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THE CONJUGACY CLASS RANKS OF M_{24}

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Dedicated to Professor Jamshid Moori on the occasion of his seventieth birthday

Communicated by Robert Turner Curtis

Abstract. M_{24} is the largest Mathieu sporadic simple group of order $244823040 = 2^{10}{\cdot}3^3{\cdot}5{\cdot}7{\cdot}11{\cdot}23$ and contains all the other Mathieu sporadic simple groups as subgroups. The object in this paper is to study the ranks of M_{24} with respect to the conjugacy classes of all its nonidentity elements.

1. Introduction

M_{24} is the largest Mathieu sporadic simple group of order $244823040 = 2^{10}{\cdot}3^3{\cdot}5{\cdot}7{\cdot}11{\cdot}23$ and contains all the other Mathieu sporadic simple groups as subgroups. It is a 5-transitive permutation group on a set of 24 points such that:
(i) the stabilizer of a point is M_{23} which is 4-transitive
(ii) the stabilizer of two points is M_{22} which is 3-transitive
(iii) the stabilizer of a dodecad is M_{12} which is 5-transitive
(iv) the stabilizer of a dodecad and a point is M_{11} which is 4-transitive

M_{24} has a trivial Schur multiplier, a trivial outer automorphism group and it is the automorphism group of the Steiner system of type S(5,8,24) which is used to describe the Leech lattice on which M_{24} acts. M_{24} has nine conjugacy classes of maximal subgroups which are listed in [4], its ordinary character table is found in [4], its blocks and its Brauer character tables corresponding to the various primes dividing its order are found in [9] and [10] respectively and its complete prime spectrum is given by $2, 3, 5, 7, 11, 23$.

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In [13] we studied some of the triple generations of M_{24}. In the present paper, we shall study the ranks of M_{24} with respect to the conjugacy classes of all its nonidentity elements. The computations were carried out using the computer algebra system GAP [8] which is running on a LINUX machine in the Department of Mathematical Sciences at the University of South Africa.

Throughout, our groups G are finite, a conjugacy class is of elements of G and p, q, r denote primes that divide the order of G unless otherwise specified to the contrary. Reference to the various maximal subgroups of M_{24} should be understood to mean up to isomorphism.

2. Preliminaries

For a finite group G and nX a conjugacy class of nonidentity elements of G, we define $rank(G:nX)$ to be the minimum number of elements of G in nX that generate G. We call $rank(G:nX)$ the **RANK** of G with respect to the conjugacy class nX. If $C_1, C_2, \ldots, C_n$ are conjugacy classes of elements of G and $g_n \in C_n$ is a fixed representative, then $\xi_G(C_1, C_2, \ldots, C_n)$ denotes the number of distinct tuples $(g_1, g_2, \ldots, g_{n-1})$, where $g_i \in C_i$ for $1 \leq i \leq n-1$ such that $g_1 g_2 \cdots g_{n-1} = g_n$. This nonnegative integer is known as the **STRUCTURE CONSTANT** of the group algebra $\mathbb{C}G$ and can be computed from the ordinary character table of G using the formula

$$\xi_G(C_1, C_2, \ldots, C_n) = \frac{|C_1||C_2|\cdots|C_{n-1}|}{|G|} \sum_{j=1}^{k} \frac{\chi_j(g_1)\chi_j(g_2)\cdots\chi_j(g_{n-1})\overline{\chi_j(g_n)}}{[\chi_j(1_G)]^{n-2}}$$

where $\chi_1, \chi_2, \ldots, \chi_k \in Irr(G)$.

Suppose that $\xi_G^*(C_1, C_2, \ldots, C_n)$ denotes the number of distinct tuples $(g_1, g_2, \ldots, g_{n-1})$ with $g_i \in C_i$ for $1 \leq i \leq n-1$ for which $g_1 g_2 \ldots g_{n-1} = g_n$ such that $G = \langle g_1, g_2, \ldots, g_{n-1} \rangle$. If we get that $\xi_G^*(C_1, C_2, \ldots, C_n) > 0$, then we say that G is $(C_1, C_2, \ldots, C_n)$-generated. If $H \leq G$ containing $g_n \in C_n$ and B is a conjugacy class of elements of H such that $g_n \in B$, then $\sum_H(C_1, C_2, \ldots, C_{n-1}, B)$ denotes the number of distinct tuples $(g_1, g_2, \ldots, g_{n-1})$ with $g_i \in C_i$ for $1 \leq i \leq n-1$ for which $g_1 g_2 \cdots g_{n-1} = g_n$ such that $\langle g_1, g_2, \ldots, g_{n-1} \rangle \leq H$. We observe that $H \cap C_i$ for $1 \leq i \leq n-1$ decomposes into a disjoint union of conjugacy classes of H and $\sum_H(C_1, C_2, \ldots, C_{n-1}, B)$ is obtained by summing over all such conjugacy classes of maximal subgroups H of G.

Theorem 2.1. [1],[12] *Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $(\underbrace{(lX, lX, \ldots, lX)}_{m-times}, (nZ)^m)$-generated.*

Proof. This is [1, Lemma 2] and [12, Theorem 5]. □

Corollary 2.2. [5] [7] *Let G be a simple group such that G is $(2X, sY, tZ)$-generated. Then G is $(sY, sY, (tZ)^2)$-generated.*

Proof. This is [5, Lemma 2]. □

Theorem 2.3. *Let G be a finite group and H a subgroup of G containing an element x such that $(o(x), [N_G(H) : H]) = 1$. Then the number h of conjugates of H containing x is given by $\chi_H(x)$, where*

χ_H is the permutation character of G acting on the conjugates of H. In particular, we obtain that

$$h = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_{N_G(H)}(x_i)|}$$

where $x_1, x_2, \ldots, x_m$ are representatives of the $N_G(H)$-conjugacy classes that fuse into the G-class $[x]_G$.

Proof. cf. [2],[6] □

The following result can be very useful in proving nongeneration.

Lemma 2.4. *Let G be a finite centerless group, lX, mY, nZ be conjugacy classes of G for which $\xi_G^*(lX, mY, nZ) < |C_G(z)|$ for $z \in nZ$. Then $\xi_G^*(lX, mY, nZ) = 0$ so that G is NOT (lX, mY, nZ)-generated.*

Proof. cf. [2],[5] □

3. The ranks of M_{24}

We study here the ranks of M_{24} with respect to the various conjugacy classes of all its nonidentity elements.

Lemma 3.1. [3] *Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then $rank(G : lX) \leq m$.*

Proof. Since G is (lX, mY, nZ)-generated, by Theorem 2.1 above, we get that G is generated by m elements from the class lX. Hence the result follows thus completing the proof. □

Proposition 3.2. *$rank(M_{24} : 2A) = 3 = rank(M_{24} : 2B)$*

Proof. We have by [13] that M_{24} is $(2A, 3B, 23A)$ and $(2B, 3A, 23A)$-generated. Thus we must have by Lemma 3.1 above that $rank(M_{24} : 2A) \leq 3$ and $rank(M_{24} : 2B) \leq 3$. By Theorem 2.1, we have that M_{24} is $(2A, 2A, 2A, (23A)^3)$ and $(2B, 2B, 2B, (23A)^3)$-generated. However by [8] we have that $(23A)^3 = 23A$ so that M_{24} becomes $(2A, 2A, 2A, 23A)$ and $(2B, 2B, 2B, 23A)$-generated. Also by the proof of [1, Lemma 3.1] and by [11, Proposition 1], the result follows and the proof is complete. □

Proposition 3.3. *$rank(M_{24} : 3A) = 2 = rank(M_{24} : 3B)$*

Proof. According to [13], we have that M_{24} is $(2A, 3B, 23A)$ and $(2B, 3A, 23B)$-generated. Only the maximal subgroups M_{23} and $L_2(23)$ of M_{24} have elements of order 23. We obtain that $\xi_{M_{24}}(3A, 3A, 23B) = 161$ and $\xi_{M_{24}}(3B, 3B, 23A) = 552$. We have that M_{23} meets the $3A$ and not the $3B$ class whereas $L_2(23)$ meets the $3B$ and not the $3A$ class of M_{24}. We thus obtain that $\xi_{M_{23}}(3a, 3a, 23b) = 138$ and $\xi_{L_2(23)}(3a, 3a, 23a) = 23$. By Theorem 2.3 above, the classes $23b$ of M_{23} and $23a$ of $L_2(23)$ are contained in one conjugate each of M_{23} and $L_2(23)$ respectively. Thus we obtain that $\xi^*_{M_{24}}(3A, 3A, 23B) = 161 - 138 = 23 = |C_{M_{24}}(23B)|$ and $\xi^*_{M_{24}}(3B, 3B, 23A) = 552 - 23 = 529 > |C_{M_{24}}(23A)| = 23$. Thus we have that M_{24} is $(3B, 3B, 23A)$ and $(3A, 3A, 23B)$-generated. Hence the result follows. □

Proposition 3.4. $rank(M_{24} : 4A) = 2 = rank(M_{24} : 4B) = rank(M_{24} : 4C)$

Proof. We obtain that $\xi_{M_{24}}(4A, 4A, 23A) = 1035, \xi_{M_{24}}(4B, 4B, 23A) = 13501$ and $\xi_{M_{24}}(4C, 4C, 23A) = 27508$. Of the maximal subgroups M_{23} and $L_2(23)$ of M_{24} which have elements of order 23, $4a$ of M_{23} fuses into $4B$ of M_{24} and $4a$ of $L_2(23)$ fuses into $4C$ of M_{24} and none of them meets the $4A$ class of M_{24}. Thus we have that $\xi_{M_{23}}(4a, 4a, 23a) = 7866$ and $\xi_{L_2(23)}(4a, 4a, 23a) = 23$. By [7] and [14], $23a$ of M_{23} and $23a$ of $L_2(23)$ are contained in one conjugate each of M_{23} and $L_2(23)$. Thus we obtain that $\xi^*_{M_{24}}(4A, 4A, 23A) = 1035$, $\xi^*_{M_{24}}(4B, 4B, 23A) = 13501 - 7866 = 5635$ and $\xi^*_{M_{24}}(4C, 4C, 23A) = 27508 - 23 = 27485$, so that M_{24} is $(4A, 4A, 23A)$, $(4B, 4B, 23A)$, $(4C, 4C, 23A)$-generated. Hence the result follows. □

Lemma 3.5 immediately following here below, will be used to prove some of the subsequent results.

Lemma 3.5. *Let G be a finite simple group such that G is $(2X, mY, nZ)$-generated. Then $rank(G : mY) = 2$.*

Proof. The result follows immediately from Corollary 2.2 above. □

Proposition 3.6. $rank(M_{24} : 5A) = 2$

Proof. We obtained from [13] that M_{24} is $(2B, 5A, 7A)$-generated. Thus by Lemma 3.5 above, the result follows immediately and the proof is complete. □

Proposition 3.7. $rank(M_{24} : 6A) = 2 = rank(M_{24} : 6B)$

Proof. We obtain that $\xi_{M_{24}}(6A, 6A, 23A) = 427041$ and $\xi_{M_{24}}(6B, 6B, 23A) = 406456$. We obtain that $6a$ of M_{23} fuses into $6A$ of M_{24} whereas $6a$ of $L_2(23)$ fuses into $6B$ of M_{24}. Thus we get that $\xi_{M_{23}}(6a, 6a, 23a) = 72588$ and $\xi_{L_2(23)}(6a, 6a, 23a) = 23$ so that $\xi^*_{M_{24}}(6A, 6A, 23A) = 427041 - 72588 = 354453$ and $\xi^*_{M_{24}}(6B, 6B, 23A) = 406456 - 23 = 406433$. Thus we obtain that M_{24} is $(6A, 6A, 23A)$, $(6B, 6B, 23A)$-generated. Hence the result follows thus completing the proof. □

Proposition 3.8. $rank(M_{24} : 7A) = 2 = rank(M_{24} : 7B)$

Proof. In [13] it has been proved that M_{24} is $(2, 7, 23)$-generated and so the result follows immediately by Lemma 3.5 above and the proof is complete. □

Proposition 3.9. $rank(M_{24} : 8A) = 2$

Proof. We obtain that $\xi_{M_{24}}(8A, 8A, 23A) = 922024$ and of the maximal subgroups of M_{24} having elements of order 23, only M_{23} meets the $8A$ class of M_{24}. So we obtain that $\xi_{M_{23}}(8a, 8a, 23a) = 154376$ so that $\xi^*_{M_{24}}(8A, 8A, 23A) = 922024 - 154376 = 767648$. Thus we obtain that M_{24} is $(8A, 8A, 23A)$-generated and hence the result follows completing the proof. □

Proposition 3.10. $rank(M_{24} : 10A) = 2$

Proof. We obtain that $\xi_{M_{24}}(10A, 10A, 23A) = 607752$ and of the maximal subgroups of M_{24} having elements of order 23, none meets the $10A$ class of M_{24}. Thus we obtain that $\xi^*_{M_{24}}(10A, 10A, 23A) = 607752$ so that M_{24} is $(10A, 10A, 23A)$-generated and the result follows. □

Proposition 3.11. $rank(M_{24} : 11A) = 2$

Proof. From [13] we have that M_{24} is $(2, 11, 23)$-generated. Therefore by Lemma 3.5 above, the result follows immediately and the proof is complete. □

Proposition 3.12. $rank(M_{24} : 12A) = 2 = rank(M_{24} : 12B)$

Proof. We obtain that $\xi_{M_{24}}(12A, 12A, 23A) = 1706232$ and $\xi_{M_{24}}(12B, 12B, 23A) = 1625824$. Only the maximal subgroup $L_2(23)$ meets the $12B$ class of M_{24} i.e. both $12a$ and $12b$ of $L_2(23)$ fuse into the $12B$ class of M_{24}. We thus obtain that $\xi_{L_2(23)}(12a, 12a, 23a) = 23 = \xi_{L_2(23)}(12b, 12b, 23a)$. So $\xi^*_{M_{24}}(12A, 12A, 23A) = 1706232$ and $\xi^*_{M_{24}}(12B, 12B, 23A) = 1625824 - 46 = 1625778$ so that M_{24} is $(12A, 12A, 23A)$, $(12B, 12B, 23A)$-generated. Hence the result follows. □

Proposition 3.13. $rank(M_{24} : 14A) = 2 = rank(M_{24} : 14B)$

Proof. We obtain that $\xi_{M_{24}}(14A, 14A, 23A) = 1328940 = \xi_{M_{24}}(14B, 14B, 23A)$. We have that $14a$ and $14b$ of M_{23} fuse into $14A$ and $14B$ of M_{24} respectively. Thus we obtain that $\xi_{M_{23}}(14a, 14a, 23a) = 52992 = \xi_{M_{23}}(14b, 14b, 23a)$ so that $\xi^*_{M_{24}}(14A, 14A, 23A) = 1328940 - 52992 = 1275948 = \xi^*_{M_{24}}(14B, 14B, 23A)$. So M_{24} is $(14A, 14A, 23A)$, $(14B, 14B, 23A)$-generated. Hence the result follows. □

Proposition 3.14. $rank(M_{24} : 15A) = 2 = rank(M_{24} : 15B)$

Proof. We obtain that $\xi_{M_{24}}(15A, 15A, 23A) = 1051008 = \xi_{M_{24}}(15B, 15B, 23A)$. We have that $15a$ and $15b$ of M_{23} fuse into $15A$ and $15B$ of M_{24} respectively. Thus we obtain that $\xi_{M_{23}}(15a, 15a, 23a) = 41998 = \xi_{M_{23}}(15b, 15b, 23a)$ so that $\xi^*_{M_{24}}(15A, 15A, 23A) = 1051008 - 41998 = 1009010 = \xi^*_{M_{24}}(15B, 15B, 23A)$. So M_{24} is $(15A, 15A, 23A)$, $(15B, 15B, 23A)$-generated and hence the result follows. □

Proposition 3.15. $rank(M_{24} : 21A) = 2 = rank(M_{24} : 21B)$

Proof. We obtain that $\xi_{M_{24}}(21A, 21A, 23A) = 590640 = \xi_{M_{24}}(21B, 21B, 23A)$. In this case there is no contribution from any of the maximal subgroups of M_{24} containing elements of order 23. This then renders $\xi^*_{M_{24}}(21A, 21A, 23A) = 590640 = \xi^*_{M_{24}}(21B, 21B, 23A)$ so that M_{24} is $(21A, 21A, 23A)$, $(21B, 21B, 23A)$-generated. Hence the result follows. □

Proposition 3.16. $rank(M_{24} : 23A) = 2 = rank(M_{24} : 23B)$

Proof. We obtain that $\xi_{M_{24}}(23A, 23A, 23A) = 441904$ and $\xi_{M_{24}}(23B, 23B, 23A) = 455728$. We have that $23a$ and $23b$ of both M_{23} and $L_2(23)$ fuse into $23A$ and $23B$ of M_{24} respectively. Their contributions are thus $\xi_{M_{23}}(23a, 23a, 23a) = 17646$, $\xi_{M_{23}}(23b, 23b, 23a) = 18222$ and $\xi_{L_2(23)}(23a, 23a, 23a) = 5, \xi_{L_2(23)}(23b, 23b, 23a) = 29$. Thus we obtain that $\xi^*_{M_{24}}(23A, 23A, 23A) = 441904 - 17646 - 5 = 424253$ and $\xi^*_{M_{24}}(23B, 23B, 23A) = 455728 - 18222 - 29 = 437477$. These then render M_{24} to be $(23A, 23A, 23A)$, $(23B, 23B, 23A)$-generated which thus give the desired results. □

We thus observe from all the proofs appearing here above that $rank(M_{24} : nX) \in \{2, 3\}$ for any conjugacy class nX of nonidentity elements of M_{24}.

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