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RIGHT AMENABLE LEFT GROUP SETS AND THE TARSKI-FØLNER THEOREM

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ABSTRACT. We introduce right amenability, right Følner nets, and right paradoxical decompositions for left homogeneous spaces and prove the Tarski-Følner theorem for left homogeneous spaces with finite stabilisers. It states that right amenability, the existence of right Følner nets, and the non-existence of right paradoxical decompositions are equivalent.

1. Introduction

The notion of amenability for groups was introduced by John von Neumann in 1929, see the paper ‘Zur allgemeinen Theorie des Mas’[3]. It generalises the notion of finiteness. A group G is *left* or *right amenable* if there is a finitely additive probability measure on $\mathcal{P}(G)$ that is invariant under left and right multiplication respectively. Groups are left amenable if and only if they are right amenable. A group is *amenable* if it is left or right amenable.

The definitions of left and right amenability generalise to left and right group sets respectively. A left group set (M, G, \triangleright) is *left amenable* if there is a finitely additive probability measure on $\mathcal{P}(M)$ that is invariant under \triangleright . There is in general no natural action on the right that is to a left group action what right multiplication is to left group multiplication. Therefore, for a left group set there is no natural notion of right amenability.

A transitive left group action \triangleright of G on M induces, for each element $m_0 \in M$ and each family $\{g_{m_0, m}\}_{m \in M}$ of elements in G such that, for each point $m \in M$, we have $g_{m_0, m} \triangleright m_0 = m$, a right

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quotient set semi-action \triangleleft of G/G_0 on M with defect G_0 given by $m \triangleleft gG_0 = g_{m_0,m} g g_{m_0,m}^{-1} \triangleright m$, where G_0 is the stabiliser of m_0 under \triangleright . Each of these right semi-actions is to the left group action what right multiplication is to left group multiplication. They occur in the definition of global transition functions of cellular automata over left homogeneous spaces as defined in [5]. A *coordinate system* is a choice of m_0 and $\{g_{m_0,m}\}_{m \in M}$.

A left homogeneous space is *right amenable* if there is a coordinate system such that there is a finitely additive probability measure on $\mathcal{P}(M)$ that is semi-invariant under \triangleleft . For example finite left homogeneous spaces, abelian groups, and finitely right generated left homogeneous spaces of sub-exponential growth are right amenable, in particular, quotients of finitely generated groups of sub-exponential growth by finite subgroups acted on by left multiplication.

A net of non-empty and finite subsets of M is a *right Følner net* if, broadly speaking, these subsets are asymptotically invariant under \triangleleft . A finite subset E of G/G_0 and two partitions $\{A_e\}_{e \in E}$ and $\{B_e\}_{e \in E}$ of M constitute a *right paradoxical decomposition* if the map $_ \triangleleft e$ is injective on A_e and B_e , and the family $\{(A_e \triangleleft e) \cup (B_e \triangleleft e)\}_{e \in E}$ is a partition of M . The Tarski-Følner theorem states that right amenability, the existence of right Følner nets, and the non-existence of right paradoxical decompositions are equivalent.

The Tarski alternative theorem and the theorem of Følner, which constitute the Tarski-Følner theorem, are famous theorems by Alfred Tarski and Erling Følner from 1938 and 1955, see the papers ‘Algebraische Fassung des Maßproblems’[4] and ‘On groups with full Banach mean value’[2]. This paper is greatly inspired by the monograph ‘Cellular Automata and Groups’[1] by Tullio Ceccherini-Silberstein and Michel Coornaert.

For a right amenable left homogeneous space with finite stabilisers we may choose a right Følner net. Using this net we show in [6] that the Garden of Eden theorem holds for such spaces. It states that a cellular automaton with finite set of states and finite neighbourhood over such a space is surjective if and only if it is pre-injective.

In Sect. 2 we introduce finitely additive probability measures and means, and kind of right semi-actions on them. In Sect. 3 we introduce right amenability. In Sect. 4 we introduce right Følner nets. In Sect. 5 we introduce right paradoxical decompositions. In Sect. 6 we prove the Tarski alternative theorem and the theorem of Følner. And in Sect. 7 we show under which assumptions left implies right amenability and give two examples of right amenable left homogeneous spaces.

Preliminary Notions. A *left group set* is a triple (M, G, \triangleright) , where M is a set, G is a group, and \triangleright is a map from $G \times M$ to M , called *left group action of G on M* , such that $G \rightarrow \text{Sym}(M)$, $g \mapsto [g \triangleright _]$, is a group homomorphism. The action \triangleright is *transitive* if M is non-empty and for each $m \in M$ the map $_ \triangleright m$ is surjective; and *free* if for each $m \in M$ the map $_ \triangleright m$ is injective. For each $m \in M$, the set $G \triangleright m$ is the *orbit of m* , the set $G_m = (_ \triangleright m)^{-1}(m)$ is the *stabiliser of m* , and, for each $m' \in M$, the set $G_{m,m'} = (_ \triangleright m)^{-1}(m')$ is the *transporter of m to m'* .

A *left homogeneous space* is a left group set $\mathcal{M} = (M, G, \triangleright)$ such that \triangleright is transitive. A *coordinate system* for \mathcal{M} is a tuple $\mathcal{K} = (m_0, \{g_{m_0, m}\}_{m \in M})$, where $m_0 \in M$ and, for each $m \in M$, we have $g_{m_0, m} \triangleright m_0 = m$. The stabiliser G_{m_0} is denoted by G_0 . The tuple $\mathcal{R} = (\mathcal{M}, \mathcal{K})$ is a *cell space*. The set $\{gG_0 \mid g \in G\}$ of left cosets of G_0 in G is denoted by G/G_0 . The map $\triangleleft: M \times G/G_0 \rightarrow M$, $(m, gG_0) \mapsto g_{m_0, m}g \triangleright m_0$ is a *right semi-action of G/G_0 on M with defect G_0* , which means that

$$\forall m \in M : m \triangleleft G_0 = m,$$

$$\forall m \in M \forall g \in G \exists g_0 \in G_0 : \forall \mathbf{g}' \in G/G_0 : m \triangleleft g \cdot \mathbf{g}' = (m \triangleleft gG_0) \triangleleft g_0 \cdot \mathbf{g}'.$$

It is *transitive*, which means that the set M is non-empty and for each $m \in M$ the map $m \triangleleft _$ is surjective; and *free*, which means that for each $m \in M$ the map $m \triangleleft _$ is injective; and *semi-commutes with \triangleright* , which means that

$$\forall m \in M \forall g \in G \exists g_0 \in G_0 : \forall \mathbf{g}' \in G/G_0 : (g \triangleright m) \triangleleft \mathbf{g}' = g \triangleright (m \triangleleft g_0 \cdot \mathbf{g}').$$

For each $A \subseteq M$, let $\mathbf{1}_A: M \rightarrow \{0, 1\}$ be the indicator function of A .

2. Finitely Additive Probability Measures and Means

In this section, let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0, m}\}_{m \in M}))$ be a cell space.

Definition 2.1. Let $\mu: \mathcal{P}(M) \rightarrow [0, 1]$ be a map. It is called

- (1) *normalised* if and only if $\mu(M) = 1$;
- (2) *finitely additive* if and only if,

$$\forall A \subseteq M \forall B \subseteq M : (A \cap B = \emptyset \implies \mu(A \cup B) = \mu(A) + \mu(B));$$

- (3) *finitely additive probability measure on M* if and only if it is normalised and finitely additive.

The set of all finitely additive probability measures on M is denoted by $\mathcal{PM}(M)$.

Definition 2.2. The group G acts on $[0, 1]^{\mathcal{P}(M)}$ on the left by

$$\begin{aligned} \vDash: G \times [0, 1]^{\mathcal{P}(M)} &\rightarrow [0, 1]^{\mathcal{P}(M)}, \\ (g, \varphi) &\mapsto [A \mapsto \varphi(g^{-1} \triangleright A)], \end{aligned}$$

such that $G \vDash \mathcal{PM}(M) \subseteq \mathcal{PM}(M)$.

Definition 2.3. The quotient set G/G_0 kind of semi-acts on $[0, 1]^{\mathcal{P}(M)}$ on the right by

$$\begin{aligned} \vDash: [0, 1]^{\mathcal{P}(M)} \times G/G_0 &\rightarrow [0, 1]^{\mathcal{P}(M)}, \\ (\varphi, \mathbf{g}) &\mapsto [A \mapsto \varphi(A \triangleleft \mathbf{g})]. \end{aligned}$$

Definition 2.4. Let φ be an element of $[0, 1]^{\mathcal{P}(M)}$. It is called *\vDash -semi-invariant* if and only if, for each element $\mathbf{g} \in G/G_0$ and each subset A of M such that the map $_ \triangleleft \mathbf{g}$ is injective on A , we have $(\varphi \vDash \mathbf{g})(A) = \varphi(A)$.

Remark 2.5. Let \mathcal{R} be the cell space $((G, G, \dots), (e_G, \{g\}_{g \in G}))$. Then, $G_0 = \{e_G\}$ and $\trianglelefteq = \cdot$. Hence, $\triangleright: (\varphi, g) \mapsto [A \mapsto \varphi(A \cdot g)]$. Except for g not being inverted, this is the right group action of G on $\mathcal{PM}(M)$ as defined in [1, Sect. 4.3, Paragraph 4]. Moreover, for each element $g \in G$, the map $_ \trianglelefteq g$ is injective. Hence, being \triangleright -semi-invariant is the same as being right-invariant as defined in [1, Sect. 4.4, Paragraph 2].

Definition 2.6. The vector space of bounded real-valued functions on M with pointwise addition and scalar multiplication is denoted by $\ell^\infty(M)$, the supremum norm on $\ell^\infty(M)$ is denoted by $\| _ \|_\infty$, the topological dual space of $\ell^\infty(M)$ is denoted by $\ell^\infty(M)^*$, the pointwise partial order on $\ell^\infty(M)$ is denoted by \leq , and the constant function $[m \mapsto 0]$ is denoted by 0 .

Definition 2.7. Let $\nu: \ell^\infty(M) \rightarrow \mathbb{R}$ be a map. It is called

- (1) *normalised* if and only if $\nu(\mathbb{1}_M) = 1$;
- (2) *non-negativity preserving* if and only if

$$\forall f \in \ell^\infty(M) : (f \geq 0 \implies \nu(f) \geq 0);$$

- (3) *mean on M* if and only if it is linear, normalised, and non-negativity preserving.

The set of all means on M is denoted by $\mathcal{M}(M)$.

Definition 2.8. Let Ψ be a map from $\ell^\infty(M)$ to $\ell^\infty(M)$. It is called *non-negativity preserving* if and only if

$$\forall f \in \ell^\infty(M) : (f \geq 0 \implies \Psi(f) \geq 0).$$

Lemma 2.9. Let G_0 be finite, let A be a finite subset of M , and let \mathfrak{g} be an element of G/G_0 . Then, $|(_ \trianglelefteq \mathfrak{g})^{-1}(A)| \leq |G_0| \cdot |A|$.

Proof. Let $a \in A$ such that $(_ \trianglelefteq \mathfrak{g})^{-1}(a) \neq \emptyset$. There are m and $m' \in M$ such that $G_{m_0, m} = \mathfrak{g}$ and $m' \trianglelefteq \mathfrak{g} = a$. For each $m'' \in M$, we have $m'' \trianglelefteq \mathfrak{g} = g_{m_0, m''} \triangleright m$ and hence

$$\begin{aligned} m'' \trianglelefteq \mathfrak{g} = a &\iff m'' \trianglelefteq \mathfrak{g} = m' \trianglelefteq \mathfrak{g} \\ &\iff g_{m_0, m'}^{-1} g_{m_0, m''} \triangleright m = m \\ &\iff g_{m_0, m'}^{-1} g_{m_0, m''} \in G_m \\ &\iff g_{m_0, m''} \in g_{m_0, m'} G_m. \end{aligned}$$

Moreover, for each m'' and each $m''' \in M$ with $m'' \neq m'''$, we have $g_{m_0, m''} \neq g_{m_0, m'''}$. Thus,

$$\begin{aligned} |(_ \trianglelefteq \mathfrak{g})^{-1}(a)| &= |\{m'' \in M \mid m'' \trianglelefteq \mathfrak{g} = a\}| \\ &= |\{m'' \in M \mid g_{m_0, m''} \in g_{m_0, m'} G_m\}| \\ &\leq |g_{m_0, m'} G_m| \\ &= |G_m| \end{aligned}$$

$$= |G_0|.$$

Therefore, because $(_ \trianglelefteq \mathfrak{g})^{-1}(A) = \bigcup_{a \in A} (_ \trianglelefteq \mathfrak{g})^{-1}(a)$, we have $|(_ \trianglelefteq \mathfrak{g})^{-1}(A)| \leq |G_0| \cdot |A|$. □

Definition 2.10. The group G acts on $\ell^\infty(M)$ on the left by

$$\begin{aligned} \vdash: G \times \ell^\infty(M) &\rightarrow \ell^\infty(M), \\ (g, f) &\mapsto [m \mapsto f(g^{-1} \triangleright m)]. \end{aligned}$$

Lemma 2.11. Let G_0 be finite. The quotient set G/G_0 kind of semi-acts on $\ell^\infty(M)$ on the right by

$$\begin{aligned} \dashv: \ell^\infty(M) \times G/G_0 &\rightarrow \ell^\infty(M), \\ (f, \mathfrak{g}) &\mapsto [m \mapsto \sum_{m' \in (_ \trianglelefteq \mathfrak{g})^{-1}(m)} f(m')], \end{aligned}$$

such that, for each tuple $(f, \mathfrak{g}) \in \ell^\infty(M) \times G/G_0$, we have $\|f \dashv \mathfrak{g}\|_\infty \leq |G_0| \cdot \|f\|_\infty$.

Proof. Let $\mathfrak{g} \in G/G_0$. Furthermore, let $f \in \ell^\infty(M)$. Moreover, let $m \in M$. Because G_0 is finite, according to Lemma 2.9, we have $|(_ \trianglelefteq \mathfrak{g})^{-1}(m)| \leq |G_0| < \infty$. Hence, the sum in the definition of \dashv is finite. Furthermore,

$$\begin{aligned} |(f \dashv \mathfrak{g})(m)| &\leq \sum_{m' \in (_ \trianglelefteq \mathfrak{g})^{-1}(m)} |f(m')| \\ &\leq \left(\sum_{m' \in (_ \trianglelefteq \mathfrak{g})^{-1}(m)} 1 \right) \cdot \|f\|_\infty \\ &= |(_ \trianglelefteq \mathfrak{g})^{-1}(m)| \cdot \|f\|_\infty \\ &\leq |G_0| \cdot \|f\|_\infty. \end{aligned}$$

Therefore, $f \dashv \mathfrak{g} \in \ell^\infty(M)$, $\|f \dashv \mathfrak{g}\|_\infty \leq |G_0| \cdot \|f\|_\infty$, and \dashv is well-defined. □

Remark 2.12. In the situation of Remark 2.5, we have $\dashv: (f, g) \mapsto [m \mapsto f(m \cdot g^{-1})]$. Hence, \dashv is the right group action of G on \mathbb{R}^G as defined in [1, Sect. 4.3, Paragraph 5].

Lemma 2.13. Let G_0 be finite and let \mathfrak{g} be an element of G/G_0 . The map $_ \dashv \mathfrak{g}$ is linear, continuous, and non-negativity preserving.

Proof. Linearity follows from linearity of summation, continuity follows from linearity and $\|_ \dashv \mathfrak{g}\|_\infty \leq |G_0| \cdot \|_ \|_\infty$, and non-negativity preservation follows from non-negativity preservation of summation. □

Lemma 2.14 ([1, Proposition 4.1.7]). Let ν be a mean on M . Then, $\nu \in \ell^\infty(M)^*$ and $\|\nu\|_{\ell^\infty(M)^*} = 1$. In particular, ν is continuous. □

Definition 2.15. The group G acts on $\ell^\infty(M)^*$ on the left by

$$\begin{aligned} \vDash: G \times \ell^\infty(M)^* &\rightarrow \ell^\infty(M)^*, \\ (g, \psi) &\mapsto [f \mapsto \psi(g^{-1} \vDash f)], \end{aligned}$$

such that $G \vDash \mathcal{M}(M) \subseteq \mathcal{M}(M)$.

Definition 2.16. Let G_0 be finite. The quotient set G/G_0 kind of semi-acts on $\ell^\infty(M)^*$ on the right by

$$\begin{aligned} \vDash: \ell^\infty(M)^* \times G/G_0 &\rightarrow \ell^\infty(M)^*, \\ (\psi, \mathfrak{g}) &\mapsto [f \mapsto \psi(f \vDash \mathfrak{g})]. \end{aligned}$$

Proof. Let $\psi \in \ell^\infty(M)^*$ and let $\mathfrak{g} \in G/G_0$. Then, $\psi \vDash \mathfrak{g} = \psi \circ (_ \vDash \mathfrak{g})$. Because ψ and $_ \vDash \mathfrak{g}$ are linear and continuous, so is $\psi \vDash \mathfrak{g}$. □

Definition 2.17. Let G_0 be finite and let ψ be an element of $\ell^\infty(M)^*$. It is called *\vDash -invariant* if and only if, for each element $\mathfrak{g} \in G/G_0$ and each function $f \in \ell^\infty(M)$, we have $(\psi \vDash \mathfrak{g})(f) = \psi(f)$.

Remark 2.18. In the situation of Remark 2.12, we have $\vDash: (\psi, g) \mapsto [f \mapsto \psi(f \vDash g)]$. Except for g not being inverted, this is the right group action of G on $\ell^\infty(G)^*$ as defined in [1, Sect. 4.3, Paragraph 6]. Hence, being \vDash -invariant is the same as being right-invariant as defined in [1, Sect. 4.4, Paragraph 3].

Theorem 2.19 ([1, Theorem 4.1.8]). *The map*

$$\begin{aligned} \Phi: \mathcal{M}(M) &\rightarrow \mathcal{PM}(M), \\ \nu &\mapsto [A \mapsto \nu(\mathbb{1}_A)], \end{aligned}$$

is bijective. □

Theorem 2.20 ([1, Theorem 4.2.1]). *The set $\mathcal{M}(M)$ is a convex and compact subset of $\ell^\infty(M)^*$ equipped with the weak-* topology.* □

3. Right Amenability

In Definition 3.2 we introduce the notion of right amenability using finitely additive probability measures. And in Theorem 3.7 we characterise right amenability of cell spaces with finite stabilisers using means.

Definition 3.1. Let (M, G, \triangleright) be a left group set. It is called *left amenable* if and only if there is a \vDash -invariant finitely additive probability measure on M .

Definition 3.2. Let $\mathcal{M} = (M, G, \triangleright)$ be a left homogeneous space. It is called *right amenable* if and only if there is a coordinate system $\mathcal{K} = (m_0, \{g_{m_0, m}\}_{m \in M})$ for \mathcal{M} such that there is a \vDash -semi-invariant

finitely additive probability measure on M , in which case the cell space $\mathcal{R} = (\mathcal{M}, \mathcal{K})$ is called *right amenable*.

Remark 3.3. In the situation of Remark 2.5, being right amenable is the same as being amenable as defined in [1, Definition 4.4.5].

In the remainder of this section, let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0, m}\}_{m \in M}))$ be a cell space such that the stabiliser G_0 of m_0 under \triangleright is finite.

Lemma 3.4. *Let \mathfrak{g} be an element of G/G_0 and let A be a subset of M such that the map $_ \trianglelefteq \mathfrak{g}$ is injective on A . Then, $\mathbb{1}_{A \trianglelefteq \mathfrak{g}} = \mathbb{1}_A \# \mathfrak{g}$.*

Proof. For each $m \in M$, because $_ \trianglelefteq \mathfrak{g}$ is injective on A ,

$$\begin{aligned} \mathbb{1}_{A \trianglelefteq \mathfrak{g}}(m) &= \begin{cases} 1, & \text{if } m \in A \trianglelefteq \mathfrak{g}, \\ 0, & \text{otherwise,} \end{cases} \\ &= |\{m' \in A \mid m' \trianglelefteq \mathfrak{g} = m\}| \\ &= \sum_{m' \in (_ \trianglelefteq \mathfrak{g})^{-1}(m)} \mathbb{1}_A(m') \\ &= (\mathbb{1}_A \# \mathfrak{g})(m). \end{aligned}$$

In conclusion, $\mathbb{1}_{A \trianglelefteq \mathfrak{g}} = \mathbb{1}_A \# \mathfrak{g}$. □

Lemma 3.5 ([1, Lemma 4.1.9]). *The vector space*

$$\mathcal{E}(M) = \{f: M \rightarrow \mathbb{R} \mid f(M) \text{ is finite}\} (= \text{span}\{\mathbb{1}_A \mid A \subseteq M\})$$

is dense in the Banach space $(\ell^\infty(M), \|_ \|_\infty)$. □

Lemma 3.6. *Let ψ be an element of $\ell^\infty(M)^*$ such that, for each element $\mathfrak{g} \in G/G_0$ and each subset A of M such that the map $_ \trianglelefteq \mathfrak{g}$ is injective on A , we have $(\psi \# \mathfrak{g})(\mathbb{1}_A) = \psi(\mathbb{1}_A)$. The map ψ is $\#$ -invariant.*

Proof. Let $\mathfrak{g} \in G/G_0$.

First, let $A \subseteq M$. Moreover, let $m \in M$. According to Lemma 2.9, we have $k_m = |(_ \trianglelefteq \mathfrak{g})^{-1}(m)| \leq |G_0|$. Hence, there are pairwise distinct $m_{m,1}, m_{m,2}, \dots, m_{m,k_m} \in M$ such that $(_ \trianglelefteq \mathfrak{g})^{-1}(m) = \{m_{m,1}, m_{m,2}, \dots, m_{m,k_m}\}$. For each $i \in \{1, 2, \dots, |G_0|\}$, put

$$A_i = \{m_{m,i} \mid m \in M, k_m \geq i\} \cap A.$$

Because, for each $m \in M$ and each $m' \in M$ such that $m \neq m'$, we have $(_ \trianglelefteq \mathfrak{g})^{-1}(m) \cap (_ \trianglelefteq \mathfrak{g})^{-1}(m') = \emptyset$, the sets $A_1, A_2, \dots, A_{|G_0|}$ are pairwise disjoint and the map $_ \trianglelefteq \mathfrak{g}$ is injective on each of these sets.

Moreover, because $\bigcup_{m \in M} (_ \trianglelefteq \mathfrak{g})^{-1}(m) = M$, we have $\bigcup_{i=1}^{|G_0|} A_i = A$. Therefore, $\mathbb{1}_A = \sum_{i=1}^{|G_0|} \mathbb{1}_{A_i}$. Thus, because $\psi \# \mathfrak{g}$ and ψ are linear,

$$(\psi \# \mathfrak{g})(\mathbb{1}_A) = (\psi \# \mathfrak{g})\left(\sum_{i=1}^{|G_0|} \mathbb{1}_{A_i}\right) = \sum_{i=1}^{|G_0|} (\psi \# \mathfrak{g})(\mathbb{1}_{A_i}) = \sum_{i=1}^{|G_0|} \psi(\mathbb{1}_{A_i}) = \psi(\mathbb{1}_A).$$

Therefore, $\psi \# \mathfrak{g} = \psi$ on the set of indicator functions.

Thus, because the indicator functions span $\mathcal{E}(M)$, and $\psi \# \mathfrak{g}$ and ψ are linear, $\psi \# \mathfrak{g} = \psi$ on $\mathcal{E}(M)$. Hence, because $\mathcal{E}(M)$ is dense in $\ell^\infty(M)$, and $\psi \# \mathfrak{g}$ and ψ are continuous, $\psi \# \mathfrak{g} = \psi$ on $\ell^\infty(M)$.

In conclusion, ψ is $\#$ -invariant. □

Theorem 3.7. *The cell space \mathcal{R} is right amenable if and only if there is a $\#$ -invariant mean on M .*

Proof. Let Φ be the map in Theorem 2.19.

First, let \mathcal{R} be right amenable. Then, there is $\#$ -semi-invariant finitely additive probability measure μ on M . Put $\nu = \Phi^{-1}(\mu)$. Then, for each $\mathfrak{g} \in G/G_0$ and each $A \subseteq M$ such that $_ \trianglelefteq \mathfrak{g}$ is injective on A , according to Lemma 3.4,

$$(\nu \# \mathfrak{g})(\mathbb{1}_A) = \nu(\mathbb{1}_A \# \mathfrak{g}) = \nu(\mathbb{1}_{A \trianglelefteq \mathfrak{g}}) = \mu(A \trianglelefteq \mathfrak{g}) = \mu(A) = \nu(\mathbb{1}_A).$$

Thus, according to Lemma 3.6, the mean ν is $\#$ -invariant.

Secondly, let there be a $\#$ -invariant mean ν on M . Put $\mu = \Phi(\nu)$.

Then, for each $\mathfrak{g} \in G/G_0$ and each $A \subseteq M$ such that $_ \trianglelefteq \mathfrak{g}$ is injective on A , according to Lemma 3.4,

$$(\mu \# \mathfrak{g})(A) = \mu(A \trianglelefteq \mathfrak{g}) = \nu(\mathbb{1}_{A \trianglelefteq \mathfrak{g}}) = \nu(\mathbb{1}_A \# \mathfrak{g}) = \nu(\mathbb{1}_A) = \mu(A).$$

Hence, μ is $\#$ -semi-invariant. □

4. Right Følner Nets

In this section, let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0, m}\}_{m \in M}))$ be a cell space.

Definition 4.1. Let $\{F_i\}_{i \in I}$ be a net in $\{F \subseteq M \mid F \neq \emptyset, F \text{ finite}\}$ indexed by (I, \leq) . It is called *right Følner net in \mathcal{R} indexed by (I, \leq)* if and only if

$$\forall \mathfrak{g} \in G/G_0 : \lim_{i \in I} \frac{|F_i \setminus (_ \trianglelefteq \mathfrak{g})^{-1}(F_i)|}{|F_i|} = 0.$$

Remark 4.2. In the situation of Remark 2.5, for each element $g \in G$ and each index $i \in I$, we have $(_ \trianglelefteq g)^{-1}(F_i) = F_i \cdot g^{-1}$. Hence, right Følner nets in \mathcal{R} are exactly right Følner nets for G as defined in [1, First paragraph after Definition 4.7.2].

Lemma 4.3. *Let V be a set, let W be a set, and let Ψ be a map from $V \times W$ to \mathbb{R} . There is a net $\{v_i\}_{i \in I}$ in V indexed by (I, \leq) such that*

$$(4.1) \quad \forall w \in W : \lim_{i \in I} \Psi(v_i, w) = 0,$$

if and only if, for each finite subset Q of W and each positive real number $\varepsilon \in \mathbb{R}_{>0}$, there is an element $v \in V$ such that

$$(4.2) \quad \forall q \in Q : \Psi(v, q) < \varepsilon.$$

Proof. First, let there be a net $\{v_i\}_{i \in I}$ in V indexed by (I, \leq) such that (4.1) holds. Furthermore, let $Q \subseteq W$ be finite and let $\varepsilon \in \mathbb{R}_{>0}$. Because (4.1) holds, for each $q \in Q$, there is an $i_q \in I$ such that,

$$\forall i \in I : (i \geq i_q \implies \Psi(v_i, q) < \varepsilon).$$

Because (I, \leq) is a directed set and Q is finite, there is an $i \in I$ such that, for each $q \in Q$, we have $i \geq i_q$. Put $v = v_i$. Then, (4.2) holds.

Secondly, for each finite $Q \subseteq W$ and each $\varepsilon \in \mathbb{R}_{>0}$, let there be a $v \in V$ such that (4.2) holds. Furthermore, let

$$I = \{Q \subseteq W \mid Q \text{ is finite}\} \times \mathbb{R}_{>0}$$

and let \leq be the preorder on I given by

$$\forall (Q, \varepsilon) \in I \forall (Q', \varepsilon') \in I : (Q, \varepsilon) \leq (Q', \varepsilon') \iff Q \subseteq Q' \wedge \varepsilon \geq \varepsilon'.$$

For each $(Q, \varepsilon) \in I$ and each $(Q', \varepsilon') \in I$, the element $(Q \cup Q', \min(\varepsilon, \varepsilon'))$ of I is an upper bound of (Q, ε) and of (Q', ε') . Hence, (I, \leq) is a directed set.

By precondition, for each $i = (Q, \varepsilon) \in I$, there is a $v_i \in V$ such that

$$\forall q \in Q : \Psi(v_i, q) < \varepsilon.$$

Let $w \in W$ and let $\varepsilon_0 \in \mathbb{R}_{>0}$. Put $i_0 = (\{w\}, \varepsilon_0)$. For each $i = (Q, \varepsilon) \in I$ with $i \geq i_0$, we have $w \in Q$ and $\varepsilon \leq \varepsilon_0$. Hence,

$$\forall i \in I : (i \geq i_0 \implies \Psi(v_i, w) < \varepsilon_0).$$

Therefore, $\{v_i\}_{i \in I}$ is a net in V indexed by (I, \leq) such that (4.1) holds. □

Lemma 4.4. *There is a right Følner net in \mathcal{R} if and only if, for each finite subset E of G/G_0 and each positive real number $\varepsilon \in \mathbb{R}_{>0}$, there is a non-empty and finite subset F of M such that*

$$\forall e \in E : \frac{|F \setminus (_ \triangleleft e)^{-1}(F)|}{|F|} < \varepsilon.$$

Proof. This is a direct consequence of Lemma 4.3 with

$$\begin{aligned} \Psi : \{F \subseteq M \mid F \neq \emptyset, F \text{ finite}\} \times G/G_0 &\rightarrow \mathbb{R}, \\ (F, \mathfrak{g}) &\mapsto \frac{|F \setminus (_ \triangleleft \mathfrak{g})^{-1}(F)|}{|F|}. \end{aligned} \quad \square$$

Lemma 4.5. *Let m be an element of M , and let \mathfrak{g} be an element of G/G_0 . There is an element $g \in \mathfrak{g}$ such that*

$$\forall \mathfrak{g}' \in G/G_0 : (m \triangleleft \mathfrak{g}) \triangleleft \mathfrak{g}' = m \triangleleft g \cdot \mathfrak{g}'.$$

Proof. There is a $g \in G$ such that $gG_0 = \mathfrak{g}$. Moreover, because \trianglelefteq is a semi-action with defect G_0 , there is a $g_0 \in G_0$ such that

$$\forall \mathfrak{g}' \in G/G_0 : (m \trianglelefteq gG_0) \trianglelefteq \mathfrak{g}' = m \trianglelefteq g \cdot (g_0^{-1} \cdot \mathfrak{g}').$$

Because $g \cdot (g_0^{-1} \cdot \mathfrak{g}') = gg_0^{-1} \cdot \mathfrak{g}'$ and $gg_0^{-1} \in \mathfrak{g}$, the statement holds. □

Lemma 4.6. *Let A and A' be two subsets of M , and let \mathfrak{g} and \mathfrak{g}' be two elements of G/G_0 . Then, for each element $m \in (_ \trianglelefteq \mathfrak{g})^{-1}(A) \setminus (_ \trianglelefteq \mathfrak{g}')^{-1}(A')$,*

$$m \trianglelefteq \mathfrak{g}' \in \bigcup_{g' \in \mathfrak{g}'} (_ \trianglelefteq (g')^{-1} \cdot \mathfrak{g})^{-1}(A) \setminus A'.$$

Lemma 4.6. Let $m \in (_ \trianglelefteq \mathfrak{g})^{-1}(A) \setminus (_ \trianglelefteq \mathfrak{g}')^{-1}(A')$. Then, $m \trianglelefteq \mathfrak{g} \in A$ and $m \trianglelefteq \mathfrak{g}' \notin A'$. According to Lemma 4.5, there is a $g \in \mathfrak{g}$ and a $g' \in \mathfrak{g}'$ such that $(m \trianglelefteq \mathfrak{g}) \trianglelefteq g^{-1} \cdot \mathfrak{g}' = m \trianglelefteq \mathfrak{g}' \notin A'$ and $(m \trianglelefteq \mathfrak{g}') \trianglelefteq (g')^{-1} \cdot \mathfrak{g} = m \trianglelefteq \mathfrak{g} \in A$. Hence, $m \trianglelefteq \mathfrak{g} \notin (_ \trianglelefteq g^{-1} \cdot \mathfrak{g}')^{-1}(A')$ and $m \trianglelefteq \mathfrak{g}' \in (_ \trianglelefteq (g')^{-1} \cdot \mathfrak{g})^{-1}(A)$. Therefore, $m \trianglelefteq \mathfrak{g} \in A \setminus (_ \trianglelefteq g^{-1} \cdot \mathfrak{g}')^{-1}(A')$ and $m \trianglelefteq \mathfrak{g}' \in (_ \trianglelefteq (g')^{-1} \cdot \mathfrak{g})^{-1}(A) \setminus A'$. In conclusion, $m \trianglelefteq \mathfrak{g} \in \bigcup_{g \in \mathfrak{g}} A \setminus (_ \trianglelefteq g^{-1} \cdot \mathfrak{g}')^{-1}(A')$ and $m \trianglelefteq \mathfrak{g}' \in \bigcup_{g' \in \mathfrak{g}'} (_ \trianglelefteq (g')^{-1} \cdot \mathfrak{g})^{-1}(A) \setminus A'$. □

Lemma 4.7. *Let G_0 be finite, let F and F' be two finite subsets of M , and let \mathfrak{g} and \mathfrak{g}' be two elements of G/G_0 . Then,*

$$|(_ \trianglelefteq \mathfrak{g})^{-1}(F) \setminus (_ \trianglelefteq \mathfrak{g}')^{-1}(F')| \leq \begin{cases} |G_0|^2 \cdot \max_{g \in \mathfrak{g}} |F \setminus (_ \trianglelefteq g^{-1} \cdot \mathfrak{g}')^{-1}(F')|, \\ |G_0|^2 \cdot \max_{g' \in \mathfrak{g}'} |(_ \trianglelefteq (g')^{-1} \cdot \mathfrak{g})^{-1}(F) \setminus F'|. \end{cases}$$

Proof. Put $A = (_ \trianglelefteq \mathfrak{g})^{-1}(F) \setminus (_ \trianglelefteq \mathfrak{g}')^{-1}(F')$. For each $g \in \mathfrak{g}$, put $B_g = F \setminus (_ \trianglelefteq g^{-1} \cdot \mathfrak{g}')^{-1}(F')$. For each $g' \in \mathfrak{g}'$, put $B'_{g'} = (_ \trianglelefteq (g')^{-1} \cdot \mathfrak{g})^{-1}(F) \setminus F'$.

According to Lemma 4.6, the restrictions $(_ \trianglelefteq \mathfrak{g}) \upharpoonright_{A \rightarrow \bigcup_{g \in \mathfrak{g}} B_g}$ and $(_ \trianglelefteq \mathfrak{g}') \upharpoonright_{A \rightarrow \bigcup_{g' \in \mathfrak{g}'} B'_{g'}}$ are well-defined. Moreover, for each $m \in M$, according to Lemma 2.9, we have $|(_ \trianglelefteq \mathfrak{g})^{-1}(m)| \leq |G_0|$ and $|(_ \trianglelefteq \mathfrak{g}')^{-1}(m)| \leq |G_0|$. Therefore, because $|\mathfrak{g}| = |G_0|$,

$$|A| \leq |G_0| \cdot \left| \bigcup_{g \in \mathfrak{g}} B_g \right| \leq |G_0| \cdot \sum_{g \in \mathfrak{g}} |B_g| \leq |G_0|^2 \cdot \max_{g \in \mathfrak{g}} |B_g|$$

and analogously

$$|A| \leq |G_0|^2 \cdot \max_{g' \in \mathfrak{g}'} |B'_{g'}|. \quad \square$$

Lemma 4.8. *Let G_0 be finite and let $\{F_i\}_{i \in I}$ be a net in $\{F \subseteq M \mid F \neq \emptyset, F \text{ finite}\}$ indexed by (I, \leq) . The net $\{F_i\}_{i \in I}$ is a right Følner net in \mathcal{R} if and only if*

$$(4.3) \quad \forall \mathfrak{g} \in G/G_0 : \lim_{i \in I} \frac{|(_ \trianglelefteq \mathfrak{g})^{-1}(F_i) \setminus F_i|}{|F_i|} = 0.$$

Proof. Let $\mathfrak{g} \in G/G_0$. Furthermore, let $i \in I$. Because $F_i = (_ \triangleleft G_0)^{-1}(F_i)$, according to Lemma 4.7,

$$|(_ \triangleleft \mathfrak{g})^{-1}(F_i) \setminus F_i| \leq |G_0|^2 \cdot \max_{g \in \mathfrak{g}} |F_i \setminus (_ \triangleleft g^{-1}G_0)^{-1}(F_i)|$$

and

$$|F_i \setminus (_ \triangleleft \mathfrak{g})^{-1}(F_i)| \leq |G_0|^2 \cdot \max_{g \in \mathfrak{g}} |(_ \triangleleft g^{-1}G_0)^{-1}(F_i) \setminus F_i|.$$

Moreover, $|\mathfrak{g}| = |G_0| < \infty$. Therefore, if $\{F_i\}_{i \in I}$ is a right Følner net in \mathcal{R} , then

$$\lim_{i \in I} \frac{|(_ \triangleleft \mathfrak{g})^{-1}(F_i) \setminus F_i|}{|F_i|} = 0;$$

and, if (4.3) holds, then

$$\lim_{i \in I} \frac{|F_i \setminus (_ \triangleleft \mathfrak{g})^{-1}(F_i)|}{|F_i|} = 0.$$

In conclusion, $\{F_i\}_{i \in I}$ is a right Følner net in \mathcal{R} if and only if (4.3) holds. □

Lemma 4.9. *Let G_0 be finite. There is a right Følner net in \mathcal{R} if and only if, for each finite subset E of G/G_0 and each positive real number $\varepsilon \in \mathbb{R}_{>0}$, there is a non-empty and finite subset F of M such that*

$$\forall e \in E : \frac{|(_ \triangleleft e)^{-1}(F) \setminus F|}{|F|} < \varepsilon.$$

Proof. This is a direct consequence of Lemma 4.8 and Lemma 4.3 with

$$\begin{aligned} \Psi : \{F \subseteq M \mid F \neq \emptyset, F \text{ finite}\} \times G/G_0 &\rightarrow \mathbb{R}, \\ (F, \mathfrak{g}) &\mapsto \frac{|(_ \triangleleft \mathfrak{g})^{-1}(F) \setminus F|}{|F|}. \end{aligned} \quad \square$$

5. Right Paradoxical Decompositions

In this section, let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0, m}\}_{m \in M}))$ be a cell space.

Definition 5.1. Let A and A' be two sets. The set $A \cup A'$ is denoted by $A \cupdot A'$ if and only if the sets A and A' are disjoint.

Definition 5.2. Let E be a finite subset of G/G_0 , and let $\{A_e\}_{e \in E}$ and $\{B_e\}_{e \in E}$ be two families of subsets of M indexed by E such that, for each index $e \in E$, the map $_ \triangleleft e$ is injective on A_e and on B_e , and

$$M = \bigcup_{e \in E} A_e \cupdot \bigcup_{e \in E} B_e = \left(\bigcup_{e \in E} A_e \triangleleft e \right) \cupdot \left(\bigcup_{e \in E} B_e \triangleleft e \right).$$

The triple $(N, \{A_e\}_{e \in E}, \{B_e\}_{e \in E})$ is called *right paradoxical decomposition of \mathcal{R}* .

Remark 5.3. In the situation of Remark 2.5, for each element $g \in G$, the map $_ \triangleleft g$ is injective. Hence, right paradoxical decompositions of \mathcal{R} are the same as right paradoxical decompositions of G as defined in [1, Definition 4.8.1].

Lemma 5.4. *Let G_0 be finite and let $(N, \{A_e\}_{e \in E}, \{B_e\}_{e \in E})$ be a right paradoxical decomposition of \mathcal{R} . Then,*

$$\mathbb{1}_M = \sum_{e \in E} \mathbb{1}_{A_e} = \sum_{e \in E} \mathbb{1}_{B_e} = \sum_{e \in E} (\mathbb{1}_{A_e} \dashv e) + \sum_{e \in E} (\mathbb{1}_{B_e} \dashv e).$$

Proof. This is a direct consequence of Definition 5.2 and Lemma 3.4. □

6. Tarski’s and Følner’s Theorem

In this section, let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0, m}\}_{m \in M}))$ be a cell space such that the stabiliser G_0 of m_0 under \triangleright is finite.

Lemma 6.1. *Let \mathfrak{g} be an element of G/G_0 . The map $_ \dashv \mathfrak{g}$ is continuous, where $\ell^\infty(M)^*$ is equipped with the weak-* topology.*

Proof. For each $f \in \ell^\infty(M)$, let $\text{ev}_f: \ell^\infty(M)^* \rightarrow \mathbb{R}, \psi \mapsto \psi(f)$. Furthermore, let $f \in \ell^\infty(M)$. Then, for each $\psi \in \ell^\infty(M)^*$,

$$(\text{ev}_f \circ (_ \dashv \mathfrak{g}))(\psi) = \text{ev}_f(\psi \dashv \mathfrak{g}) = (\psi \dashv \mathfrak{g})(f) = \psi(f \dashv \mathfrak{g}) = \text{ev}_{f \dashv \mathfrak{g}}(\psi).$$

Thus, $\text{ev}_f \circ (_ \dashv \mathfrak{g}) = \text{ev}_{f \dashv \mathfrak{g}}$. Hence, because $\text{ev}_{f \dashv \mathfrak{g}}$ is continuous, so is $\text{ev}_f \circ (_ \dashv \mathfrak{g})$. Therefore, the map $_ \dashv \mathfrak{g}$ is continuous. □

Lemma 6.2. *Let $\{\nu_i\}_{i \in I}$ be a net in $\mathcal{M}(M)$ such that, for each element $\mathfrak{g} \in G/G_0$, the net $\{(\nu_i \dashv \mathfrak{g}) - \nu_i\}_{i \in I}$ converges to 0 in $\ell^\infty(M)^*$ equipped with the weak-* topology. The cell space \mathcal{R} is right amenable.*

Proof. Let $\mathfrak{g} \in G/G_0$. According to Theorem 2.20, the set $\mathcal{M}(M)$ is compact in $\ell^\infty(M)^*$ equipped with the weak-* topology. Hence, there is a subnet $\{\nu_{i_j}\}_{j \in J}$ of $\{\nu_i\}_{i \in I}$ that converges to a $\nu \in \mathcal{M}(M)$. Because, according to Lemma 6.1, the map $_ \dashv \mathfrak{g}$ is continuous, the net $\{(\nu_{i_j} \dashv \mathfrak{g}) - \nu_{i_j}\}_{j \in J}$ converges to $(\nu \dashv \mathfrak{g}) - \nu$ in $\ell^\infty(M)^*$. Because it is a subnet of $\{(\nu_i \dashv \mathfrak{g}) - \nu_i\}_{i \in I}$, it also converges to 0 in $\ell^\infty(M)^*$. Because the space $\ell^\infty(M)^*$ is Hausdorff, we have $(\nu \dashv \mathfrak{g}) - \nu = 0$ and hence $\nu \dashv \mathfrak{g} = \nu$. Altogether, ν is a \dashv -invariant mean. In conclusion, according to Theorem 3.7, the cell space \mathcal{R} is right amenable. □

Lemma 6.3. *Let m be an element of M , and let E and E' be two subsets of G/G_0 . There is a subset E'' of G/G_0 such that $(m \triangleleft E) \triangleleft E' = m \triangleleft E''$; if $G_0 \in E \cap E'$, then $G_0 \in E''$; if E and E' are finite, then $|E''| \leq |E| \cdot |E'|$; and if $G_0 \cdot E' \subseteq E'$, then*

$$E'' = \{g \cdot e' \mid e \in E, e' \in E', g \in e\}.$$

Proof. For each $e \in E$, according to Lemma 4.5, there is a $g_e \in e$ such that

$$\forall \mathfrak{g} \in G/G_0 : (m \triangleleft e) \triangleleft \mathfrak{g} = m \triangleleft g_e \cdot \mathfrak{g}.$$

Put $E'' = \{g_e \cdot e' \mid e \in E, e' \in E'\}$. Then, $(m \triangleleft E) \triangleleft E' = m \triangleleft E''$. Moreover, if $G_0 \in E \cap E'$, then $G_0 = g_{G_0} \cdot G_0 \in E''$; if E and E' are finite, then $|E''| \leq |E| \cdot |E'|$; and if $G_0 \cdot E' \subseteq E'$, then E'' is as stated. □

Main Theorem 6.1. Let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0, m}\}_{m \in M}))$ be a cell space such that the stabiliser G_0 of m_0 under \triangleright is finite. The following statements are equivalent:

- (1) The cell space \mathcal{R} is not right amenable;
- (2) There is no right Følner net in \mathcal{R} ;
- (3) There is a finite subset E of G/G_0 such that $G_0 \in E$ and, for each finite subset F of M , we have $|F \triangleleft E| \geq 2|F|$;
- (4) There is a 2-to-1 surjective map $\phi: M \rightarrow M$ and there is a finite subset E of G/G_0 such that

$$\forall m \in M \exists e \in E : \phi(m) \triangleleft e = m;$$

- (5) There is a right paradoxical decomposition of \mathcal{R} .

Proof. **1 implies 2:** Let there be a right Følner net $\{F_i\}_{i \in I}$ in \mathcal{R} . Furthermore, let $i \in I$. Put

$$\begin{aligned} \nu_i: \ell^\infty(M) &\rightarrow \mathbb{R}, \\ f &\mapsto \frac{1}{|F_i|} \sum_{m \in F_i} f(m). \end{aligned}$$

Then, $\nu_i \in \mathcal{M}(M)$. Moreover, let $\mathfrak{g} \in G/G_0$ and let $f \in \ell^\infty(M)$. Then,

$$\begin{aligned} (\nu_i \triangleleft \mathfrak{g})(f) &= \nu_i(f \triangleleft \mathfrak{g}) \\ &= \frac{1}{|F_i|} \sum_{m \in F_i} (f \triangleleft \mathfrak{g})(m) \\ &= \frac{1}{|F_i|} \sum_{m \in F_i} \sum_{m' \in (_ \triangleleft \mathfrak{g})^{-1}(m)} f(m') \\ &= \frac{1}{|F_i|} \sum_{m \in (_ \triangleleft \mathfrak{g})^{-1}(F_i)} f(m). \end{aligned}$$

Hence,

$$\begin{aligned} &(\nu_i \triangleleft \mathfrak{g} - \nu_i)(f) \\ &= \frac{1}{|F_i|} \left(\sum_{m \in (_ \triangleleft \mathfrak{g})^{-1}(F_i) \setminus F_i} f(m) - \sum_{m \in F_i \setminus (_ \triangleleft \mathfrak{g})^{-1}(F_i)} f(m) \right). \end{aligned} \text{Therefore,}$$

$$\begin{aligned} |(\nu_i \triangleleft \mathfrak{g} - \nu_i)(f)| &\leq \frac{1}{|F_i|} \left(\sum_{m \in (_ \triangleleft \mathfrak{g})^{-1}(F_i) \setminus F_i} |f(m)| + \sum_{m \in F_i \setminus (_ \triangleleft \mathfrak{g})^{-1}(F_i)} |f(m)| \right) \\ &\leq \left(\frac{|(_ \triangleleft \mathfrak{g})^{-1}(F_i) \setminus F_i|}{|F_i|} + \frac{|F_i \setminus (_ \triangleleft \mathfrak{g})^{-1}(F_i)|}{|F_i|} \right) \cdot \|f\|_\infty. \end{aligned}$$

According to Definition 4.1 and Lemma 4.8, the nets $\{|(_ \triangleleft \mathfrak{g})^{-1}(F_i) \setminus F_i|/|F_i|\}_{i \in I}$ and $\{|F_i \setminus (_ \triangleleft \mathfrak{g})^{-1}(F_i)|/|F_i|\}_{i \in I}$ converge to 0. Hence, so does $\{|(\nu_i \triangleleft \mathfrak{g} - \nu_i)(f)|\}_{i \in I}$. Thus, the net $\{\nu_i \triangleleft \mathfrak{g} - \nu_i\}_{i \in I}$ converges to 0 in $\ell^\infty(M)^*$ equipped with the weak-* topology. Hence, according to Lemma 6.2, the cell space \mathcal{R} is right amenable. In conclusion, by contraposition, if \mathcal{R} is not right amenable, then there is no right Følner net in \mathcal{R} .

2 implies 3: Let there be no right Følner net in \mathcal{R} . According to Lemma 4.4, there is a finite $E_1 \subseteq G/G_0$ and an $\varepsilon \in \mathbb{R}_{>0}$ such that, for each non-empty and finite $F \subseteq M$, there is an $e_F \in E_1$ such that

$$\frac{|F \setminus (_ \triangleleft e_F)^{-1}(F)|}{|F|} \geq \varepsilon.$$

Put $E_2 = \{G_0\} \cup E_1$.

Let $F \subseteq M$ be non-empty and finite. Then, $F \subseteq F \cup (F \triangleleft E_1) = F \triangleleft E_2$. Thus,

$$\begin{aligned} |F \triangleleft E_2| - |F| &= |(F \triangleleft E_2) \setminus F| \\ &= |(F \triangleleft E_1) \setminus F| \\ &\geq |(F \triangleleft e_F) \setminus F|. \end{aligned}$$

Moreover, according to Lemma 2.9, we have $|(_ \triangleleft e_F)^{-1}((F \triangleleft e_F) \setminus F)| \leq |G_0| \cdot |(F \triangleleft e_F) \setminus F|$.

Hence,

$$|F \triangleleft E_2| - |F| \geq \frac{|(_ \triangleleft e_F)^{-1}((F \triangleleft e_F) \setminus F)|}{|G_0|}.$$

Therefore, because $F \setminus (_ \triangleleft e_F)^{-1}(F) \subseteq (_ \triangleleft e_F)^{-1}((F \triangleleft e_F) \setminus F)$,

$$\begin{aligned} |F \triangleleft E_2| - |F| &\geq \frac{|F \setminus (_ \triangleleft e_F)^{-1}(F)|}{|G_0|} \\ &\geq \frac{\varepsilon}{|G_0|} |F|. \end{aligned}$$

Put $\xi = 1 + \varepsilon/|G_0|$. Then, $|F \triangleleft E_2| \geq \xi|F|$. Because ε does not depend on F , neither does ξ . Therefore, for each non-empty and finite $F \subseteq M$, we have $|F \triangleleft E_2| \geq \xi|F|$.

Let $F \subseteq M$ be non-empty and finite. Because $\xi > 1$, there is an $n \in \mathbb{N}$ such that $\xi^n \geq 2$. Hence,

$$\begin{aligned} \underbrace{|(((F \triangleleft E_2) \triangleleft \dots) \triangleleft E_2) \triangleleft E_2|}_{n \text{ times}} &\geq \xi|((F \triangleleft E_2) \triangleleft \dots) \triangleleft E_2| \\ &\geq \dots \\ &\geq \xi^n |F| \\ &\geq 2|F|. \end{aligned}$$

Moreover, according to Lemma 6.3, there is an $E \subseteq G/G_0$ such that E is finite, $G_0 \in E$, and $F \triangleleft E = (((F \triangleleft E_2) \triangleleft \dots) \triangleleft E_2) \triangleleft E_2$. In conclusion, $|F \triangleleft E| \geq 2|F|$.

3 implies 4 (see Fig. 1):

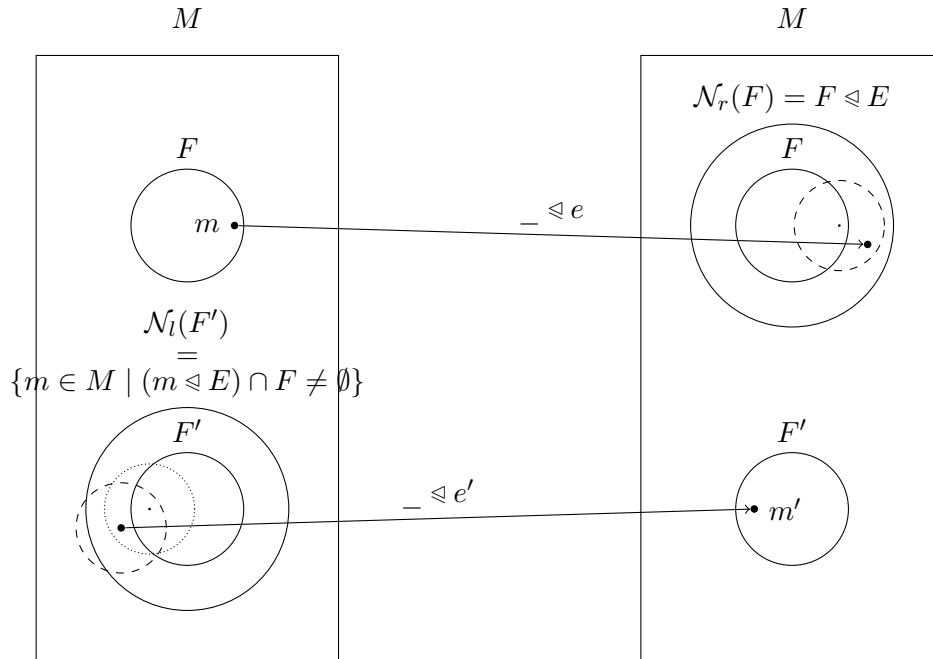


FIGURE 1. Schematic representation of the set-up of the proof of Theorem 6.1, Item 3 implies Item 4: Each region enclosed by one of the two rectangles is M ; the regions enclosed by the smaller circles with solid borders are subsets F and F' of M respectively; the regions enclosed by the circles with dashed borders are $m \triangleleft E$ and $m' \triangleleft E$ respectively; the region enclosed by the circle with dotted border is $\bigcup_{e' \in E} (_ \triangleleft e')^{-1}(m')$; the regions enclosed by the larger circles are $\mathcal{N}_r(F)$ and $\mathcal{N}_l(F')$ respectively.

Let there be a finite $E \subseteq G/G_0$ such that, for each finite $F \subseteq M$, we have $|F \triangleleft E| \geq 2|F|$. Furthermore, let \mathcal{G} be the bipartite graph

$$(M, M, \{(m, m') \in M \times M \mid \exists e \in E : m \triangleleft e = m'\}).$$

Moreover, let $F \subseteq M$ be finite. The right neighbourhood of F in \mathcal{G} is

$$\mathcal{N}_r(F) = \{m' \in M \mid \exists e \in E : F \triangleleft e \ni m'\} = F \triangleleft E$$

and the left neighbourhood of F in \mathcal{G} is

$$\mathcal{N}_l(F) = \{m \in M \mid \exists e \in E : m \triangleleft e \in F\} = \bigcup_{e \in E} (_ \triangleleft e)^{-1}(F).$$

By precondition $|\mathcal{N}_r(F)| = |F \triangleleft E| \geq 2|F|$. Moreover, because $G_0 \in E$, we have $F = (_ \triangleleft G_0)^{-1}(F) \subseteq \mathcal{N}_l(F)$ and hence $|\mathcal{N}_l(F)| \geq |F| \geq 2^{-1}|F|$. Therefore, according to the Hall harem theorem, there is a perfect $(1, 2)$ -matching for \mathcal{G} . In conclusion, there is a 2-to-1 surjective map $\phi: M \rightarrow M$ such that, for each $m \in M$, we have $(\phi(m), m) \in E$, that is, there is an $e \in E$ such that $\phi(m) \triangleleft e = m$.

4 implies 5 (see Fig. 2):

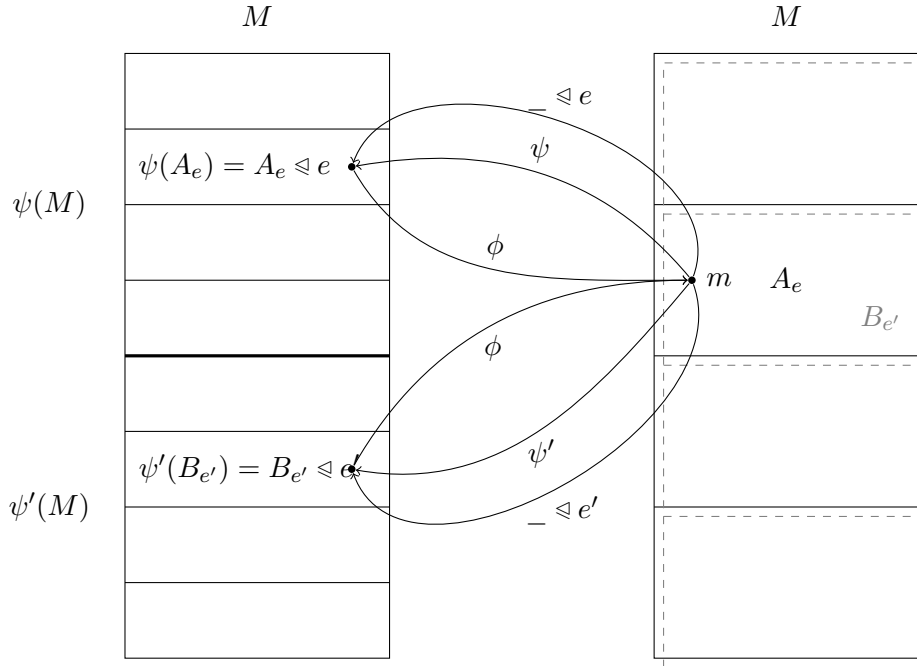


FIGURE 2. Schematic representation of the set-up of the proof of Theorem 6.1, Item 4 implies Item 5: Each region enclosed by one of the three columns, two with solid and one with dashed border, is M ; the dot in the right column, called m , is an element of M , and the two dots in the left column are its preimages under ϕ , which are its images under ψ and ψ' ; there are elements e and e' of E such that $m \triangleleft e = \psi(m)$ and $m \triangleleft e' = \psi'(m)$, in other words, $m \in A_e$ and $m \in B_{e'}$; as depicted in the right columns with solid and dashed borders, the families $\{A_e\}_{e \in E}$ and $\{B_{e'}\}_{e' \in E}$ are partitions of M ; as depicted in the left column, the set $\{\psi(M), \psi'(M)\}$ is a partition of M , the family $\{\psi(A_e)\}_{e \in E} = \{A_e \triangleleft e\}_{e \in E}$ is a partition of $\psi(M)$, and the family $\{\psi'(B_{e'})\}_{e' \in E} = \{B_{e'} \triangleleft e'\}_{e' \in E}$ is a partition of $\psi'(M)$.

Let there be a 2-to-1 surjective map $\phi: M \rightarrow M$ and a finite subset E of G/G_0 such that

$$\forall m \in M \exists e \in E : \phi(m) \triangleleft e = m.$$

By the axiom of choice, there are two injective maps ψ and $\psi': M \rightarrow M$ such that, for each $m \in M$, we have $\phi^{-1}(m) = \{\psi(m), \psi'(m)\}$. For each $e \in E$, let

$$A_e = \{m \in M \mid m \triangleleft e = \psi(m)\} \quad \text{and} \quad B_{e'} = \{m \in M \mid m \triangleleft e' = \psi'(m)\}.$$

Let $m \in M$. There is an $e \in E$ such that $\phi(\psi(m)) \triangleleft e = \psi(m)$. Because $\phi(\psi(m)) = m$, we have $m \in A_e$. And, because \triangleleft is free, for each $e' \in E \setminus \{e\}$, we have $m \triangleleft e' \neq m \triangleleft e = \psi(m)$ and thus $m \notin A_{e'}$. Therefore,

$$M = \bigcup_{e \in E} A_e \quad \text{and analogously} \quad M = \bigcup_{e \in E} B_e.$$

Moreover, $\psi(A_e) = A_e \triangleleft e$ and $\psi'(B_e) = B_e \triangleleft e$. Hence, because $M = \psi(M) \cup \psi'(M)$, and ψ and ψ' are injective,

$$M = \left(\bigcup_{e \in E} \psi(A_e) \right) \cup \left(\bigcup_{e \in E} \psi'(B_e) \right) = \left(\bigcup_{e \in E} A_e \triangleleft e \right) \cup \left(\bigcup_{e \in E} B_e \triangleleft e \right).$$

Furthermore, because ψ and ψ' are injective, for each $e \in E$, the maps $(_ \triangleleft e)|_{A_e} = \psi|_{A_e}$ and $(_ \triangleleft e)|_{B_e} = \psi'|_{B_e}$ are injective. In conclusion, $(N, \{A_e\}_{e \in E}, \{B_e\}_{e \in E})$ is a right paradoxical decomposition of \mathcal{R} .

5 implies 1: Let there be a right paradoxical decomposition $(N, \{A_e\}_{e \in E}, \{B_e\}_{e \in E})$ of \mathcal{R} . According to Lemma 5.4,

$$\mathbf{1}_M = \sum_{e \in E} \mathbf{1}_{A_e} = \sum_{e \in E} \mathbf{1}_{B_e} = \sum_{e \in E} (\mathbf{1}_{A_e} \triangleleft e) + \sum_{e \in E} (\mathbf{1}_{B_e} \triangleleft e).$$

Suppose that \mathcal{R} is right amenable. Then, according to Theorem 3.7, there is a \triangleleft -invariant mean ν on M . Because ν is linear and normalised,

$$\begin{aligned} 1 &= \nu(\mathbf{1}_M) \\ &= \sum_{e \in E} \nu(\mathbf{1}_{A_e} \triangleleft e) + \sum_{e \in E} \nu(\mathbf{1}_{B_e} \triangleleft e) \\ &= \sum_{e \in E} (\nu \triangleleft e)(\mathbf{1}_{A_e}) + \sum_{e \in E} (\nu \triangleleft e)(\mathbf{1}_{B_e}) \\ &= \sum_{e \in E} \nu(\mathbf{1}_{A_e}) + \sum_{e \in E} \nu(\mathbf{1}_{B_e}) \\ &= \nu(\mathbf{1}_M) + \nu(\mathbf{1}_M) \\ &= 1 + 1 \\ &= 2, \end{aligned}$$

which contradicts that $1 \neq 2$. In conclusion, \mathcal{R} is not right amenable. □

Corollary 6.4 (Tarski alternative theorem; Alfred Tarski, 1938). *Let \mathcal{M} be a left homogeneous space with finite stabilisers. It is right amenable if and only if there is a coordinate system \mathcal{K} for \mathcal{M} such that there is no right paradoxical decomposition of $(\mathcal{M}, \mathcal{K})$.* □

Corollary 6.5 (Theorem of Følner; Erling Følner, 1955). *Let \mathcal{M} be a left homogeneous space with finite stabilisers. It is right amenable if and only if there is a coordinate system \mathcal{K} for \mathcal{M} such that there is a right Følner net in $(\mathcal{M}, \mathcal{K})$.* □

Remark 6.6. In the situation of Remark 2.5, Corollaries 6.4 and 6.5 constitute [1, Theorem 4.9.1].

7. From Left to Right Amenability

Lemma 7.1. *Let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0, m}\}_{m \in M}))$ be a cell space and let H be a subgroup of G such that, for each element $\mathfrak{g} \in G/G_0$, there is an element $h \in H$ such that the maps $_ \triangleleft \mathfrak{g}$ and $h \triangleright _$ are inverse to each other.*

If $(M, H, \triangleright|_{H \times M})$ is left amenable, then \mathcal{R} is right amenable.

Proof. Let $\mu \in \mathcal{PM}(M)$. Furthermore, let $\mathfrak{g} \in G/G_0$. There is an $h \in H$ such that $_ \triangleleft \mathfrak{g}$ and $h \triangleright _$ are inverse to each other. Moreover, let $A \subseteq M$. Because $_ \triangleleft \mathfrak{g} = (h \triangleright _)^{-1} = h^{-1} \triangleright _$, we have $A \triangleleft \mathfrak{g} = h^{-1} \triangleright A$. Therefore,

$$\begin{aligned} (\mu \# \mathfrak{g})(A) &= \mu(A \triangleleft \mathfrak{g}) \\ &= \mu(h^{-1} \triangleright A) \\ &= (h \# \mu)(A). \end{aligned}$$

Thus, $\mu \# \mathfrak{g} = h \# \mu$. Hence, if μ is $\#|_{H \times [0,1]^{\mathcal{P}(M)}}$ -invariant, then μ is $\#$ -semi-invariant. In conclusion, if $(M, H, \triangleright|_{H \times M})$ is left amenable, then \mathcal{R} is right amenable. \square

Lemma 7.2. *Let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0, m}\}_{m \in M}))$ be a cell space and let H be a subgroup of G such that $G = G_0H$, for each element $\mathfrak{g} \in G/G_0$, the map $_ \triangleleft \mathfrak{g}$ is injective,*

$$\forall h \in H : _ \triangleleft hG_0 = h \triangleright _,$$

and

$$\forall h \in H \forall \mathfrak{g} \in G/G_0 : (_ \triangleleft hG_0) \triangleleft \mathfrak{g} = _ \triangleleft h \cdot \mathfrak{g}.$$

If $(M, H, \triangleright|_{H \times M})$ is left amenable, then \mathcal{R} is right amenable.

Proof. Let $gG_0 \in G/G_0$. Because $g^{-1} \in G = G_0H$, there is a $g_0 \in G_0$ and there is an $h \in H$ such that $g^{-1} = g_0h$. Thus, $h = g_0^{-1}g^{-1} \in H$. Hence, for each $m \in M$,

$$\begin{aligned} ((_ \triangleleft gG_0) \circ (h \triangleright _))(m) &= (h \triangleright m) \triangleleft gG_0 \\ &= (m \triangleleft hG_0) \triangleleft gG_0 \\ &= m \triangleleft hgG_0 \\ &= m \triangleleft g_0^{-1}g^{-1}gG_0 \\ &= m \triangleleft G_0 \\ &= m. \end{aligned}$$

Therefore, $h \triangleright _$ is right inverse to $_ \triangleleft gG_0$. Hence, $_ \triangleleft gG_0$ is surjective and thus, because it is injective by precondition, bijective. Therefore, $_ \triangleleft gG_0$ and $h \triangleright _$ are inverse to each other. In conclusion, according to Lemma 7.1, if $(M, H, \triangleright|_{H \times M})$ is left amenable, then \mathcal{R} is right amenable. \square

Definition 7.3. Let G be a group. The set

$$Z(G) = \{z \in G \mid \forall g \in G : zg = gz\}$$

is called *centre of G* .

Lemma 7.4. Let G be a group. The centre of G is a subgroup of G . □

Lemma 7.5. Let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0, m}\}_{m \in M}))$ be a cell space and let H be a subgroup of G such that G is equal to G_0H , $\triangleright|_{H \times M}$ is free, and $\{g_{m_0, m}\}_{m \in M}$ is included in $Z(H)$. If $(M, H, \triangleright|_{H \times M})$ is left amenable, then \mathcal{R} is right amenable.

Proof. Let $g \in G$. For each $m \in M$,

$$m \triangleleft gG_0 = g_{m_0, m}g \triangleright m_0 = g_{m_0, m} \triangleright (g \triangleright m_0).$$

Let $m \in M$. For each $m' \in M$, because $\triangleright|_{Z(H) \times M}$ is free and $g_{m_0, m}, g_{m_0, m'} \in Z(H)$,

$$\begin{aligned} m' \triangleleft gG_0 = m \triangleleft gG_0 &\iff g_{m_0, m'} = g_{m_0, m} \\ &\iff m' = m. \end{aligned}$$

Therefore, $_ \triangleleft gG_0$ is injective.

Let $m \in M$ and let $h \in H$. Because $g_{m_0, m} \in Z(G)$,

$$\begin{aligned} m \triangleleft hG_0 &= g_{m_0, m}h \triangleright m_0 \\ &= hg_{m_0, m} \triangleright m_0 \\ &= h \triangleright m. \end{aligned}$$

Put $m' = m \triangleleft hG_0$. Then,

$$\begin{aligned} g_{m_0, m}h \triangleright m_0 &= hg_{m_0, m} \triangleright m_0 \\ &= h \triangleright m \\ &= m'. \end{aligned}$$

Hence, because $g_{m_0, m'} \triangleright m_0 = m'$ also and $\triangleright|_{H \times M}$ is free, $g_{m_0, m'} = g_{m_0, m}h$. Therefore,

$$\begin{aligned} (m \triangleleft hG_0) \triangleleft gG_0 &= m' \triangleleft gG_0 \\ &= g_{m_0, m'}g \triangleright m_0 \\ &= g_{m_0, m}hg \triangleright m_0 \\ &= m \triangleleft hgG_0. \end{aligned}$$

In conclusion, according to Lemma 7.2, if $(M, H, \triangleright|_{H \times M})$ is left amenable, then \mathcal{R} is right amenable. □

Example 7.6. Let $M = \mathbb{K}$ be a field, let

$$G = \{f: M \rightarrow M, x \mapsto ax + b \mid a, b \in M, a \neq 0\}$$

be the group of affine functions with composition as group multiplication, and let

$$H = \{f: M \rightarrow M, x \mapsto x + b \mid b \in M\}$$

be the group of translations also with composition as group multiplication. The group H is an abelian subgroup of G , which in turn is a non-abelian subgroup of the symmetry group of M . Moreover, according to [1, Example 4.6.2 and Theorem 4.6.3], the group G is left amenable and hence, according to [1, Proposition 4.5.1], so is its subgroup H . Furthermore, the group G acts transitively on M by function application by \triangleright and so does H by $\triangleright|_{H \times M}$, even freely so. Because the groups G and H are left amenable, so are the left group sets (M, G, \triangleright) and $(M, H, \triangleright|_{H \times M})$. The stabiliser of $m_0 = 0$ is the group of dilations

$$G_0 = \{f: M \rightarrow M, x \mapsto ax \mid a \in M \setminus \{0\}\}.$$

We have $G = G_0H$. For each $m \in M$, let

$$\begin{aligned} g_{m_0, m}: M &\rightarrow M, \\ x &\mapsto x + m, \end{aligned}$$

be the translation by m . Then, $\{g_{m_0, m}\}_{m \in M}$ is included in $Z(H) = H$. Hence, according to Lemma 7.5, the cell space $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0, m}\}_{m \in M}))$ is right amenable.

Lemma 7.7. Let H and N be two groups, let $\phi: H \rightarrow \text{Aut}(N)$ be a group homomorphism, let G be the Cartesian product $H \times N$, and let

$$\begin{aligned} \cdot: G \times G &\rightarrow G, \\ ((h, n), (h', n')) &\mapsto (hh', n\phi(h)(n')). \end{aligned}$$

The tuple (G, \cdot) is a group, called semi-direct product of H and N with respect to ϕ , and denoted by $H \rtimes_{\phi} N$. □

Lemma 7.8. Let G be a semi-direct product of H and N with respect to ϕ . The neutral element of G is (e_H, e_N) and, for each element $(h, n) \in G$, the inverse of (h, n) is $(h^{-1}, \phi(h^{-1})(n))$. □

Definition 7.9. Let (M, G, \triangleright) be a left homogeneous space. It is called *principal* if and only if the action \triangleright is free.

Lemma 7.10. Let (M, H, \triangleright_H) be a principal left homogeneous space. Furthermore, let G_0 be a group, let $\phi: G_0 \rightarrow \text{Aut}(H)$ be a group homomorphism, let m_0 be an element of M , for each element $m \in M$, let $h_{m_0, m}$ be the unique element of H such that $h_{m_0, m} \triangleright m_0 = m$, and let

$$\triangleright_{G_0}: G_0 \times M \rightarrow M,$$

$$(g_0, m) \mapsto \phi(g_0)(h_{m_0, m}) \triangleright_H m_0.$$

Moreover, let G be the semi-direct product of G_0 and H with respect to ϕ , and let

$$\begin{aligned} \triangleright &: G \times M \rightarrow M, \\ ((g_0, h), m) &\mapsto h \triangleright_H (g_0 \triangleright_{G_0} m). \end{aligned}$$

The triple $(M, G_0, \triangleright_{G_0})$ is a left group set and the group G_0 is the stabiliser of m_0 under \triangleright_{G_0} . Furthermore, the tuple $\mathcal{R} = ((M, G, \triangleright), (m_0, \{(e_{G_0}, h_{m_0, m})\}_{m \in M}))$ is a cell space and the group $G_0 \times \{e_H\}$ is the stabiliser of m_0 under \triangleright . Moreover, under the identification of G_0 with $G_0 \times \{e_H\}$ and of H with $\{e_{G_0}\} \times H$, the left group sets $(M, G_0, \triangleright_{G_0})$ and (M, H, \triangleright_H) are left group subsets of (M, G, \triangleright) .

Proof. Because $\phi(e_{G_0}) = \text{id}_{\text{Aut}(H)}$, for each $m \in M$,

$$\begin{aligned} e_{G_0} \triangleright_{G_0} m &= \phi(e_{G_0})(h_{m_0, m}) \triangleright_H m_0 \\ &= h_{m_0, m} \triangleright_H m_0 \\ &= m. \end{aligned}$$

Let g_0 and $g'_0 \in G_0$, and let $m \in M$. Because \triangleright_H is free and $h_{m_0, \phi(g'_0)(h_{m_0, m}) \triangleright_H m_0} \triangleright_H m_0 = \phi(g'_0)(h_{m_0, m}) \triangleright_H m_0$, we have $h_{m_0, \phi(g'_0)(h_{m_0, m}) \triangleright_H m_0} = \phi(g'_0)(h_{m_0, m})$. Therefore,

$$\begin{aligned} g_0 g'_0 \triangleright_{G_0} m &= \phi(g_0 g'_0)(h_{m_0, m}) \triangleright_H m_0 \\ &= (\phi(g_0) \circ \phi(g'_0))(h_{m_0, m}) \triangleright_H m_0 \\ &= \phi(g_0)(\phi(g'_0)(h_{m_0, m})) \triangleright_H m_0 \\ &= \phi(g_0)(h_{m_0, \phi(g'_0)(h_{m_0, m}) \triangleright_H m_0}) \triangleright_H m_0 \\ &= g_0 \triangleright_{G_0} (\phi(g'_0)(h_{m_0, m}) \triangleright_H m_0) \\ &= g_0 \triangleright_{G_0} (g'_0 \triangleright_{G_0} m). \end{aligned}$$

In conclusion, $(M, G_0, \triangleright_{G_0})$ is a left group set.

Because $h_{m_0, m_0} = e_H$, for each $g_0 \in G_0$,

$$\begin{aligned} g_0 \triangleright_{G_0} m_0 &= \phi(g_0)(e_H) \triangleright_H m_0 \\ &= e_H \triangleright_H m_0 \\ &= m_0. \end{aligned}$$

In conclusion, G_0 is the stabiliser of m_0 under \triangleright_{G_0} .

For each $m \in M$,

$$(e_{G_0}, e_H) \triangleright m = e_H \triangleright_H (e_{G_0} \triangleright_{G_0} m) = m.$$

Let $g_0 \in G_0$, let $h \in H$, and let $m \in M$. Because $hh_{m_0,m} \triangleright_H m_0 = h \triangleright_H m$, we have $hh_{m_0,m} = h_{m_0,h \triangleright_H m}$. Hence,

$$\begin{aligned} \phi(g_0)(h) \triangleright_H (g_0 \triangleright_{G_0} m) &= \phi(g_0)(h) \triangleright_H (\phi(g_0)(h_{m_0,m}) \triangleright_H m_0) \\ &= \phi(g_0)(h) \phi(g_0)(h_{m_0,m}) \triangleright_H m_0 \\ &= \phi(g_0)(hh_{m_0,m}) \triangleright_H m_0 \\ &= \phi(g_0)(h_{m_0,h \triangleright_H m}) \triangleright_H m_0 \\ &= g_0 \triangleright_{G_0} (h \triangleright_H m). \end{aligned}$$

Therefore, for each $g_0 \in G_0$, each $g'_0 \in G_0$, each $h \in H$, each $h' \in H$, and each $m \in M$,

$$\begin{aligned} (g_0, h)(g'_0, h') \triangleright m &= (g_0 g'_0, h \phi(g_0)(h')) \triangleright m \\ &= h \phi(g_0)(h') \triangleright_H (g_0 g'_0 \triangleright_{G_0} m) \\ &= h \triangleright_H \left(\phi(g_0)(h') \triangleright_H (g_0 \triangleright_{G_0} (g'_0 \triangleright_{G_0} m)) \right) \\ &= h \triangleright_H \left(g_0 \triangleright_{G_0} (h' \triangleright_H (g'_0 \triangleright_{G_0} m)) \right) \\ &= (g_0, h) \triangleright (h' \triangleright_H (g'_0 \triangleright_{G_0} m)) \\ &= (g_0, h) \triangleright ((g'_0, h') \triangleright m). \end{aligned}$$

In conclusion, (M, G, \triangleright) is a left group action.

Because \triangleright_H is transitive and, for each $h \in H$ and each $m \in M$, we have $(e_{G_0}, h) \triangleright m = h \triangleright m$, the left group action \triangleright is transitive and hence $\mathcal{M} = (M, G, \triangleright)$ is a left homogeneous space. Moreover, because, for each $m \in M$,

$$\begin{aligned} (e_{G_0}, h_{m_0,m}) \triangleright m_0 &= h_{m_0,m} \triangleright_H (e_{G_0} \triangleright_{G_0} m_0) \\ &= h_{m_0,m} \triangleright_H m_0 \\ &= m, \end{aligned}$$

the tuple $\mathcal{K} = (m_0, \{(e_{G_0}, h_{m_0,m})\}_{m \in M})$ is a coordinate system for \mathcal{M} . Therefore, $\mathcal{R} = (\mathcal{M}, \mathcal{K})$ is a cell space.

Because G_0 is the stabiliser of m_0 under \triangleright_{G_0} , for each $(g_0, h) \in G$, we have $(g_0, h) \triangleright m_0 = h \triangleright_H (g_0 \triangleright_{G_0} m_0) = h \triangleright m_0$. Because \triangleright_H is free, $G_0 \times \{e_H\}$ is the stabiliser of m_0 under \triangleright .

Under the identification of G_0 with $G_0 \times \{e_H\}$ and of H with $\{e_{G_0}\} \times H$, we have $\triangleright|_{G_0 \times M} = \triangleright_{G_0}$ and $\triangleright|_{H \times M} = \triangleright_H$. \square

Corollary 7.11. *In the situation of Lemma 7.10, let H be abelian. The cell space \mathcal{R} is right amenable.*

Proof. According to [1, Theorem 4.6.1], because H is abelian, it is left amenable. Therefore, (M, H, \triangleright_H) is left amenable. Identify G_0 with $G_0 \times \{e_H\}$ and identify H with $\{e_{G_0}\} \times H$. Then, H is a subgroup of

G , and $G = G_0H$, and $\triangleright|_{H \times M} = \triangleright_H$ is free, and, for each $m \in M$, we have $(e_{G_0}, h_{m_0, m}) \in H = Z(H)$. Hence, according to Lemma 7.5, the cell space \mathcal{R} is right amenable. \square

Example 7.12. Let d be a positive integer; let E be the d -dimensional Euclidean group, that is, the symmetry group of the d -dimensional Euclidean space, in other words, the isometries of \mathbb{R}^d with respect to the Euclidean metric with function composition; let T be the d -dimensional translation group; and let O be the d -dimensional orthogonal group. The group T is abelian, a normal subgroup of E , and isomorphic to \mathbb{R}^d with addition; the group O is isomorphic to the quotient E/T and to the $(d \times d)$ -dimensional orthogonal matrices with matrix multiplication; the group E is isomorphic to the semi-direct product $O \rtimes_{\iota} T$, where $\iota: O \rightarrow \text{Aut}(\mathbb{R}^d)$ is the inclusion map. The groups T , O , and E act on \mathbb{R}^d on the left by function application, denoted by \triangleright_T , \triangleright_O , and \triangleright , respectively; under the identification of T with \mathbb{R}^d by $t \mapsto [v \mapsto v + t]$, of O with the orthogonal matrices of $\mathbb{R}^{d \times d}$ by $A \mapsto [v \mapsto Av]$, and of E with $O \rtimes_{\iota} T$ by $(A, t) \mapsto [v \mapsto Av + t]$, we have

$$\begin{aligned} \triangleright_T: T \times \mathbb{R}^d &\rightarrow \mathbb{R}^d, \\ (t, v) &\mapsto v + t, \end{aligned}$$

and

$$\begin{aligned} \triangleright_O: O \times \mathbb{R}^d &\rightarrow \mathbb{R}^d, \\ (A, v) &\mapsto Av, \end{aligned}$$

and

$$\begin{aligned} \triangleright: E \times \mathbb{R}^d &\rightarrow \mathbb{R}^d, \\ ((A, t), v) &\mapsto Av + t, \end{aligned}$$

and

$$\begin{aligned} \iota: O &\rightarrow \text{Aut}(\mathbb{R}^d), \\ A &\mapsto [v \mapsto Av]. \end{aligned}$$

Hence, for each vector $v \in \mathbb{R}^d$, we have $v \triangleright_T 0 = v$, therefore, $\triangleright_O = [(A, v) \mapsto \iota(A)(v) \triangleright_T 0]$, and thus $\triangleright = [((A, t), v) \mapsto t \triangleright_T (A \triangleright_O v)]$. Moreover, because the group $(T, \circ) \cong (\mathbb{R}^d, +)$ is abelian, according to [1, Theorem 4.6.1], it is left amenable and so is $(\mathbb{R}^d, \mathbb{R}^d, +) \cong (\mathbb{R}^d, T, \triangleright)$. In conclusion, according to Corollary 7.11, the cell space $((\mathbb{R}^d, E, \triangleright), (0, \{-v\}_{v \in \mathbb{R}^d}))$ is right amenable.

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