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# ON EMBEDDING OF PARTIALLY COMMUTATIVE METABELIAN GROUPS TO MATRIX GROUPS

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Abstract. The Magnus embedding of a free metabelian group induces the embedding of partially commutative metabelian group $S_\Gamma$ in a group of matrices $M_\Gamma$. Properties and the universal theory of the group $M_\Gamma$ are studied.

## 1. Introduction

Let $\Gamma$ be a finite undirected graph with no loops and no multiple edges, $V = \{v_1, \dots, v_r\}$ is the set of the vertices of $\Gamma$, and $E$ the set of edges of this graph.

$$F_\Gamma = \langle v_1, \dots, v_r \mid v_i v_j = v_j v_i \text{ if } (v_i, v_j) \in E \rangle$$

is a free partially commutative group.

A partially commutative metabelian group $S_\Gamma$ has the same defining relations as the group $F_\Gamma$ and the identity $[[x, y], [u, v]] = 1$. Properties of groups $S_\Gamma$ and their universal theories were studied in [1]–[5].

The Magnus embedding (see, for example [6, 7]) enables us to consider a free metabelian group $S$ as a subgroup of a matrix group $M$.

The universal theory of a free metabelian group $S$ coincides with the universal theory of matrix group $M$. This fact helps Chapuis [8] to show that the universal theory of a free metabelian group is solvable.

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The problem on solvability of the universal theory of partially commutative metabelian group has not been solved yet. This problem is included in "The Kourovka Notebook" [9] with number 17.104.

The Magnus embedding induces an embedding of $S_\Gamma$ into a semidirect product of a free abelian group $A$ of finite rank and an abelian normal subgroup equipped with a $\mathbb{Z}[A]$-module structure. By $M_\Gamma$ we denote this semidirect product.

We study properties of $M_\Gamma$ and its universal theory. In [1] it was shown that the universal theory of $M_\Gamma$ is solvable. We consider some transformations of a defining graph $\Gamma$ and show that these transformations do not change the universal theory of $M_\Gamma$ (Theorem 3.3). Next, we use this result to study the universal theories of so called metabelian graph products. The notion of graph product appeared in the paper [10]. After that graph products were studied in several papers (see for example [11, 12]). We consider a graph product $S(\Gamma, A_1, \ldots, A_r)$ of free abelian groups $A_1, \ldots, A_r$ in the variety of metabelian groups. The group $S(\Gamma, A_1, \ldots, A_r)$ is embedded to a matrix group denoted by $M(\Gamma, A_1, \ldots, A_r)$. The embedding is induced by the Magnus embedding. By Theorem 3.3 the universal theories of all matrix groups $M(\Gamma, A_1, \ldots, A_r)$ coincide for all free abelian groups $A_1, \ldots, A_r$ (Corollary 4).

However, given a graph $\Gamma$ the universal theories of the groups $S_\Gamma$ and $M_\Gamma$ are different in general. For example, if $L_3$ is the linear graph of the length 3 then the universal theories of the partially commutative group $S_{L_3}$ and the group of matrix $M_{L_3}$ do not coincide. We obtain this result using the notion of centralizer dimension (Proposition 5).

Nevertheless, some common properties for the universal theories of the groups $S_\Gamma$ and $M_\Gamma$ are established by Theorem 4.1. This theorem states that if an equation in one variable has coefficients in $S_\Gamma$ then this equation is solvable in $S_\Gamma$ iff it is solvable in $M_\Gamma$.

Notice that there is the analogous result for groups of kind $F/[R, R]$, where $F$ is a free group and $R$ is its normal subgroup such that the ring $\mathbb{Z}[F/R]$ has no zero divisors. This result has been proved in [13].

## 2. Preliminaries and Notation

Let us introduce notation we use throughout this paper.

Let $V = \{v_1, \ldots, v_r\}$ be the set of the vertices of a graph $\Gamma$, and $E$ the set of edges of this graph. We consider only finite undirected graphs with no loops and no multiple edges.

Let $A$ be the free abelian group of rank $r$ with a basis $\{a_1, \ldots, a_r\}$ and $\mathbb{Z}[A]$ the integer group ring of $A$. Denote by $T$ the right free $\mathbb{Z}[A]$-module with a basis $\{t_1, \ldots, t_r\}$ and by

$$M = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$$

the matrix group.

The submodule $\widetilde{T}_\Gamma$ of the module $T$ is generated by the elements

$$t_{ij} = t_i(a_j - 1) + t_j(1 - a_i)$$

such that the vertices $v_i$ and $v_j$ of $\Gamma$ are adjacent and $i < j$. $T_\Gamma$ is the factor module $T/\widetilde{T}_\Gamma$.

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Let $S$ be the free metabelian group of rank $r$ with a basis $\{s_1, \dots, s_r\}$ and $R$ the normal subgroup generated by the commutators $[s_i, s_j] = s_i^{-1}s_j^{-1}s_is_j$ such that $(v_i, v_j) \in E$. Then the partially commutative metabelian group $S_\Gamma$ is isomorphic to the factor group $S/R$.

The Magnus embedding $\mu$ of $S$ to $M$ extends the map

$$s_i \longmapsto \begin{pmatrix} a_i & 0 \\ t_i & 1 \end{pmatrix}, \quad i = 1, \dots, r.$$

Define the epimorphism $d$ of the module $T$ to the fundamental ideal $\Delta$ of the ring $\mathbb{Z}[A]$ as follows. If $t = t_1\beta_1 + \cdots + t_r\beta_r$ is in $T$ then

$$d(t) = (a_1 - 1)\beta_1 + \cdots + (a_r - 1)\beta_r.$$

In [6] it was shown that a matrix

$$\begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix}$$

is in the image of the group $S$ under the Magnus embedding iff

$$d(t) = a - 1.$$

In particular, the matrix

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

is in the image of the commutant $[S, S]$ of the group $S$ iff

$$d(t) = 0. \tag{2.1}$$

It is easy to derive that the Magnus embedding maps $[s_i, s_j]$ to

$$\begin{pmatrix} 1 & 0 \\ t_{ij} & 1 \end{pmatrix}.$$

So, the subgroup $R$ is mapped onto the subgroup

$$\begin{pmatrix} 1 & 0 \\ \widetilde{T}_\Gamma & 1 \end{pmatrix}.$$

Therefore, the Magnus embedding $\mu$ induces the embedding $\mu_\Gamma$ of the group $S_\Gamma$ into the matrix group

$$M_\Gamma = \begin{pmatrix} A & 0 \\ T_\Gamma & 1 \end{pmatrix}. \tag{2.2}$$

A matrix

$$\begin{pmatrix} a & 0 \\ t + \widetilde{T}_\Gamma & 1 \end{pmatrix} \in M_\Gamma,$$

is in the image of the group $S_\Gamma$ in matrix group (2) if and only if

$$d(t) = a - 1. \tag{2.3}$$

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Note that we can choose any element $t$ in the corresponding adjacent class of $T_\Gamma$ to check if condition (2.3) is satisfied.

## 3. Universal Equivalence of Groups $M_\Gamma$

Let $v_1$ be a vertex of a graph $\Gamma$, $W'$ be the set of vertices in this graph such that these vertices are adjacent to $v_1$, and $W = W' \sqcup \{v_1\}$. Denote by $\Gamma_0$ the graph obtained from $\Gamma$ by adding the vertex $v_0$ and the edges connecting $v_0$ with all vertices in $W$.

So, the set $V_0$ of vertices in $\Gamma_0$ is $V \bigsqcup \{v_0\}$. By $E_0$ denote the set of edges of $\Gamma_0$.

In [3] the vertices $v_0$ and $v_1$ were called equivalent. In this paper it was also proved that the universal theories of partially commutative metabelian groups $S_\Gamma$ and $S_{\Gamma_0}$ coincide (Theorem 4).

If the vertices $v$ and $w$ are equivalent we use notation $v \sim_\perp w$.

Let $A_0$ be the free abelian group with the basis $\{a_0, a_1, \cdots, a_r\}$ such that $A$ is a subgroup of $A_0$ and $T_0$ the free $\mathbb{Z}[A_0]$-module with the basis $\{t_0, t_1, \cdots, t_r\}$,

$$M_0 = \begin{pmatrix} A_0 & 0 \\ T_0 & 1 \end{pmatrix} \quad M_{\Gamma_0} = \begin{pmatrix} A_0 & 0 \\ T_{\Gamma_0} & 1 \end{pmatrix}.$$

For a natural number $n$ consider the map $\psi_n$ of $M_0$ to $M$ that takes each matrix

$$\mathfrak{m}_0 = \begin{pmatrix} a_0^{l_0} a_1^{l_1} \cdots a_r^{l_r} & 0 \\ \sum_{i=0}^r t_i \alpha_i(a_0, \ldots, a_r) & 1 \end{pmatrix} \in M_0$$

to the matrix

$$\mathfrak{m} = \begin{pmatrix} a_1^{nl_0+l_1} a_2^{l_2} \cdots a_r^{l_r} & 0 \\ t_1(\frac{a_1^n - 1}{a_1 - 1}\alpha_0(a_1^n, a_1, \cdots, a_r) + \alpha_1(a_1^n, a_1, \ldots, a_r)) + \sum_{i=2}^r t_i \alpha_i(a_1^n, a_1 \ldots, a_r) & 1 \end{pmatrix}.$$

**Lemma 3.1.** *The map $\psi_n$ defines a retraction of $M_0$ to $M$ and $\widetilde{T_0}\psi_n = \widetilde{T}$.*

*Proof.* For short let us use the following notation

$$\alpha_i = \alpha_i(a_0, a_1, \ldots, a_r), \;\; \alpha_i' = \alpha_i(a_1^n, a_1, \ldots, a_r),$$

$$\beta_i = \beta_i(a_0, a_1, \ldots, a_r), \;\; \beta_i' = \beta_i(a_1^n, a_1, \ldots, a_r).$$

Let

$$\mathfrak{n}_0 = \begin{pmatrix} a_0^{q_0} a_1^{q_1} \cdots a_r^{q_r} & 0 \\ \sum_{i=1}^r t_i \beta_i & 1 \end{pmatrix} \in M_0.$$

Then

$$\mathfrak{m}_0\psi_n = \begin{pmatrix} a_1^{nl_0+l_1} a_2^{l_2} \cdots a_r^{l_r} & 0 \\ t_1(\frac{a_1^n-1}{a_1-1}\alpha_0' + \alpha_1') + \sum_{i=2}^r t_i \alpha_i' & 1 \end{pmatrix},$$

$$\mathfrak{n}_0\psi_n = \begin{pmatrix} a_1^{nq_0+q_1} a_2^{q_2} \cdots a_r^{q_r} & 0 \\ t_1(\frac{a_1^n-1}{a_1-1}\beta_0' + \beta_1') + \sum_{i=2}^r t_i \beta_i' & 1 \end{pmatrix},$$

$$\mathfrak{m}_0\mathfrak{n}_0 = \begin{pmatrix} a_0^{l_0+q_0} a_1^{l_1+q_1} \cdots a_r^{l_r+q_r} & 0 \\ \sum_{i=0}^r t_i(\alpha_i a_0^{q_0} \cdots a_r^{q_r} + \beta_i) & 1 \end{pmatrix}.$$

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Applying $\psi_n$ to the last matrix we obtain

$$(\mathfrak{m}_0\mathfrak{n}_0)\psi_n = \begin{pmatrix} a_1^{n(l_0+q_0)+l_1+q_1} a_2^{l_2+q_2} \cdots a_r^{l_r+q_r} & 0 \\ \tau & 1 \end{pmatrix}, \tag{3.1}$$

where

$$\tau = t_1\left(\frac{a_1^n-1}{a_1-1}(\alpha_0' a_1^{nq_0+q_1} a_2^{q_2}\cdots a_r^{q_r} + \beta_0') + \alpha_1' a_1^{nq_0+q_1} a_2^{q_2}\cdots a_r^{q_r} + \beta_1'\right) + \sum_{i=2}^{r} t_i(\alpha_i' a_1^{nq_0+q_1} a_2^{q_2}\cdots a_r^{q_r} + \beta_i'\Big).$$

Compute the $(2,1)$-entry of the matrix $(\mathfrak{m}_0\psi_n)(\mathfrak{n}_0\psi_n)$. It coincides with $\tau$. Comparing the matrices $(\mathfrak{m}_0\mathfrak{n}_0)\psi_n$ and $(\mathfrak{m}_0\psi_n)(\mathfrak{n}_0\psi_n)$ we see that they are equal.

Therefore $\psi_n$ is a homomorphism acting identically on $M$. So, $\psi_n$ is a retraction.

Let us show that $\widetilde{T}_0\psi_n = \widetilde{T}$.

If $i \neq 0$ then $t_{ij}\psi_n = t_{ij}$.

Consider the images of the elements $t_{0j}$ under $\psi_n$. We obtain

$$t_{01}\psi_n = (t_0(a_1-1) + t_1(1-a_0))\psi_n = t_1\left(\frac{a_1^n-1}{a_1-1}(a_1-1) - a_1^n + 1\right) = 0.$$

If $j \neq 1$ then we have

$$t_{0j}\psi_n = (t_0(1-a_j) + t_j(a_0-1))\psi_n = t_1\frac{a_1^n-1}{a_1-1}(1-a_j) + t_j(a_1^n-1) = \frac{a_1^n-1}{1-a_1}t_{1j}.$$

But if $(v_0, v_j) \in E_0$ then $(v_1, v_j) \in E$. Consequently $t_{0j}\psi_n$ is in $\widetilde{T}$. This completes the proof of the lemma. □

We identify $S_\Gamma$ and $S_{\Gamma_0}$ with their images in $M_\Gamma$ and $M_{\Gamma_0}$ respectively. By Lemma 3.1 the homomorphism $\psi_n$ induces the homomorphism $\varphi_n$ of $M_{\Gamma_0}$ onto $M_\Gamma$.

**Lemma 3.2.** *The homomorphism $\varphi_n$ has the following properties*

1. *$\varphi_n$ is a retraction;*
2. *$\varphi_n$ maps $S_{\Gamma_0}$ onto $S_\Gamma$;*
3. *For any element $1 \neq g \in M_{\Gamma_0}$ there exists $n_0$ such that $g\varphi_n \neq 1$ whenever $n \geq n_0$.*

*Proof.* The first property follows from the definition of $\psi_n$.

Let us prove the second property. Let $S_0$ be the free metabelian group with the basis $\{s_0, s_1, \ldots, s_r\}$ and $S$ its subgroup generated by $s_1, \ldots, s_r$. Take all commutators $[s_i, s_j]$ such that $(v_i, v_j) \in E_0$ and $0 \leq i < j \leq r$. Denote by $R_0$ the normal closure of these commutators in $S_0$.

We have

$$s_0R_0 = \begin{pmatrix} a_0 & 0 \\ t_0 + \widetilde{T}_0 & 1 \end{pmatrix} \stackrel{\varphi_n}{\longmapsto} \begin{pmatrix} a_1^n & 0 \\ \frac{a_1^n-1}{a_1-1}t_1 + \widetilde{T} & 1 \end{pmatrix} = $$
$$= \begin{pmatrix} a_1 & 0 \\ t_1 + \widetilde{T} & 1 \end{pmatrix}^n = (s_1R)^n.$$

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Clearly, $(s_iR_0)\varphi_n = s_iR$ for $1 \le i \le r$. Therefore, $S_{\Gamma_0}\psi_n = S_\Gamma$.

We are left to prove the third property.

Let

$$1 \neq g = \begin{pmatrix} a & 0 \\ t_0\alpha_0 + \cdots + t_r\alpha_r + \widetilde{T}_0 & 1 \end{pmatrix}, \ a \in A_0, \ \alpha_i \in \mathbb{Z}[A_0].$$

Case 1: $a \neq 1$.

If $a$ does not depend on $a_0$ then any number can be chosen for $n$.

Let $a$ depend on $a_0$, i.e. $a = a_0^{l_0} \cdots a_r^{l_r}$, $l_0 \neq 0$. Choose $n_0$ such that $l_0n_0 + l_1 > 0$ for $l_0 > 0$ and $l_0n_0 + l_1 < 0$ for $l_0 < 0$. Obviously, for all $n \ge n_0$ we get $g\varphi_n \neq 1$.

Case 2a: $a = 1$, $\sum_{i=0}^r \alpha_i(a_i - 1) = 0$.

In this case the matrix $g$ is in the commutant of $S_{\Gamma_0}$. The homomorphism $\varphi_n$ induces the homomorphism $\overline{\varphi}_n$ of $S_{\Gamma_0}$ onto $S_\Gamma$ and for $\varphi_n$ we have

$$s_0R_0 \longmapsto s_1^nR, \ s_iR_0 \longmapsto s_iR, \ i = 1, \dots, r.$$

In [3], Theorem 4, it was shown that there exists $n_0$ such that for any $n \ge n_0$ the image of $g \in [S_{\Gamma_0}, S_{\Gamma_0}]$ is not equal to the unit. This value of $n_0$ satisfies the condition of the lemma.

Case 2b: $a = 1$, $\sum_{i=0}^r \alpha_i(a_i - 1) \neq 0$.

Let $\alpha = \sum_{i=0}^r \alpha_i(a_i - 1)$. Define a homomorphism $\chi_n : \mathbb{Z}[A_0] \longrightarrow \mathbb{Z}[A]$ on the basis of $A_0$ as follows.

$$\chi_n = \{a_0 \longmapsto a_1^n, a_1 \longmapsto a_1, \dots, a_r \longmapsto a_r\}.$$

Choose $n_0$ in such a way that the image of $\alpha$ under $\chi_n$ is non-zero. Show that for any $n \ge n_0$ the element $g\varphi_n$ is not equal to the unit.

We have

$$g\varphi_n = \begin{pmatrix} 1 & 0 \\ t_1(\frac{a_1^n - 1}{a_1 - 1}\alpha_0' + \alpha_1') + t_2\alpha_2' + \cdots + t_r\alpha_r' & 1 \end{pmatrix}.$$

Therefore

$$(a_1 - 1)\left(\frac{a_1^n - 1}{a_1 - 1}\alpha_0' + \alpha_1'\right) + (a_2 - 1)\alpha_2' + \cdots + (a_r - 1)\alpha_r' =$$
$$(a_1^n - 1)\alpha_0' + (a_1 - 1)\alpha_1' + \cdots + (a_r - 1)\alpha_r' = \alpha\chi_n \neq 0.$$

The lemma is proved completely. □

A group $G$ *is discriminated by* a group $H$ if for any sat of non-unit elements $\{g_1, \cdots, g_n\}$ in $G$ there exists a homomorphism $\varphi : G \longrightarrow H$ such that $\varphi(g_i) \neq 1$ for all $i = 1, \dots, n$.

It is well known that if $G$ is discriminated by $H$ and $H$ is discriminated by $G$ then the universal theories of $G$ and $H$ coincide.

**Theorem 3.3.** *Let $v_0$ and $v_1$ be equivalent $v_0 \sim_\perp v_1$ in a graph $\Gamma_0$ and let the graph $\Gamma$ be obtained from a graph $\Gamma_0$ by deleting a vertex $v_0$ and all edges incident to $v_0$. Then the universal theories of $M_\Gamma$ and $M_{\Gamma_0}$ coincide.*

*Proof.* It is clear that the group $M_\Gamma$ is embedded in the group $M_{\Gamma_0}$. On the other hand, Lemma 3.2 implies that $M_{\Gamma_0}$ is discriminated by $M_\Gamma$. So, the assertion follows. □

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Let $\Gamma$ be a graph with the set of vertices $V = \{v_1, \dots, v_r\}$ and the set of edges $E$. Consider a family of non-trivial groups $\{G_v \,|\, v \in V\}$. The *graph product* of these groups is the factor group of the free product $\prod_{v\in V}^{*} G_v$ by the normal subgroup generated by commutants $[G_v, G_w]$ such that $(v, w) \in E$.

Let us define the notion of metabelian graph product. Actually we need to change the free product of groups by the metabelian product. We are going to use the definition of metabelian product for free abelian groups.

Let $A_i = A_{v_i}$ be a free abelian group, $i = 1, \dots, r$.

*Metabelian product* $\mathfrak{A}^2 \prod A_i$ is the factor group of the free product $F = \prod^{*} A_i$ by the second commutant

$$F^{(2)} = [[F, F], [F, F]].$$

*Metabelian graph product* $S(\Gamma, A_1, \dots, A_r)$ is the factor of $\mathfrak{A}^2 \prod A_i$ by the normal subgroup generated by the commutants $[A_i, A_j]$, such that $(v_i, v_j) \in E$.

Metabelian graph product of free abelian groups of finite ranks can be obtained as follows. Let $A_i$ be the free group of rank $ri \geq 2$. For $i = 1, \dots, r$ we add the vertices $v_{i_2}, \dots, v_{i_{ri}}$ to the set $V$ of vertices of the graph $\Gamma$. We connect all vertices $v_i, v_{i_2}, \dots, v_{i_{ri}}$ pairwise. In addition, we connect all vertexes $v_{i_2}, \dots, v_{i_{ri}}$ with $v$ whenever $(v_i, v) \in E$. Let $\Delta$ be the obtained graph. It is clear that $S_\Delta$ and $S(\Gamma, A_1, \dots, A_r)$ are isomorphic.

Denote $M_\Delta$ by $M(\Gamma, A_1, \dots, A_r)$.

Since $v_{i_2}, \dots, v_{i_{ri}}$ are equivalent to $v_i$ Theorem 3.3 implies the following corollary.

**Corollary 3.4.** *Let $A_i$, $i = 1, \dots, r$, be free abelian groups of finite ranks and $\Gamma$ a graph. Then the universal theories of $M_\Gamma$ and $M(\Gamma, A_1, \dots, A_r)$ coincide.*

Let $\Gamma$ be a totally disconnected graph. Then $S_\Gamma \simeq S$ is the free metabelian group of rank $r$ and $M_\Gamma \simeq AwrB$ is a wreath product of two abelian groups of rank $r$. In [8] it was shown that the universal theories of the groups $S$ and $AwrB$ coincide.

But there exists a graph $\Gamma$ such that the universal theories of $S_\Gamma$ and $M_\Gamma$ differ.

**Proposition 3.5.** *Let $\Gamma = L_3$ be the linear graph on three vertices. Then the universal theories of $S_\Gamma$ and $M_\Gamma$ do not coincide.*

*Proof.* Recall that *centralizer dimension* $CdimG$ of a group $G$ is equal to $n$ if there exist subsets

$$A_1 \subset A_2 \subset \cdots \subset A_n,$$

in $G$ such that their centralizers

$$C(A_1) > C(A_2) \cdots > C(A_n)$$

are strictly decreasing and $n$ is the largest number such that this property holds for $n$.

If there is no largest $n$ then set $Cdim(G) = \infty$.

Notice that coincidence of universal (equivalently, existential) theories of two groups implies coincidence of their centralizer dimensions.

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First, let us find centralizer dimension of the group $S_\Gamma$ which is isomorphic to the direct product of the free metabelian group $S_2\langle x_1, x_3\rangle$ of rank 2 and the infinite cyclic group $\langle x_2\rangle$. In [14], the following formula for centralizer dimension of a direct product of two groups was proved

$$Cdim(S_2 \times \langle x\rangle) = Cdim(S_2) + Cdim(\langle x\rangle) - 1.$$

So, it is easy to see that $Cdim(S_\Gamma) = 3$.

Now let us find $Cdim(M_\Gamma)$. Let

$$\widehat{t}_{13} = \begin{pmatrix} 1 & 0 \\ t_1(a_3-1) + t_3(1-a_1) + \widetilde{T}_\Gamma & 1 \end{pmatrix},$$

$$\widehat{t}_1 = \begin{pmatrix} 1 & 0 \\ t_1 + \widetilde{T}_\Gamma & 1 \end{pmatrix}, \; \widehat{t}_2 = \begin{pmatrix} 1 & 0 \\ t_2 + \widetilde{T}_\Gamma & 1 \end{pmatrix}$$

$$\widehat{a}_1 = \begin{pmatrix} a_1 & 0 \\ \widetilde{T}_\Gamma & 1 \end{pmatrix}, \; \widehat{a}_2 = \begin{pmatrix} a_2 & 0 \\ \widetilde{T}_\Gamma & 1 \end{pmatrix}.$$

We obtain the chain of centralizers

$$M_\Gamma \overset{\widehat{a}_1}{>} C(\widehat{t}_{13}) \overset{\widehat{a}_2}{>} C(\widehat{t}_{13}, \widehat{t}_2) \overset{\widehat{t}_1}{>} C(\widehat{t}_{13}, \widehat{t}_2, \widehat{a}_2).$$

In detail:

1. $[\widehat{a}_1, \widehat{t}_{13}] = [x_1, [x_1, x_3]] \neq 1$ in $S_2$.

2. $[\widehat{a}_2, \widehat{t}_{13}] = [x_2, [x_1, x_3]] = 1$. If $[\widehat{a}_2, \widehat{t}_2] = 1$ then $t_2(a_2 - 1)$ is in the submodule $\widetilde{T}_\Gamma$ of the free module $T$ generated by the elements

$$(a_1 - 1)t_2 + (1 - a_2)t_1, \; (a_3 - 1)t_2 + (1 - a_2)t_3.$$

But this is impossible.

3. For the same reason $\widehat{t}_1 \in C(\widehat{t}_{13}, \widehat{t}_2) \backslash C(\widehat{t}_{13}, \widehat{t}_2, \widehat{a}_2)$.

Therefore $Cdim(M_\Gamma) \geq 4$. This concludes the proof. □

## 4. Equations in one unknown

**Theorem 4.1.** *An equation*

$$g_1 x^{m_1} \cdots g_l x^{m_l} = 1, \; g_i \in S_\Gamma, \quad (4.1)$$

*is solvable in $S_\Gamma$ iff it is solvable in $M_\Gamma$.*

*Proof.* Let

$$\mu_\Gamma : g_j \longmapsto \widehat{g}_j = \begin{pmatrix} b_j & 0 \\ \tau_j + \widetilde{T}_\Gamma & 1 \end{pmatrix}, \; j = 1, \ldots, l, \; b_j \in A, \; \tau_j \in T.$$

Suppose that equation (4.1) is solvable in $M_\Gamma$ and

$$\widehat{x} = \begin{pmatrix} x & 0 \\ \tau & 1 \end{pmatrix}$$

is its solution.

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One can compute that the left-hand side of the equation is equal to

$$\begin{pmatrix} b_1\cdots b_l x^{m_1+\cdots+m_l} & 0 \\ \gamma & 1 \end{pmatrix},$$

where

$$\gamma = \tau_l x^{m_l} + \sum_{j=1}^{l-1} \tau_j b_{j+1}\cdots b_l x^{m_j+\cdots+m_l} +$$

$$\tau(\frac{x^{m_l}-1}{x-1} + \sum_{j=1}^{l-1} b_{j+1}\cdots b_l \frac{x^{m_j}-1}{x-1} x^{m_{j+1}+\cdots+m_l}) + \widetilde{T}_\Gamma,$$

and $\frac{x^m-1}{x-1} = m$ for $x = 1$.

Equation (4.1) is solvable in $M_\Gamma$ iff the system of equations

$$b_1\cdots b_l x^{m_1+\cdots+m_l} = 1 \ \bigwedge \ \gamma = 0 \tag{4.2}$$

is solvable with respect to $\tau \in T$ and $x \in A$.

Let us use the following notation

$$B = \frac{x^{m_l}-1}{x-1} + \sum_{j=1}^{l-1} b_{j+1}\cdots b_l \frac{x^{m_j}-1}{x-1} x^{m_{j+1}+\cdots+m_l}.$$

Since $d$ is a module homomorphism and $d(\tau_j) = b_j - 1$ we have

$$d(\gamma) = (b_l - 1)x^{m_l} + \sum_{j=1}^{l-1} (b_j - 1) b_{j+1}\cdots b_l x^{m_j+\cdots+m_l} + d(\tau)B.$$

Since $(x, \tau)$ is a solution of (4.2), we obtain

$$B(x-1) + (b_l - 1)x^{m_l} + \sum_{j=1}^{l-1} (b_j - 1) b_{j+1}\cdots b_l x^{m_j+\cdots+m_l} = 0.$$

We obviously have

$$Bd(\tau) + (b_l - 1)x^{m_l} + \sum_{j=1}^{l-1} (b_j - 1) b_{j+1}\cdots b_l x^{m_j+\cdots+m_l} = d(\gamma) = 0.$$

Consequently $B(x-1) = Bd(\tau)$.

If $B = 0$ then system (4.2) does not depend on $\tau$. Therefore for any $y \in T_\Gamma$ the matrix

$$\begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} \tag{4.3}$$

is a solution of (4.2). Evidently, for $x \in A$ one can find $y$ such that the matrix (4.3) is in $S_\Gamma$.

If $B \neq 0$ then $d(\tau) = x - 1$. This means that the matrix $\widehat{x}$ is $S_\Gamma$. This completes the proof. □

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