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FINITE GROUPS WITH THE SAME CONJUGACY CLASS SIZES AS A FINITE SIMPLE GROUP

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Abstract. For a finite group H, let $cs(H)$ denote the set of non-trivial conjugacy class sizes of H and $OC(H)$ be the set of the order components of H. In this paper, we show that if S is a finite simple group with the disconnected prime graph and G is a finite group such that $cs(S) = cs(G)$, then $|S| = |G/Z(G)|$ and $OC(S) = OC(G/Z(G))$. In particular, we show that for some finite simple group S, $G \cong S \times Z(G)$.

1. Introduction

In this paper, all groups are finite and for a group G and $x \in G$, $C_G(x)$ and $cl_G(x)$ are the centralizer of x in G and the conjugacy class in G containing x, respectively and $cs(G)$ denotes the set of non-trivial conjugacy class sizes of G.

A. Camina and R. Camina in [10] found a nilpotent group G and a non-nilpotent group H such that $cs(G) = cs(H) = \{20, 10, 5, 4, 2\}$. This examples show that nilpotency can not be determined by cs.

In [33], Navarro by constructing some examples showed that solvability can not be recognized by cs.

J. Thompson (see [30, Problem 12.38]) conjectured that simplicity can be determined by cs in the class of finite centerless groups. In a series of papers [2], [4]-[7], [17] and [34], the veracity of Thompson's conjecture for some finite simple groups has been studied . In [9], A. Camina and R. Camina asked about the structure of a group with the same cs as a simple group. In the other word, it is interesting to know that whether simplicity can be determined by cs (up to an abelian direct factor).

In 2015, it has been shown that if G is a group with $cs(G) = cs(PSL_2(q))$, where q is a prime power, then $G \cong PSL_2(q) \times Z(G)$ [8]. In this paper, we continue this investigation for some other finite simple groups.

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Throughout this paper, we use the following notation: For a natural number n, let $\pi(n)$ be the set of prime divisors of n and for a group H, let $\pi(H) = \pi(|H|)$. Also, $n.H$ denotes a central extension of H by a cyclic group of order n. For a prime r and natural numbers a and b, $|a|_r$ is the r-part of a, i.e., $|a|_r = r^t$ when $r^t \mid a$ and $r^{t+1} \nmid a$ and, $\gcd(a, b)$ and $\mathrm{lcm}(a, b)$ are the greatest common divisor of a and b and the lowest common multiple of a and b, respectively. For the set of primes π, x is named a π-element (π'-element) of a group H if $\pi(O(x)) \subseteq \pi$ ($\pi(O(x)) \subseteq \pi(H) - \pi$). For a group G, the prime graph $GK(G)$ of G is a simple graph whose vertices are the prime divisors of the order of G and two distinct prime numbers p and q are joined by an edge if G contains an element of order pq. Denote by $t(G)$ the number of connected components of the graph $GK(G)$ and denote by $\pi_i = \pi_i(G)$, $i = 1, \dots, t(G)$, the i-th connected component of $GK(G)$. For a group G of an even order, let $2 \in \pi_1$. If $GK(G)$ is disconnected, then $|G|$ can be expressed as a product of co-prime positive integers $m_i(G)$, $i = 1, 2, \dots, t(G)$, where $\pi(m_i(G)) = \pi_i(G)$, and if there is no ambiguity write m_i for showing $m_i(G)$. These m_i's are called the order components of G and the set of order components of G will be denoted by $OC(G)$. List of all finite simple groups with disconnected prime graph and their sets of order components have been obtained in [32] and [35].

2. The main results

Throughout this section, let S be a simple group with the disconnected prime graph and G be a group with $cs(S) = cs(G)$. Also, fix $\bar{G} = G/Z(G)$ and for $x \in G$, let $\bar{x}$ denote the image of x in $\bar{G}$.

Lemma 2.1. [18, Proposition 4] *Let H be a group. If there exists $p \in \pi(H)$ such that p does not divide any conjugacy class sizes of H, then the p-Sylow subgroup of H is an abelian direct factor of H.*

Lemma 2.2. *Let H be a group and $x, y \in H$.*

(i) *If $xy = yx$ and $\gcd(O(x), O(y)) = 1$, then $C_H(xy) = C_H(x) \cap C_H(y)$. In particular, $C_H(xy) \leq C_H(x)$ and $|cl_H(x)|$ divides $|cl_H(xy)|$.*

(ii) *If $K \leq H$, then $cs(KZ(H)) = cs(K)$.*

Proof. The proof of (i) is straightforward and we obtain (ii) from [3, Lemma 2.2] $\square$

Definition 2.3. *A subgroup K of H is called isolated if for every $h \in H$, $K \cap K^h = 1$ or K and for all x in $K - \{1\}$, $C_H(x) \leq K$.*

Lemma 2.4. *For every $i \geq 2$,*

(i) *$\frac{|S|}{m_i} \in cs(S)$ and $\frac{|S|}{m_i}$ is maximal and minimal in $cs(S)$ by divisibility.*

(ii) *For every $\alpha \in cs(S)$, either $\alpha = \frac{|S|}{m_i}$ or $m_i \mid \alpha$.*

Proof. Since S is a simple group with the odd order component m_i, [35] shows that S contains a subgroup, namely H_i, of the order m_i which is abelian and isolated. So for every $x \in H_i - \{1\}$, $C_S(x) = H_i$ and hence, $|cl_S(x)| = \frac{|S|}{m_i} \in cs(S)$. If $y \in S - \{1\}$ such that $|cl_S(y)| \mid \frac{|S|}{m_i}$, then $m_i \mid |C_S(y)|$. Since m_i is an odd order component of S, we get that $|C_S(y)| = m_i$ and hence, $|cl_S(y)| = \frac{|S|}{m_i}$, so $\frac{|S|}{m_i}$ is minimal in $cs(S)$ by divisibility. Also, since H_i is an abelian π_i-Hall subgroup of S, we can see at once that $\frac{|S|}{m_i}$ is maximal in $cs(S)$ by divisibility and hence, (i) follows.

For proving (ii), let $\alpha \neq \frac{|S|}{m_i}$. Since $\alpha \in cs(S)$, there exists an element $y \in S - \{1\}$ such that $|cl_S(y)| = \alpha$. On the contradiction, suppose that $m_i \nmid \alpha$. Thus $\gcd(|C_S(y)|, m_i) \neq 1$. So there exists a prime q such that $q \mid \gcd(|C_S(y)|, m_i)$ and hence, $q \in \pi_i$ and $C_S(y)$ contains a non-trivial element x of order q. Therefore, $O(xy) = \mathrm{lcm}(O(y), q)$. Since π_i is a connected component of $GK(S)$, we get that y is a π_i-element of S and hence, as mentioned in (i), y can be considered as an element of H_i which is an abelian π_i-Hall subgroup of S. Therefore, $m_i \mid |C_S(y)|$, so $|cl_S(y)| \mid \frac{|S|}{m_i}$ and hence, by minimality of $\frac{|S|}{m_i}$ in $cs(S)$, we get that $|cl_S(y)| = \frac{|S|}{m_i}$, which is a contradiction. This shows that $m_i \mid \alpha$, as wanted in (ii). $\square$

Lemma 2.5. $|S| \mid |G/Z(G)|$.

Proof. By Lemma 2.4(i), for every $i \geq 2$, $\frac{|S|}{m_i} \in cs(S)$. Also, since π_1 is a connected component of $GK(S)$, for every non-trivial π_1-element $x_1 \in S$, $|C_S(x_1)| \mid m_1$ and hence $\Pi_{i=2}^{t(S)} m_i \mid |cl_S(x_1)|$. Therefore, $\mathrm{lcm}\{\alpha : \alpha \in cs(S)\} = |S|$. On the other hand, for every $y \in G$, $Z(G) \leq C_G(y)$ and hence, $|cl_G(y)| \mid [G : Z(G)] = |G/Z(G)|$. Therefore, $|S| = \mathrm{lcm}\{\alpha : \alpha \in cs(S) = cs(G)\} \mid |G/Z(G)|$, as wanted. $\square$

Chen in [17] proved that if G is centerless, then $|S| = |G|$ and $OC(S) = OC(G)$. In the following, we are going to prove the similar facts for an arbitrary group G with $cs(S) = cs(G)$.

Lemma 2.6. *Let $i \geq 2$.*

(i) *For every π_i-element $x \in G - Z(G)$, $|cl_G(x)| = \frac{|S|}{m_i}$.*
(ii) *For every π_i-element $\bar{x} \in \bar{G}$, $|cl_{\bar{G}}(\bar{x})| = \frac{|S|}{m_i}$.*

Proof. If there exists a non-central π_i'-element $y \in G$ such that $|cl_G(y)| = \frac{|S|}{m_i}$, then

$$\gcd(m_i, |cl_G(y)|) = 1$$

and Lemma 2.5 shows that $m_i \mid |C_G(y)/Z(G)|$ and hence, for every $q \in \pi_i$, $C_G(y)$ contains a q-Sylow subgroup of G, so we can assume that for a given non-central q-element z, $z \in C_G(y)$ (up to conjugation). Thus Lemma 2.2(i) guarantees that $|cl_G(y)| = \frac{|S|}{m_i}, |cl_G(z)| \mid |cl_G(yz)|$ and hence, by maximality and minimality of $\frac{|S|}{m_i}$ in $cs(S) = cs(G)$, $|cl_G(z)| = |cl_G(y)| = |cl_G(yz)| = \frac{|S|}{m_i}$. On the other hand, every non-central π_i-element x of G can be written as a product of some π_i-elements of the prime power orders which their orders are co-prime and at least one of them is non-central, so by Lemma 2.2(i), $\frac{|S|}{m_i} \mid |cl_G(x)|$. Thus maximality of $\frac{|S|}{m_i}$ in $cs(G)$ by divisibility forces $|cl_G(x)| = \frac{|S|}{m_i}$, as desired.

Now assume that for every non-central π_i'-element $y \in G$, $|cl_G(y)| \neq \frac{|S|}{m_i}$. Thus since $\frac{|S|}{m_i} \in cs(S) = cs(G)$, there exists a non-central π_i-element $z \in G$ such that $|cl_G(z)| = \frac{|S|}{m_i}$. Lemma 2.2(i) allows us to assume that z is an element of order p^{α} for some $p \in \pi_i$. If there exists $q \in \pi_i - \{p\}$, then since $\gcd(m_i, |cl_G(z)|) = 1$, we deduce that $C_G(z)$ contains every q-element of G (up to conjugation). Also, $|S| \mid |G/Z(G)|$ and hence, G contains some non-central q-elements. Let w be a non-central q-element of $C_G(z)$. Then by Lemma 2.2(i), $|cl_G(z)| = \frac{|S|}{m_i}, |cl_G(w)|$ divide $|cl_G(zw)|$, so maximality and minimality of $\frac{|S|}{m_i}$ in $cs(S) = cs(G)$ forces $|cl_G(w)| = |cl_G(zw)| = |cl_G(z)| = \frac{|S|}{m_i}$. Now let u be a non-central p-element of G. Since $|cl_G(w)| = \frac{|S|}{m_i}$ and $p \in \pi_i$, we have $C_G(w)$ contains a p-Sylow subgroup of G, so we can assume that $u \in C_G(w)$. Thus by Lemma 2.2(i), $|cl_G(w)| = \frac{|S|}{m_i}, |cl_G(u)|$ divide $|cl_G(uw)|$, so maximality and minimality of $\frac{|S|}{m_i}$ in $cs(S) = cs(G)$ forces $|cl_G(u)| = |cl_G(uw)| = |cl_G(u)| = \frac{|S|}{m_i}$. The same reasoning as above shows that for every non-central π_i-element x of a prime power order, $|cl_G(x)| = \frac{|S|}{m_i}$. Also since every non-central π_i-element x of G can be written as a product of some π_i-elements of the prime power orders which their orders are co-prime and at least one of them is non-central, by Lemma 2.2(i), $\frac{|S|}{m_i} \mid |cl_G(x)|$. Thus maximality of $\frac{|S|}{m_i}$ in $cs(G)$ by divisibility forces $|cl_G(x)| = \frac{|S|}{m_i}$, as desired.

Now let $\pi_i = \{p\}$. Since $\gcd(p, |cl_G(z)|) = 1$, we deduce that $C_G(z)$ contains a p-Sylow subgroup P of G and hence, $z \in Z(P) - Z(G)$. If there exists a non-central p-element u of G such that $|cl_G(u)| \neq \frac{|S|}{m_i}$, then Lemma 2.4(ii) shows that $m_i \mid |cl_G(u)|$ and hence,

$$|G/Z(G)|_p = |C_G(u)/Z(G)|_p |cl_G(u)|_p > |S|_p, \tag{2.1}$$

because $u \in C_G(u) - Z(G)$. On the other hand, our assumption implies that if y is a non-central π_i'-element of G, then $|cl_G(y)| \neq \frac{|S|}{m_i}$. Thus Lemma 2.4(ii) shows that $m_i \mid |cl_G(y)|$. Also by (2.1), $p \mid |C_G(y)/Z(G)|$, so $C_G(y)$ contains a non-central p-element v. Thus Lemma 2.2(i) shows that $|cl_G(y)|, |cl_G(v)| \mid |cl_G(yv)|$, so $m_i \mid |cl_G(yv)|$ and hence, $|cl_G(v)| \neq \frac{|S|}{m_i}$. Consequently, Lemma 2.4(ii) forces $m_i \mid |cl_G(v)|$. This shows that $|cl_G(v)|_p = |cl_G(vy)|_p = |cl_G(y)|_p = |m_i|_p = |S|_p$ and hence, $|C_G(v)|_p = |C_G(vy)|_p = |C_G(y)|_p \geq |G/Z(G)|_p/|S|_p \neq 1$, by (2.1). On the other hand, Lemma 2.2(i) yields that $C_G(vy) = C_G(v) \cap C_G(y) \leq C_G(v), C_G(y)$ and without loss of generality, we can assume that $C_P(vy) \in \mathrm{Syl}_p(C_G(vy))$. Thus the facts that "$C_G(vy) \leq C_G(v), C_G(y)$ and $|C_G(vy)|_p = |C_G(v)|_p = |C_G(y)|_p$" guarantee that $C_P(v) = C_P(yv) = C_P(y)$. But v is a central p-element in $C_G(vy)$, so $v \in C_P(vy) \leq P$. Therefore, $Z(P) \leq C_P(v)$ and hence, $z \in Z(P) \leq C_P(y)$. Note that $\gcd(O(y), p) = 1$. Thus Lemma 2.2(i) gives $|cl_G(y)|, |cl_G(z)| \mid |cl_G(yz)|$, so the above statements show that $m_i, \frac{|S|}{m_i} \mid |cl_G(yz)|$. Therefore, $|S| \mid |cl_G(yz)|$ and consequently, $|cl_G(yz)| \notin cs(S)$, which is a contradiction. This contradiction shows that for every non-central π_i-element $u \in G$, $|cl_G(u)| = \frac{|S|}{m_i}$, as wanted in (i).

Now let $\bar{x}$ be a π_i-element of $\bar{G}$. So there exists a non-central π_i-element $y \in G$ such that $\bar{x} = \bar{y}$. Therefore, (i) shows that $|cl_G(y)| = \frac{|S|}{m_i}$. Fix $C_{\bar{G}}(\bar{y}) = C/Z(G)$. Note that $C_G(y)$ is a normal subgroup in C, because for every $g \in C$, $(\bar{g})^{-1}\bar{y}\bar{g} = \bar{y}$, so there exists $z \in Z(G)$ such that $g^{-1}yg = yz$ and hence, for every $h \in C_G(y)$, $(g^{-1}hg)^{-1}y(g^{-1}hg) = y$ which means that $g^{-1}hg \in C_G(y)$. If there exists a π_i'-element $\bar{g} \in C_{\bar{G}}(\bar{y})$, then we can assume that g is a π_i'-element of G and $g^{-1}yg = yz$, for some $z \in Z(G)$. Since $O(y) = O(yz) = \mathrm{lcm}(O(y), O(z))$, we get that z is a π_i-element of $Z(G)$. Also, $ygy^{-1} = gz$ and hence, $O(g) = O(gz) = \mathrm{lcm}(O(g), O(z))$, so $O(z) \mid O(g)$. This forces $z = 1$ and hence, $g \in C_G(y)$. If there exists $g \in G$ such that $\gcd(O(g), m_i), \gcd(O(g), \frac{|S|}{m_i}) \neq 1$ and $\bar{g} \in C_{\bar{G}}(\bar{y})$, then we have $g = g_1g_2 = g_2g_1$, where g_1 is a π_i-element and g_2 is a π_i'-element of G. Therefore, $\bar{y} \in C_{\bar{G}}(\bar{g}_1) \cap C_{\bar{G}}(\bar{g}_2)$. So the above statements show that $g_2 \in C_G(y)$. Also, if $g_1 \notin C_G(y)$, then we have $g_1 \in C - C_G(y)$ and hence, $p \mid |C/C_G(y)|$, for some $p \in \pi_i$. Since $C \leq G$, we get that $|C/C_G(y)| \mid |cl_G(y)| = \frac{|S|}{m_i}$, so $p \mid \frac{|S|}{m_i}$, which is a contradiction. This shows that $g_1 \in C_G(y)$. The same reasoning shows that every π_i-element of C lies in $C_G(y)$ and hence, $C = C_G(y)$, so $|cl_{\bar{G}}(\bar{y})| = |cl_G(y)| = \frac{|S|}{m_i}$, as wanted in (ii). $\square$

Remark 2.7. *Let $q \in \pi(S)$ and Q be a q-Sylow subgroup of S. Then since S is simple and $Z(Q) \neq 1$, we deduce that there exists a non-trivial element $\alpha \in cs(S) = cs(G)$ such that $q \nmid \alpha$, so if $\alpha = |cl_G(y)|$ for some $y \in G$, then y is a non-central element of G and $C_G(y)$ contains a q-Sylow subgroup of G.*

Theorem 2.8. (i) $|G/Z(G)| = |S|$.
(ii) $OC(G/Z(G)) = OC(S)$.

Proof. (i) By Lemma 2.5, $|S| \mid |G/Z(G)|$. Now let there exist $q \in \pi(G/Z(G)) - \pi(S)$. Then since $q \nmid |S|$, we get that q does not divide any conjugacy class sizes of G and hence, Lemma 2.1 forces the q-Sylow subgroup of G to be a subgroup of $Z(G)$, so $q \notin \pi(G/Z(G))$, which is a contradiction. This yields that $\pi(G/Z(G)) = \pi(S)$. If $q \in \pi(G/Z(G)) = \pi(S)$ such that $|G/Z(G)|_q \neq |S|_q$, then we have $|G/Z(G)|_q > |S|_q$, by Lemma 2.5. If $q \notin \pi_2$, then since for a non-central π_2-element $y \in G$, $q \mid |C_G(y)/Z(G)|$, we can assume that $C_G(y)$ contains a non-central q-element z. But $|cl_G(y)| = \frac{|S|}{m_2}$, by Lemma 2.6(i) and hence, Lemmas 2.2(i) and 2.4(i) show that $|cl_G(z)| = |cl_G(yz)| = \frac{|S|}{m_2}$, so $C_G(y) = C_G(z)$. Now let Q be a q-Sylow subgroup of G containing z. Then Remark 2.7 forces $C_G(Q)$ to contain a non-central element w. Without loss of generality, we can assume that w is of a prime power order. Since $C_G(Q) \leq C_G(z) = C_G(y)$, we get that $w \in C_G(y)$. On the other hand, $q \nmid |cl_G(w)|$ and hence, $|cl_G(w)| \neq \frac{|S|}{m_2}$, so it can be concluded from Lemmas 2.4(ii) and 2.6(i) that $m_2 \mid |cl_G(w)|$ and w is a π_2'-element of $C_G(y)$. Thus Lemma 2.2(i) shows that $|cl_G(w)|, |cl_G(y)| = \frac{|S|}{m_2} \mid |cl_G(wy)|$ and hence, $|S| \mid |cl_G(wy)|$, so $|cl_G(wy)| \notin cs(S) = cs(G)$, which is a contradiction. This guarantees that if $q \notin \pi_2$, then $|G/Z(G)|_q = |S|_q$. Now assume that $q \in \pi_2$. By considering the elements of $cs(S) = cs(G)$, we can find a non-central element $v \in G$ of a prime power order such that $|cl_G(v)| = \frac{|S|}{k_1}$, where k_1 is a divisor of m_1. By Lemma 2.6(i), we can assume that v is a π_1-element of G and by our assumption, $q \mid |C_G(v)/Z(G)|$ and hence, we can assume that $C_G(v)$ contains a non-central q-element x. Since $q \in \pi_2$, we conclude from Lemma 2.6(i) that $|cl_G(x)| = \frac{|S|}{m_2}$ and by Lemma 2.2(i), $|cl_G(x)|, |cl_G(v)| \mid |cl_G(vx)|$, so $|S| \mid |cl_G(vx)|$ and hence, $|cl_G(vx)| \notin cs(S) = cs(G)$, which is a contradiction. This shows that $|G/Z(G)|_q = |S|_q$, as wanted in (i). Now we are going to prove (ii). Let $1 \leq i, j \leq t(S)$ such that $i \neq 1, j$. If there exist $p \in \pi_i$ and $q \in \pi_j$ such that p and q are adjacent in $GK(\bar{G})$, then $\bar{G}$ contains a p-element $\bar{x}$ such that $q \mid |C_{\bar{G}}(\bar{x})|$. Also by Lemma 2.6(ii), $|cl_{\bar{G}}(\bar{x})| = \frac{|S|}{m_i}$, so $|cl_{\bar{G}}(\bar{x})|_q = |S|_q = |\bar{G}|_q$, and hence, $|C_{\bar{G}}(\bar{x})|_q = 1$, which is a contradiction. Thus p and q are not adjacent in $GK(\bar{G})$. On the other hand, (i) and Lemma 2.6(ii) show that the order of the centralizer of every non-trivial π_i-element of $\bar{G}$ is m_i and hence, π_i is a connected component of $\bar{G}$. Now let $p, q \in \pi_1$ such that p and q are adjacent in $GK(S)$. So $cs(S) = cs(G)$ contains an element α such that $|\alpha|_p < |S|_p$ and $|\alpha|_q < |S|_q$. Let y be an element of G such that $|cl_G(y)| = \alpha$. Then since $|G/Z(G)| = |S|$, we get that $p, q \mid |C_G(y)/Z(G)|$ and since $C_G(y)/Z(G) \leq C_{\bar{G}}(\bar{y}) \leq C_{\bar{G}}(\bar{y}^m)$, for every natural number m, we can assume that $O(\bar{y})$ is a prime power and $p, q \mid |C_{\bar{G}}(\bar{y})|$. So if p or $q \mid O(\bar{y})$, then p and q are adjacent in $GK(\bar{G})$ and if $O(\bar{y})$ is a power of a prime r, where $r \notin \{p, q\}$, then we have p, r and q, r are adjacent in $GK(\bar{G})$. This shows that there exists a path between p and q and hence, for every path in $GK(S)$ between elements of π_1, there exists a path in $GK(\bar{G})$ between elements of π_1, so since π_js, for $j \geq 2$, are connected components of $GK(\bar{G})$, we get that π_1 is a connected component of $GK(\bar{G})$, too. Also $|\bar{G}| = |S|$ and hence, $OC(\bar{G}) = OC(S)$, as desired in (ii). $\square$

Definition 2.9. [17] *For a group H, the number of isomorphism classes of groups with the same set $OC(H)$ of order components is denoted by $h(OC(H))$. If $h(OC(H)) = k$, then H is called k-recognizable by the set of its order components and if $k = 1$, then H is simply called OC-characterizable or OC-recognizable.*

In many papers, it has been shown that many finite simple groups with disconnected prime graphs are OC-characterizable, for example see [11]-[29].

Corollary 2.10. *If S is OC-characterizable, then $G/Z(G) \cong S$.*

Proof. Since by Theorem 2.8(ii), $OC(\bar{G}) = OC(S)$, the result follows from the OC-recognizability of S. $\square$

Definition 2.11. [31] *A central extension of a group H is a group K such that $K/Z(K) \cong H$. A central extension of H which is perfect is called a covering group of H. Also, if K is a covering group of H such that $K \ncong H$, then K is named a proper covering group of H. It was shown by Schur that there is a unique covering group of the maximal order, called the full covering group of H. The center of the full covering group of H is denoted by $M(H)$ and it is called the Schur multiplier of H.*

Theorem 2.12. *If S is OC-characterizable and there is no proper covering group of S with the same cs as $cs(S)$, then $G \cong S \times Z(G)$.*

Proof. On the contrary, suppose that G is the smallest group such that $cs(S) = cs(G)$ and $G \ncong S \times Z(G)$. Then since by Corollary 2.10, $G/Z(G) \cong S$ and $G'Z(G)/Z(G)$ is a normal subgroup of $G/Z(G)$, we get that $G'Z(G)/Z(G) = 1$ or $G/Z(G)$. Also G is non-solvable and hence, the former case can not occur. Thus $G'Z(G)/Z(G) = G/Z(G)$ and hence, $G'Z(G) = G$, so Lemma 2.2(ii) shows that $cs(S) = cs(G) = cs(G'Z(G)) = cs(G')$. Thus our assumption forces $G' = G$, so G is perfect and hence, G is a proper covering group of S with the same cs as $cs(S)$. This is a contradiction with our assumption. Therefore $G \cong S \times Z(G)$, as desired. (Note that some part of this proof is similar to that of in [8].) $\square$

Theorem 2.13. *If S is OC-characterizable and $M(S) = 1$, then $G \cong S \times Z(G)$.*

Proof. It follows immediately from Theorem 2.12. $\square$

Theorem 2.14. *If S is one of the groups in Table 1 (up to isomorphism), then $G \cong S \times Z(G)$.*

Proof. Since S is OC-characterizable, by the references stated in the third column of Table 1, we deduce from Theorem 2.13 that if $M(S) = 1$, then $G \cong S \times Z(G)$, as desired. In the following, we are going to study the remaining cases, with the help of [1]. For this aim let H be a covering group of S such that $S \ncong H$. Then considering $M(S)$ shows that:

- If $S = M_{12}$, then $H = 2.M_{12}$ and hence, $792 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = J_2$, then $H = 2.J_2$ and hence, $5040 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.

- If $S = HS$, then $H = 2.HS$ and hence, $30800 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = RU$, then $H = 2.RU$ and hence, $57002400 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = J_3$, then $H = 3.J_3$ and hence, $620160 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = McL$, then $H = 3.McL$ and hence, $99792000 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = Suz$, then $H = 2.Suz, 3.Suz$ or $6.Suz$ and hence, 4670265600 or $415134720 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = M_{22}$, then $H = 2.M_{22}, 3.M_{22}, 4.M_{22}, 6.M_{22}$ or $12.M_{22}$ and hence, $27720 \in cs(S) - cs(H)$, $12320 \in cs(S) - cs(H)$ or $2310 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = Suz(8)$, then $H = 2.Suz(8)$ or $H = 4.Suz(8)$ and hence, $455 \in cs(S) - cs(H)$, so $cs(H) \neq cs(S)$.
- If $S = G_2(3)$, then $H = 3.G_2(3)$ and hence, $2184 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = G_2(4)$, then $H = 2.G_2(4)$ and hence, $131040 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = PSL_n(q)$, where $n > 2$ is prime, $(n, q) \neq (3, 2), (3, 4)$ and $\gcd(n, q-1) = n$, then $H = n.PSL_n(q) \cong SL_n(q)$. Let $GF(q)$ be a field with q elements and $(GF(q))^* = GF(q) - \{0\}$. Then $(GF(q))^*$ is a cyclic group of order $q-1$. Since $n \mid q-1$, we can assume that $(GF(q))^*$ contains an element ξ of the order n. Set $x = \mathrm{diag}(1, \xi, \xi^2, \dots, \xi^{n-1}) \in SL_n(q)$ and $Z = Z(SL_n(q))$, where $\mathrm{diag}(1, \xi, \xi^2, \dots, \xi^{n-1})$ means a diagonal matrix with numbers $1, \xi, \xi^2, \dots, \xi^{n-1}$ on a diagonal. We can check at once that $C_{PSL_n(q)}(xZ) = \langle \tau_1 Z \rangle \rtimes \langle \tau_2 Z \rangle$, where for $a_1, \dots, a_{n-1} \in (GF(q))^*$, $\tau_1 = \mathrm{diag}(a_1, \dots, a_{n-1}, (a_1 \cdots a_{n-1})^{-1})$ and
$$\tau_2 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix},$$
and hence, $|cl_{PSL_n(q)}(xZ)| = \frac{|SL_n(q)|}{n(q-1)^{n-1}} \in cs(PSL_n(q))$. We claim that $\frac{|SL_n(q)|}{n(q-1)^{n-1}} \notin cs(SL_n(q))$. If not, then there exists $y \in SL_n(q)$ such that $|C_{SL_n(q)}(y)| = n(q-1)^{n-1}$. Since $y \in C_{SL_n(q)}(y)$ and $\gcd(|C_{SL_n(q)}(y)|, q) = 1$, we get that y is a semi-simple element of $SL_n(q)$, so there exists a maximal torus T of $SL_n(q)$ containing y. Thus $T \leq C_{SL_n(q)}(y)$ and hence, $|T|$ divides $n(q-1)^{n-1}$. This forces $|T| = (q-1)^{n-1}$ and hence, we can assume that $y = \mathrm{diag}(y_1, \dots, y_{n-1}, y_n = (y_1 \cdots y_{n-1})^{-1})$ for some $y_1, \dots, y_{n-1} \in (GF(q))^*$. Since $|C_{SL_n(q)}(y)| = n(q-1)^{n-1}$, we can check at once that y_is are distinct and hence, $C_{SL_n(q)}(y) = T$, so $|C_{SL_n(q)}(y)| = (q-1)^{n-1}$, which is a contradiction. This shows that $cs(S) \neq cs(H)$.
- If $S = PSU_n(q)$, where $n > 2$ is prime, $(n, q) \neq (3, 2), (5, 2)$ and $\gcd(n, q+1) = n$, then $H = n.PSU_n(q) \cong SU_n(q)$ and hence, by replacing $GF(q)$ with $GF(q^2)$ in the case $S = PSL_n(q)$, we can see that $cs(H) \neq cs(S)$.

The above consideration shows that if S is one of the groups mentioned in Table 1 with $M(S) \neq 1$, then there is no proper covering group of S with the same cs as $cs(S)$ and hence, Theorem 2.12 forces $G \cong S \times Z(G)$, as desired. Note that if $S \cong PSL_3(2) \cong PSL_2(7)$, then it has been shown in [8] that $G \cong S \times Z(G)$. So theorem follows. □

TABLE 1.

Group S	Conditions	References for OC-recognition	$\lvert M(S)\rvert$
M_{11}, M_{23}, M_{24} J_1, J_4, He, HN, Ly, Th Co_2, Co_3, Fi_{23}, M		[14]	1
M_{12}, J_2, HS, RU		[14]	2
J_3, McL		[14]	3
Suz		[14]	6
M_{22}		[14]	12
${}^2G_2(3^{2n+1}), {}^2F_4(2^{2n+1})$ $Sz(2^{2n+1})$	$n \geq 1$ $n \geq 2$	[13]	1 1
$Sz(8)$		[13]	4
$G_2(q)$	$q \neq 2, 3, 4$	[12],[15]	1
$G_2(3)$		[12],[15]	3
$G_2(4)$		[12],[15]	2
$E_8(q)$		[16]	1
$E_6(q)$	$\gcd(3, q-1) = 1$	[29]	1
$F_4(q)$	$q > 2$	[21],[23]	1
$PSL_n(q)$	$n > 2$ is prime and $(n, q) \neq (3, 2), (3, 4)$	[26]	$\gcd(n, q-1)$
$PSL_3(2) \cong PSL_2(7)$		[8]	2
$PSU_n(q)$	$n > 2$ is prime and $(n, q) \neq (3, 2)$, $(5, 2)$	[24]	$\gcd(n, q+1)$
$C_n(2^m)$	either $n = 2$ and $m > 2$ or $n = 2^u \geq 4$	[22] [25]	1
$D_{n+1}(2)$	$n > 3$ is prime	[19]	1
${}^2D_n(2^m)$	either $m = 1$ and $n = 2^u + 1 \geq 5$ or $n = 2^u \geq 4$	[20] [27]	1
${}^2E_6(q)$	$\gcd(3, q+1) = 1$ and $q \neq 2$	[28]	1
${}^3D_4(q^3)$		[11]	1

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