. In fact, he proved that a group G has finite conjugacy classes of subgroups (or equivalently each subgroup of G is normal in a subgroup of finite index) if and only if the centre $Z(G)$ has finite index in G, while in a group G every subgroup has finite index in its normal closure if and only if the

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commutator subgroup G' is finite, and so if and only if G is a BFC-group. A third natural normality condition was considered forty years later and it is in some sense much more difficult to handle. A group G is called a *CF-group* if the index $|X : X_G|$ is finite for each subgroup X of G. The consideration of the locally dihedral 2-group shows that locally finite groups with the CF-property need not be FC-groups. The CF-property has been introduced in [2], where it was proved that any locally finite CF-group contains an abelian subgroup of finite index; this result was later extended to locally (soluble-by-finite) CF-groups (see [23]), but it cannot be proved in the general case, as Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) have obviously the CF-property.

A group G is said to have the *BCF-property* if there exists a positive integer k such that $|X : X_G| \leq k$ for all subgroups X of G. It can be proved that locally finite CF-groups have the BCF-property (see [2]), and that locally graded BCF-groups are abelian-by-finite (see [23]). Recall here that a group G is *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index; in particular, all locally (soluble-by-finite) groups are locally graded.

Let G be a group, and let X be a subgroup of G. The *normal oscillation* of X in G is the cardinal number

$$\min\{|X : X_G|, |X^G : X|\}.$$

Clearly, X is normal in G if and only if it has normal oscillation 1. Moreover, X has finite normal oscillation in G if and only if either X has finite index in its normal closure X^G or it is finite over its core X_G; in particular, finite subgroups and subgroups of finite index have finite normal oscillation. We shall say that a group G is an *FNO-group* if every subgroup of G has finite normal oscillation. Then all groups with finite commutator subgroup and all CF-groups have the FNO-property. It has recently been proved that any locally finite FNO-group is nilpotent-by-finite (see [7]).

Let $\mathfrak{X}$ be a class of groups. Recall that a group G is said to be an *$\mathfrak{X}C$-group* (or to have *$\mathfrak{X}$-conjugacy classes*) if the factor group $G/C_G(\langle g\rangle^G)$ belongs to $\mathfrak{X}$ for each element g of G. Thus $\mathfrak{F}C$ is precisely the class of FC-groups. If $\mathfrak{X}$ is chosen to be either the class $\mathfrak{C}$ of Černikov groups or the class $\mathfrak{P}$ of polycyclic-by-finite groups, we obtain the relevant classes of *CC-groups* and *PC-groups*, introduced in [18] and [5], respectively. We mention here that a periodic group G has the CC-property if and only if $\langle x\rangle^G$ is a Černikov group for each element x of G, while PC-groups can be characterized as those groups which can be covered by their polycyclic-by-finite normal subgroups. This means that $\boldsymbol{M}\mathfrak{C}$ is the class of periodic CC-groups and $\boldsymbol{M}\mathfrak{P}$ is the class of all PC-groups.

Further classes of generalized FC-groups have been considered by several authors. Among the most interesting ones, we mention the class of FCI-groups and that of FNI-groups. A group G is said to be an *FCI-group* if every cyclic non-normal subgroup has finite index in its centralizer, while G is an *FNI-group* if each non-normal subgroup of G has finite index in its normalizer. These group classes have been completely described in the locally (soluble-by-finite) case by D. J. S. Robinson [20].

3. Countable recognizability

If x is an element of a group G admitting infinitely many conjugates, it is obvious that x belongs to some countable subgroup of G where it has again infinitely many conjugates. It follows that the class of FC-groups is countably recognizable. It is also clear that groups with finite commutator subgroup (which are precisely the groups with the BFC-property) form a countably recognizable group class. In this section it will be proved that most of the relevant classes of generalized FC-groups are countably recognizable. It was proved by Baer [1] that the class of FC-hypercentral groups is countably recognizable, and it is also known that the property of being nilpotent-by-finite can be detected from the behaviour of countable subgroups (see for instance [7]). Our first aim is to prove that also the intermediate class of FC-nilpotent groups is countably recognizable.

Note first that the class FC_n, consisting of all FC-nilpotent groups of class at most n, is not local for any positive integer n. To see this, it is enough to consider any locally finite group with trivial FC-centre, like for instance an infinite simple locally finite group or one of the periodic metabelian groups constructed by V. S. Čarin (see [19, Part 1, p.152]).

Lemma 3.1. *Let G be a group, and let X be a countable subgroup of G. Then for each non-negative integer n there exists a countable subgroup H_n of G containing X such that $H_n \cap FC_n(G) = FC_n(H_n)$.*

Proof. The proof is by induction on n, the statement being obvious for $n = 0$. If the subgroup X is contained in $FC_{n+1}(G)$, it is enough to put $H_{n+1} = X$. Assume now that X is not contained in $FC_{n+1}(G)$, and let x be any element of $X \setminus FC_{n+1}(G)$. Then the coset $xFC_n(G)$ has infinitely many conjugates in $G/FC_n(G)$, and so there exists a countably infinite subset Y_x of G such that $x^{y_1}FC_n(G) \neq x^{y_2}FC_n(G)$ for all elements y_1 and y_2 of Y_x such that $y_1 \neq y_2$. Clearly, the subgroup

$$K = \langle X, Y_x \mid x \in X \setminus FC_{n+1}(G)\rangle$$

is countable, and so by induction we can find a countable subgroup U_1 of G containing K such that $U_1 \cap FC_n(G) = FC_n(U_1)$. Apply now the same argument to U_1 in order to obtain a new countable subgroup U_2 containing U_1, with $U_2 \cap FC_n(G) = FC_n(U_2)$ and such that for every $a \in U_1 \setminus FC_{n+1}(G)$ there is a countably infinite subset Z_a of U_2 for which $a^{z_1}FC_n(G) \neq a^{z_2}FC_n(G)$ whenever z_1 and z_2 are different elements of Z_a. In this way we can construct an increasing sequence $(U_k)_{k\in\mathbb{N}}$ of countable subgroups of G such that $U_k \cap FC_n(G) = FC_n(U_k)$.

Consider the countable subgroup

$$U = \bigcup_{k\in\mathbb{N}} U_k.$$

If u is any element of $FC_n(U)$ and k is a positive integer such that $u \in U_k$, then u belongs to $FC_n(U_k)$ and so also to $FC_n(G)$. Therefore $U \cap FC_n(G) = FC_n(U)$. Let v be any element of $FC_{n+1}(U)$, and assume for a contradiction that v does not belong to $FC_{n+1}(G)$. Fix a positive integer k such that $v \in U_k$. It follows from our construction that U_{k+1} contains a countably infinite subset $W = \{w_i \mid i \in \mathbb{N}\}$ such that

$v^{w_i} FC_n(G) \neq v^{w_j} FC_n(G)$ if $i \neq j$. As $FC_n(U)$ is contained in $FC_n(G)$, we have also that $v^{w_i} FC_n(U) \neq v^{w_j} FC_n(U)$ if $i \neq j$, contradicting the assumption that v belongs to $FC_{n+1}(U)$. Therefore $FC_{n+1}(U)$ is contained in $FC_{n+1}(G)$, and the proof of the statement can be completed by choosing $H_{n+1} = U$. □

Theorem 3.2. *For each positive integer n, the class of FC-nilpotent groups of class at most n is countably recognizable.*

Proof. Let G be a group whose countable subgroups are FC-nilpotent with class at most n. It follows from Lemma 3.1 that every countable subgroup of G is contained in $FC_n(G)$, so that $FC_n(G) = G$ and G is FC-nilpotent with class at most n. □

The following elementary lemma is due to Baer, and shows in particular that the union of a countable collection of local classes is countably recognizable; it can be used for instance to prove that soluble groups and nilpotent groups form countably recognizable group classes.

Lemma 3.3. *Let $(\mathfrak{X}_n)_{n\in\mathbb{N}}$ be a countable collection of subgroup closed and countably recognizable group classes. Then also the class*

$$\mathfrak{X} = \bigcup_{n\in\mathbb{N}} \mathfrak{X}_n$$

is countably recognizable.

Proof. Let G be a group whose countable subgroups belong to $\mathfrak{X}$, and assume for a contradiction that G is not an $\mathfrak{X}$-group. Then for each positive integer n there exists a countable subgroup H_n of G which is not in $\mathfrak{X}_n$. As all classes $\mathfrak{X}_n$ are subgroup closed, it follows that the countable subgroup

$$\langle H_n \mid n \in \mathbb{N} \rangle$$

cannot be in $\mathfrak{X}$, and this contradiction proves the statement. □

Corollary 3.4. *The class of FC-nilpotent groups is countably recognizable.*

Proof. As the class of FC-nilpotent groups can be decomposed as

$$FC_* = \bigcup_{n\in\mathbb{N}} FC_n,$$

the statement is a direct consequence of Theorem 3.2 and Lemma 3.3. □

Lemma 3.5. *Let G be a group, and let X be a countable subgroup of G. Then for every element g of G, there exists a countable subgroup Y of G such that $\langle g, X \rangle \leq Y$ and $C_X(\langle g \rangle^G) = C_X(\langle g \rangle^Y)$.*

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Proof. It can obviously be assumed that $C_X(\langle g\rangle^G) \neq X$, so that in particular $X = \langle X \setminus C_X(\langle g\rangle^G)\rangle$. For each element x of $X \setminus C_X(\langle g\rangle^G)$, choose a finite subset E_x of G such that x does not belong to $C_X(\langle g\rangle^{E_x})$, and put

$$Y = \langle g, x, E_x \mid x \in X \setminus C_X(\langle g\rangle^G)\rangle.$$

Then Y is a countable subgroup of G such that $\langle g, X\rangle \leq Y$ and $C_X(\langle g\rangle^G) = C_X(\langle g\rangle^Y)$. The statement is proved. □

As we mentioned in Section 2, the class of periodic FC-groups is precisely the class $\boldsymbol{M}\mathfrak{F}$. Our next result shows that many classes of the form $\boldsymbol{M}\mathfrak{X}$ are countably recognizable.

Theorem 3.6. *Let $\mathfrak{X}$ be a subgroup closed and countably recognizable group class. Then the class $\boldsymbol{M}\mathfrak{X}$ is countably recognizable.*

Proof. Let G be a group whose countable subgroups belong to $\boldsymbol{M}\mathfrak{X}$, and assume for a contradiction that G is not in $\boldsymbol{M}\mathfrak{X}$. Then there exists a finitely generated subgroup E of G such that the normal closure E^G does not belong to $\mathfrak{X}$. But $\mathfrak{X}$ is countably recognizable, and so E^G contains a countable subgroup U which is not in $\mathfrak{X}$. Let X be a countable subgroup of G such that $U \leq E^X$. Then $H = \langle E, X\rangle$ is a countable subgroup of G, and the normal closure E^H is not in $\mathfrak{X}$, because $\mathfrak{X}$ is subgroup closed and $U \leq E^X \leq E^H$. This contradiction shows that G lies in $\boldsymbol{M}\mathfrak{X}$, and hence $\boldsymbol{M}\mathfrak{X}$ is countably recognizable. □

Since it is known that both the class $\mathfrak{P}$ of all polycyclic-by-finite groups and the class $\mathfrak{C}$ of all Černikov groups are countably recognizable, it follows from the above theorem that the class $\boldsymbol{M}\mathfrak{P}$ (which coincides with the class of all PC-groups) and the class $\boldsymbol{M}\mathfrak{C}$ are countably recognizable. Our next result shows that many other similar classes, and in particular that of arbitrary CC-groups, have countable character.

Theorem 3.7. *Let $\mathfrak{X}$ be a subgroup closed and countably recognizable class of groups. Then the class $\mathfrak{X}C$, consisting of all groups with $\mathfrak{X}$-conjugacy classes, is countably recognizable.*

Proof. Let G be a group whose countable subgroups belong to $\mathfrak{X}C$, and assume for a contradiction that G contains an element g such that $G/C_G(\langle g\rangle^G)$ is not an $\mathfrak{X}$-group. As the class $\mathfrak{X}$ is countably recognizable, there exists a countable subgroup $H/C_G(\langle g\rangle^G)$ of $G/C_G(\langle g\rangle^G)$ which is not in $\mathfrak{X}$. Clearly $H = XC_G(\langle g\rangle^G)$, where X is a suitable countable subgroup, and

$$X/C_X(\langle g\rangle^G) \simeq H/C_G(\langle g\rangle^G)$$

is not in $\mathfrak{X}$. By Lemma 3.5 there exists a countable subgroup Y of G containing $\langle g, X\rangle$ and such that $C_X(\langle g\rangle^G) = C_X(\langle g\rangle^Y)$. Then

$$X/C_X(\langle g\rangle^G) = X/C_X(\langle g\rangle^Y) \simeq XC_Y(\langle g\rangle^Y)/C_Y(\langle g\rangle^Y) \leq Y/C_Y(\langle g\rangle^Y),$$

a contradiction, because $\mathfrak{X}$ is $\boldsymbol{S}$-closed and $Y/C_Y(\langle g\rangle^Y)$ belongs to $\mathfrak{X}$. Therefore G is an $\mathfrak{X}C$-group and the class $\mathfrak{X}C$ is countably recognizable. □

It was claimed in [21], Lemma 5, that the class FC^n is countably recognizable for each non-negative integer n, but unfortunately the proof of this result contains a mistake. However, as the class $FC^0 = \mathfrak{F}$ is obviously countably recognizable, this statement can now be obtained by induction as a consequence of Theorem 3.7. Therefore also Theorem 2 of [21] remains true.

Corollary 3.8. *The class FC^n is countably recognizable for each non-negative integer n.*

Of course, it follows from Lemma 3.3 and Corollary 3.8 that the class

$$FC^* = \bigcup_{n\in\mathbb{N}_0} FC^n$$

is countably recognizable. Our next result shows that also the FC^∞-property can be detected from the behaviour of countable subgroups.

Theorem 3.9. *The class FC^∞ is countably recognizable.*

Proof. Let G be a group whose countable subgroups are FC^∞-groups, and assume for a contradiction that there exists an element g of G such that the factor group $G/C_G(\langle g\rangle^G)$ does not belong to the class FC^*. If n is any non-negative integer n, the class FC^n is countably recognizable class, and so there exists a countable subgroup X_n of G such that $X_nC_G(\langle g\rangle^G)/C_G(\langle g\rangle^G)$ is not in the class FC^n. It follows from Lemma 3.5 that for each n there is a countable subgroup Y_n of G such that $\langle g, X_n\rangle \leq Y_n$ and $C_{X_n}(\langle g\rangle^G) = C_{X_n}(\langle g\rangle^{Y_n})$. The subgroup

$$Y = \langle Y_n \mid n \in \mathbb{N}_0\rangle$$

is countable, so that $Y/C_Y(\langle g\rangle^Y)$ is an FC^k-group for some non-negative integer k, and hence also $X_kC_Y(\langle g\rangle^Y)/C_Y(\langle g$ belongs to FC^k. On the other hand,

$$C_{X_k}(\langle g\rangle^Y) \leq C_{X_k}(\langle g\rangle^{Y_k}) = C_{X_k}(\langle g\rangle^G),$$

and so $C_{X_k}(\langle g\rangle^Y) = C_{X_k}(\langle g\rangle^G)$. Therefore $X_kC_G(\langle g\rangle^G)/C_G(\langle g\rangle^G)$ is an FC^k-group, and this contradiction proves the statement. □

It is easy to show that, like the class of finite-by-abelian groups, also the class of groups which are finite over the centre is countably recognizable. Our next result proves that the third class of groups considered by Neumann has the same property.

Theorem 3.10. *The class of CF-groups is countably recognizable.*

Proof. Let G be a group whose countable subgroups have the CF-property. Assume for a contradiction that G is not a CF-group, and let X be a subgroup of G such that the index $|X : X_G|$ is infinite. Then X/X_G contains a countably infinite subgroup Y/X_G. Let W be a transversal to X_G in Y. If y and z are distinct elements of W, the product $y^{-1}z$ does not belong to $X_G = Y_G$, and so there exists an element $g(y,z)$ of G such that $y^{-1}z$ is not in $Y^{g(y,z)}$. Put $K = \langle W\rangle$, and consider the countable subgroup

$$H = \langle K, g(y,z) \mid y,z \in W, y \neq z\rangle.$$

Then H is a CF-group, so that the index $|K : K_H|$ must be finite, and hence there exist distinct elements y,z of W such that $y^{-1}z$ lies in K_H, a contradiction because

$$K_H \leq K^{g(y,z)} \leq Y^{g(y,z)}.$$

Therefore G has the CF-property, and the class CF is countably recognizable. □

If n is any positive integer, we shall say that a group G has the *CF_n-property* if $|X : X_G| \leq n$ for all subgroups X of G. The same argument used in the proof of Theorem 3.10 shows that the class of CF_n-groups is countably recognizable for each n. Therefore the class

$$BCF = \bigcup_{n\in\mathbb{N}} CF_n$$

is likewise countably recognizable by Lemma 3.3.

Groups with the FNO-property are of course closely related to CF-groups, and they form another countably recognizable group class.

Theorem 3.11. *The class of FNO-groups is countably recognizable.*

Proof. Let G be a group whose countable subgroups have the FNO-property, and assume for a contradiction that G contains a subgroup X such that both indices $|X^G : X|$ and $|X : X_G|$ are infinite. Let $(x_n)_{n\in\mathbb{N}}$ be a countably infinite collection of elements of X such that $x_iX_G \neq x_jX_G$ if $i \neq j$, and put $Y = \langle x_n \mid n \in \mathbb{N}\rangle$. For all positive integers i and j such that $i \neq j$ there exists an element $g(i,j)$ of G such that $x_i^{-1}x_j$ does not belong to $X^{g(i,j)}$. On the other hand, as the index $|X^G : X|$ is infinite, there exist countable subgroups Z of X and U of G such that $Y \leq Z$ and the normal closure Z^U contains an infinite subset W for which $w_1X \neq w_2X$, whenever w_1, w_2 are elements of W and $w_1 \neq w_2$. Then

$$H = \langle Z, U, g(i,j) \mid i \neq j\rangle$$

is a countable subgroup of G, and $x_i^{-1}x_j$ is not in $Z^{g(i,j)}$ if $i \neq j$. It follows that $x_iZ_H \neq x_jZ_H$ for all $i \neq j$, and so the index $|Z : Z_H|$ is infinite. Moreover, $Z^H \geq Z^U \geq W$ and hence also the index $|Z^H : Z|$ is infinite, a contradiction because H is an FNO-group. Therefore G is an FNO-group, and the class FNO is countably recognizable. □

We will now consider the classes of groups with restricted conjugacy classes studied by Robinson in [20].

Theorem 3.12. *The classes FCI and FNI are countably recognizable.*

Proof. Suppose first that G is a group whose countable subgroups have the FCI-property, and assume for a contradiction that G contains a cyclic non-normal subgroup $\langle x\rangle$ such that the index $|C_G(x):\langle x\rangle|$ is infinite. Let g be an element of G such that $\langle x\rangle^g \neq \langle x\rangle$, and consider a countably infinite subgroup $U/\langle x\rangle$ of $C_G(x)/\langle x\rangle$. Then $H = \langle U, g\rangle$ is a countable subgroup of G, and $\langle x\rangle$ is a non-normal subgroup of H which has infinite index in the centralizer $C_H(x)$. This contradiction shows that G is an FCI-group, and so the class FCI is countably recognizable.

Suppose now that G is a group whose countable subgroups belong to FNI, and assume that G contains a non-normal subgroup X such that the index $|N_G(X) : X|$ is infinite. Let $x \in X$ and $g \in G$ be elements such that x^g is not in X, and consider a countable subgroup U of $N_G(X)$ such that x belongs to U and UX/X is infinite. Then $H = \langle U, g\rangle$ is a countable subgroup of G, and $X \cap U$ is a non-normal subgroup of H. Moreover, U is contained in the normalizer $N_H(X \cap U)$, and hence $X \cap U$ has infinite index in $N_H(X \cap U)$. This contradiction proves that G is an FNI-group, and so FNI is a countably recognizable class. □

A central problem in the theory of groups with finite conjugacy classes is to establish conditions under which a periodic residually finite FC-group can be embedded into the direct product of a collection of finite groups. We point out here that the relevant class $\boldsymbol{SD}\mathfrak{F}$, consisting of all groups which are isomorphic to subgroups of direct products of finite groups, is not countably recognizable. Since P. Hall proved that any countable periodic residually finite FC-group belongs to $\boldsymbol{SD}\mathfrak{F}$ (see for instance [3, Theorem 1.5.1]), it is enough to show that there exists an uncountable periodic residually finite FC-group which is not isomorphic to a subgroup of a direct product of finite groups. To prove this, fix a prime number p and let C_n be a cyclic group of order p^n, for each positive integer n. Consider the cartesian product C of the sequence $(C_n)_{n\in\mathbb{N}}$, and let G be the subgroup of all elements of finite order of C. Then G is residually finite, but it cannot be embedded into a direct product of finite groups. Therefore the class $\boldsymbol{SD}\mathfrak{F}$ is not countably recognizable.

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