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ON NONSOLVABLE GROUPS WHOSE PRIME DEGREE GRAPHS HAVE FOUR VERTICES AND ONE TRIANGLE

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Abstract. Let G be a finite group. The prime degree graph of G, denoted by $\Delta(G)$, is an undirected graph whose vertex set is $\rho(G)$ and there is an edge between two distinct primes p and q if and only if pq divides some irreducible character degree of G. In general, it seems that the prime graphs contain many edges and thus they should have many triangles, so one of the cases that would be interesting is to consider those finite groups whose prime degree graphs have a small number of triangles. In this paper we consider the case where for a nonsolvable group G, $\Delta(G)$ is a connected graph which has only one triangle and four vertices.

1. Introduction

Let G be a finite group. We consider the set of the irreducible complex characters of G, namely $Irr(G)$, and the related degree set $cd(G) = \{\chi(1) : \chi \in Irr(G)\}$. Let $\rho(G)$ be the set of all primes which divide some character degree of G. There is a large literature which is devoted to study the ways in which one can associate a graph with a group, for the purpose of investigating the algebraic structure using properties of the associated graph. One of these graphs is the prime degree graph of G which is denoted by $\Delta(G)$. It is an undirected graph with vertex set $\rho(G)$, where $p, q \in \rho(G)$ are joined by an edge if there exists an irreducible character degree $\chi(1) \in cd(G) \setminus \{1\}$ which is divisible by pq.

It is a subject of interest to determine which finite simple graphs can occur as the prime degree graphs of finite groups. This question has attracted many researchers over the years. (For more

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information see the survey paper [2].) In general, it seems that the prime degree graphs contain many edges and thus they should have many triangles, so one of the cases that would be interesting is to consider those finite groups whose prime degree graphs have a small number of triangles. In [4], the author studied finite groups whose prime degree graphs have no triangle. In particular, he proved that if $\Delta(G)$ has no triangle, then $|\rho(G)| \leq 5$. He also obtained a complete classification of all finite groups whose prime degree graphs contain no triangle with five vertices. In [5], the author studied finite groups whose prime degree graphs have at most two triangles. In particular, in [5, Theorem A], he considered the case where $\Delta(G)$ has one triangle and proved that $\Delta(G)$ has at most six vertices and if $\Delta(G)$ has six vertices, then $G \simeq PSL(2, 2^f) \times A$, where A is abelian, $|\pi(2^f - \delta)| = 2$, and $|\pi(2^f + \delta)| = 3$ for some $\delta = \pm 1$ with $f \geq 10$. Furthermore, if $\Delta(G)$ has five vertices, he described all possible cases for such a graph.

Suppose that G is a finite nonsolvable nonsimple group whose prime degree graph is a connected graph with four vertices and only one triangle. Such a group exists. For instance, one may consider $G = M_{11} \times A$, where A is an abelian group. This implies that $cd(G) = cd(M_{11}) = \{1, 10, 11, 16, 44, 45, 55\}$ and $\Delta(G)$ is the graph in Figure 1. Let N be the solvable radical of G. It is clear that $N \neq 1$. In this paper, our goal is to prove that for such a group, either $\Delta(G/N)$ contains a triangle, or there exists a normal subgroup M of G such that either $\Delta(M)$ contains a triangle; or M/N is isomorphic with one the following groups:

$$\{PSL(2,4), PSL(2,8), PSL(2,7), PSL(2,9), PSL(2,17)\}$$

Theorem 4 is the main theorem of this paper.

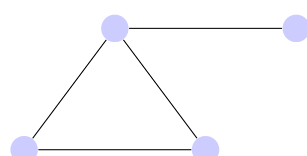

FIGURE 1. a connected graph with four vertices and one triangle

Notation 1. *We denote the number of connected components of a graph $\mathcal{G}$ by $n(\mathcal{G})$. Let a and b be two vertices of the graph $\mathcal{G}$. We denote the edge between a and b by a-b. Let x be a positive integer. By $\pi(x)$ we mean the set of prime divisors of x. We denote the set of prime divisors of the order of a finite group G as $\pi(G)$. Let K be a normal subgroup of G, $\chi \in Irr(G)$ and θ be an irreducible constituent of χ_K. Let $e(\theta) = [\chi_K, \theta]$, $I_G(\theta) = \{g \in G : \theta^g = \theta\}$ be the inertia group of θ in G, and $t(\theta) = |G : I_G(\theta)|$. Other notations throughout the paper are standard.*

2. Connected degree graph with four vertices and one triangle

Suppose that G is a finite group whose prime degree graph has four vertices and there exists a nonabelian simple group S such that $S \leq G \leq Aut(S)$. Since S is simple, we have $\rho(S) = \pi(S)$. By

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Burnside $p^{\alpha}q^{\beta}$-theorem it can be verified that $3 \leq |\pi(S)|$. As $\rho(S) \subseteq \rho(G)$, we have $3 \leq |\rho(S) = \pi(S)| \leq 4$. Finite simple groups with orders divisible by three or four primes are classified in [1]. Assume that $S = PSL(2, p^n)$, where p is a prime number. It is a famous result that

$$\rho(S) = \{p\} \cup \pi(p^n - 1) \cup \pi(p^n + 1).$$

As $\Delta(S)$ is a subgraph of $\Delta(G)$, $p^{2n} - 1$ is divisible by exactly two or three primes. First consider the case where $p^{2n} - 1$ is divisible by exactly two primes.

Lemma 1. [3] *Let $S = PSL(2, p^n)$, where p is a prime number and $p^n \geq 4$. The graph $\Delta(S)$ has exactly three vertices if and only if $p^n \in \{2^2, 2^3, 3^2, 5, 7, 17\}$. In this case $\Delta(S)$ has at most one edge and if $S < G \leq Aut(S)$, then $\Delta(G)$ has exactly one edge. In particular, $\Delta(G)$ is a subgraph of a square for all $S \leq G \leq Aut(S)$.*

Now consider the case when $p^{2n} - 1$ is divisible by exactly three primes. Either $p = 2$ or p is an odd prime. Thus we have the following lemmas:

Lemma 2. [3] *If $S = PSL(2, 2^n)$ and $|\rho(S)| = 4$, then either*

(i) *$n = 4$, $2^n + 1 = 17$, and $2^n - 1 = 3.5$, or*

(ii) *$n \geq 5$ is a prime, $2^n - 1 = r$ is prime, and $2^n + 1 = 3.t^b$, with t an odd prime and $b \geq 1$ odd.*

Thus one of $2^n \pm 1$ is a prime r and the other is $3.t^b$ with t prime, $\rho(S) = \{2, 3, r, t\}$, and 3-t is the only edge.

Lemma 3. *Let $S = PSL(2, p^n)$, where p is an odd prime and assume that $p^{2n} - 1$ is divisible by exactly three primes. Then one of the following holds:*

(1) *$q \in \{3^4, 5^2, 7^2\}$.*

(2) *$p = 3$ and n is an odd prime.*

(3) *$p \geq 11$, $n = 1$.*

In particular, if neither $p^n - 1$ nor $p^n + 1$ is a power of 2 and if $p \geq 11$ with $n = 1$, then p is neither a Mersenne nor a Fermat prime.

Proof. By [3, Lemma 2.5] it is clear that q satisfies one of the above parts. Suppose that neither $p^n - 1$ nor $p^n + 1$ is a power of 2. We have this property for the first and the second case. By the definition, it is clear this property holds for part (3) unless p is a Mersenne or a Fermat prime. □

Now we can discuss the main theorem of this paper.

Theorem 4. *Suppose that G is a nonsolvable nonsimple group with solvable radical N. Suppose that $\Delta(G)$ is a connected graph with four vertices and one triangle (as in Figure 1). Then either $\Delta(G/N)$ contains a triangle or there exists a normal subgroup M of G such that $\rho(M) = \rho(G)$, $M/N \simeq PSL(2, q)$, where $q = p^n \geq 4$ for a prime p and a positive integer n, and G/N is an almost simple group with socle M/N. Furthermore, we have one of the following cases:*

(i) *$\Delta(M)$ contains a triangle and $p \neq 2$.*

(ii) *M/N is isomorphic with one the following groups:*

$$\{PSL(2,4), PSL(2,8), PSL(2,7), PSL(2,9), PSL(2,17)\}$$

Proof. Suppose that $\Delta(G/N)$ is triangle free. Let M be a normal subgroup of G such that M/N is a chief factor of G. This verifies that $M/N \simeq \underbrace{S \times ... \times S}_{k-times}$. As M/N is nonsolvable, $\Delta(G/N)$ is triangle free and $\Delta(M/N)$ is a subgraph of $\Delta(G/N)$, it is easy to see that $k = 1$, M/N is a simple group and $C_{G/N}(M/N)$ is trivial. Thus G/N is an almost simple group with socle M/N. As $\Delta(G/N)$ is triangle free, [4, Lemma 3.2] implies that $M/N \simeq PSL(2,q)$, where $q = p^n \geq 4$ for a prime p and a positive integer n and $\pi(G/N) = \pi(M/N)$. It is clear that $\rho(M) \subseteq \rho(G)$. Let $p \in \rho(G)$ and $\chi \in Irr(G)$ with $p|\chi(1)$. Let θ be an irreducible constituent of χ_N. Either $p|\theta(1)$ or $p|\frac{\chi(1)}{\theta(1)}$. In each case we conclude that $p \in \rho(M)$, so $\rho(M) = \rho(G)$. By Burnside $p^\alpha q^\beta$-theorem we conclude that $3 \leq |\rho(M/N)| \leq 4$. First suppose that $|\rho(M/N)| = 3$. Since $M/N \simeq PSL(2,q)$, Lemma 1 implies that M/N is isomorphic with one the following groups:

$$\{PSL(2,4), PSL(2,8), PSL(2,7), PSL(2,9), PSL(2,17)\}.$$

So suppose that $|\rho(M/N)| = 4$.

Step 1. q is not even.

If $p = 2$, then $cd(M/N) = \{1, 2^n, 2^n + 1, 2^n - 1\}$ and the outer automorphism group of M/N is a cyclic group of order n, generated by a field automorphism. Let $|G/N : M/N| = d > 1$ which is a divisor of n. By Lemma 2, either $n = 4$ or $n \geq 5$ is a prime number, $\rho(S) = \{2, 3, r, t\}$ for some primes r and t and 3-t is the only edge of $\Delta(S)$. If $n = 4$, then d is either two or four. By [6, Theorem A] we deduce that $3.5.d \in cd(G/N)$. This implies that the primes 2, 3, and 5 form a triangle in $\Delta(G/N)$ which is impossible. Hence consider the case $n \geq 5$ is a prime which implies that $d = n$. By Fermat's little theorem we have $2^n \equiv 2 (mod \quad n)$, so $2^n + 1 \equiv 3 (mod \quad n)$ and $2^n - 1 \equiv 1 (mod \quad n)$. These relations imply that n is neither r nor t. Thus $\rho(G/N) = \{2, 3, r, t, n\}$, a contradiction. Hence p is an odd prime.

Step 2. $|G/N : M/N|$ is a power of 2. This verifies that $\Delta(G/N) = \Delta(M/N)$ is the first graph in Figure 2.

Since q is not even, $|\rho(M/N)| = 4$ and $M/N \simeq PSL(2,q)$, we conclude that $q^2 - 1$ is divisible by exactly three primes. If either $q - 1$ or $q + 1$ is a power of two, then the other one will be divisible by three distinct primes which is impossible since $\Delta(G/N)$ is triangle free. So we may assume that $\pi(q - 1) = \{2, r\}$ and $\pi(q + 1) = \{2, t\}$, for some distinct odd primes r and t. Now by Lemma 3 we have one of the following cases:

(1) $q \in \{3^4, 5^2, 7^2\}$.

(2) $p = 3$ and n is an odd prime.

(3) $p \geq 11$, $n = 1$ and p is neither a Mersenne nor a Fermat prime.

As $q = p^n$ is not even, the outer automorphism group of M/N is a group of order $2n$, generated by a field automorphism φ of order n and a diagonal automorphism δ of order 2. It is easy to see that in

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cases (1) and (3), $|G/N : M/N|$ is a power of 2. So suppose $p = 3$ and n is an odd prime. We claim that $G/N \simeq PGL(2, q)$. Suppose it is not true. Since $|G/N : M/N|$ is a divisor of $2n$ and n is an odd prime, we have either $|G/N : M/N| = n$ or $|G/N : M/N| = 2n$. In each case we conclude that $n \in \rho(G/N)$. On the other hand, neither $3^n - 1$ nor $3^n + 1$ is divisible by 3, as $3^n \equiv 3(mod \quad n)$. If $n \neq 3$, then $\rho(G/N) = \{2, 3, r, t, n\}$, a contradiction. If $n = 3$, then $M/N \simeq PSL(2, 27)$. Since G/N is not isomorphic with $PGL(2, 27)$, [6, Theorem A] verifies that $3(q-1)$ and $3(q+1)$ are elements of $cd(G/N)$. Hence 2, 3 and r generate a triangle in $\Delta(G/N)$, a contradiction. Thus $G/N \simeq PGL(2, q)$ which implies that $|G/N : M/N|$ is a power of 2.

Therefore $|G/N : M/N|$ is a power of 2. Now [6, Theorem A] implies that $\Delta(G/N) = \Delta(M/N)$ is the first graph in Figure 2.

Since M is a nonsolvable group, [2] implies that $n(\Delta(M)) \leq 3$. As $p \neq 2$ and $cd(M/N) \subseteq cd(M)$, we can see that $n(\Delta(M)) \neq 3$. Hence $\Delta(M)$ is either connected or disconnected with two connected components.

Step 3. If $n(\Delta(M)) = 2$, then $\Delta(M)$ contains a triangle.

Suppose that $n(\Delta(M)) = 2$. Since M is nonsolvable and $|\rho(M) = \rho(G)| = 4$, [3, Theorem A] implies that $\Delta(M)$ is one of the disconnected graphs in Figure 2.

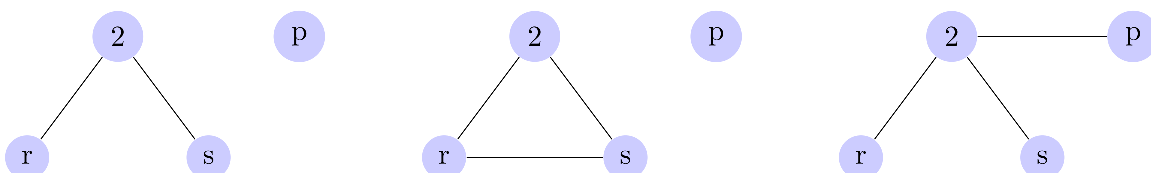

FIGURE 2.

Suppose that $\Delta(M)$ is the first graph. If 2 is not a neighbor of p in $\Delta(G)$, then without loss of generality, we may assume that s-p is an edge in $\Delta(G)$. Let $\chi \in Irr(G)$ with ps divides $\chi(1)$. As $\Delta(G)$ contains only one triangle and 2 is not a neighbor of p, neither r nor 2 divides $\chi(1)$. Let θ be an irreducible constituent of χ_M. Then $\chi(1) = e(\theta)t(\theta)\theta(1)$, where $e(\theta)t(\theta)$ divides $|G : M|$. Since $2 \nmid \chi(1)$ and $|G/N : M/N|$ is a power of two, we deduce that $e(\theta)t(\theta) = 1$, so $\chi(1) = \theta(1)$. This verifies that s-p is an edge in $\Delta(M)$, which is impossible. Thus 2-p is an edge in $\Delta(G)$. Without loss of generality, we may assume that s-p is an edge in $\Delta(G)$. Similar argument verifies that p-s is an edge in $\Delta(M)$, which is impossible. Hence $\Delta(M)$ is the second graph, so it contains a triangle.

Step 4. If $n(\Delta(M)) = 1$, then $\Delta(M) = \Delta(G)$.

Suppose that $\Delta(M)$ is a connected graph. As $\rho(M) = \rho(G)$ and $\Delta(M/N) \leq \Delta(M) \leq \Delta(G)$, we may have the following cases for $\Delta(M)$:

(1) $\Delta(M)$ is a path of length three.
(2) $\Delta(M)$ is the third graph in Figure 2.
(3) $\Delta(M) = \Delta(G)$.

As M is a nonsolvable group, [3, Theorem B] implies that case (1) is impossible. Similar arguments as in step 3 verifes that the second case is impossible, so $\Delta(M) = \Delta(G)$. In particular, $\Delta(M)$ contains a triangle. □

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