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ON THE DIMENSION OF THE PRODUCT $[L_2, L_2, L_1]$ IN FREE LIE ALGEBRAS

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ABSTRACT. Let L be a free Lie algebra of rank $r \geq 2$ over a field F and let L_n denote the degree n homogeneous component of L. By using the dimensions of the corresponding homogeneous and fine homogeneous components of the second derived ideal of free centre-by-metabelian Lie algebra over a field F, we determine the dimension of $[L_2, L_2, L_1]$. Moreover, by this method, we show that the dimension of $[L_2, L_2, L_1]$ over a field of characteristic 2 is different from the dimension over a field of characteristic other than 2.

1. Introduction

Let L be a free Lie algebra of rank $r \geq 2$ over a field F on $X = \{x_1, x_2, \ldots, x_r\}$ which is a set ordered by $x_1 < x_2 < \cdots < x_r$. We let L_n denote the degree n homogeneous component of L. Then

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_n \oplus \cdots .$$

In this paper we use the left normed convention for Lie brackets. Namely,

$$[a_1, a_2, \ldots, a_i] = [[a_1, a_2, \ldots, a_{i-1}], a_i].$$

The dimension of L_n is calculated by Witt's formula

$$\dim L_n = f(n, r) = \frac{1}{n} \sum_{d|n} \mu(d) r^{\frac{n}{d}},$$

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where μ is the Möbius function (see [9], [3, Theorem 5.11]). In [8] R. Stöhr and M. Vaughan-Lee investigated the dimension of subspaces of the form $[L_m, L_n]$ for all $m, n \geq 1$. In the author's master thesis [4] and in [6], the author and R. Stöhr obtained formulae for the dimensions of the subspaces of the form $[L_m, L_n, L_k] = [[L_m, L_n], L_k]$ for some $m, n, k \geq 1$. They showed that the dimension of $[L_2, L_2, L_1]$ over fields of characteristic 2 is different from the dimension over fields of characteristic other than 2 by using the theorem 4 in Kuz'min's pioneering paper [2].

In this paper, in conjunction with the result by the author and R. Stöhr (see [7, Theorem 6.1.],[5]) and the result by L.G. Kovács and R. Stöhr (see [1, Theorem 7.2.]), we investigate the dimension of $[L_2, L_2, L_1]$. By this approach, we prove that the dimension of $[L_2, L_2, L_1]$ depends on the characteristic of F.

2. Preliminary

We use some notation and some preliminary notions which were introduced in [5] and [7]. Let G denote the free centre-by-metabelian Lie algebra of finite rank r over a field F on a free generating set X, $|X| = r > 1$. This algebra is

$$G = L/[L'', L],$$

where L'' is the second derived ideal of L. The second derived ideal G'' of G is

$$G'' = L''/[L'', L].$$

Let G_n denote the degree n homogeneous component of G, that is spanned by Lie products of degree n in the free generators of G, and we write G''_n for the degree n homogeneous component of the second derived ideal:

$$G''_n = G'' \cap G_n.$$

It is easy to see that G''_4 is isomorphic to $G_2 \wedge G_2$, a free abelian group. For $n \geq 5$ there is a direct decomposition

$$G''_n = F_n \oplus T_n,$$

where T_n is an elementary abelian 2-group and F_n is a free abelian group (see [2, Theorem 4], [7, Theorem 7.2.]).

Let G_q denote the fine homogeneous component of G of multidegree q for a fixed composition $q = (q_1, q_2, \ldots, q_r)$ of n, that is the subspace of G generated by all Lie products of partial degree q_i with respect to x_i for $1 \leq i \leq r$. Each of the homogeneous components G_n can be written as a direct sum of fine homogeneous components,

$$G_n = \bigoplus_{q \models n} G_q.$$

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If all non-zero parts of q are equal to 1, a fine homogeneous component G_q of multidegree q is called multilinear. We define Kuz'min elements as Lie monomials of the form

$$[[y_1, y_2], [y_3, y_4, y_5, \dots, y_n]]$$

for all $y_i \in X$ with $i \in 1, 2, \dots, n$ such that

$$y_1 > y_2, y_3 > y_4, y_1 \geq y_3, y_4 \leq y_2 \leq y_5 \leq \cdots \leq y_n$$

and t-elements are defined as

$$\begin{aligned} w(y_1, y_2, y_3, y_4, y_5, \dots, y_n) &= [[y_1, y_2], [y_3, y_4, y_{5,n}]] \\ &+ [[y_2, y_3], [y_1, y_4, y_5, \dots, y_n]] + [[y_3, y_1], [y_2, y_4, y_5, \dots, y_n]] \end{aligned}$$

for all $y_i \in X$ with $i = 1, 2, \dots, n$.

Theorem 2.1. [7, Theorem 6.1] *Let G be the free centre-by-metabelian Lie algebra of rank $r > 1$ over a field F of characteristic other than 2. Then the dimensions of the homogeneous components and the fine homogeneous components of the second derived algebra G'' are as follows:*

(i) *If $n \geq 5$ is odd, then*

$$\dim(G''_n) = \frac{1}{2} r(n-3) \binom{n+r-3}{n-1}.$$

Moreover, if $q \models n$ is a composition of n in r parts such that k of the parts are non-zero and m of the parts are 1, then

$$\dim(G''_q) = \binom{k}{2} - m.$$

(ii) *If $n \geq 6$ is even, then*

$$\dim(G''_n) = \binom{n-1}{2} \binom{n+r-3}{n}.$$

Moreover, if $q \models n$ is a composition of n in r parts such that k of the parts are non-zero, then

$$\dim(G''_q) = \binom{k-1}{2}.$$

Theorem 2.2. [1, Theorem 7.2] *Let G be the free centre-by-metabelian Lie algebra of rank $r > 1$ over a field F of characteristic 2, and $q \models n$ with $n \geq 5$ be a composition of n in r parts such that k of the parts are non-zero and m of the parts are equal to 1.*

(i) *If $m = k$, that is, if q is multilinear, then*

$$\dim(G''_q) = \binom{k-1}{2}.$$

(ii) *If at least one of the parts of q is greater than 1, then*

$$\dim(G''_q) = \begin{cases} \binom{k-1}{2}, & \text{if all parts of } q \text{ are even} \\ \\ \binom{k}{2} - m, & \text{otherwise.} \end{cases}$$

3. Main Result

Let $q \models 5$ be a composition of 5 in r parts such that k of the parts are non-zero and m of the parts are 1. The homogeneous component G''_5 is the sum of the fine homogeneous components G''_q, namely,

$$G''_5 = \bigoplus_{q \models 5} G''_q.$$

Lemma 3.1. *Over any field F, let G''_5 be the degree 5 homogeneous component of the second derived ideal G''. Then*

$$\dim[L_2, L_2, L_1] = \dim[L_3, L_2] - \dim(G''_5).$$

Proof. Recall that the free centre-by-metabelian Lie algebra G is the quotient $L/[L'', L]$, where L'' is the second derived ideal of L. Then G is a graded algebra, and we denote its degree n homogeneous component by G_n. Here $G_n \cong L_n/(L_n \cap [L'', L])$. Moreover, the second derived ideal of G is the quotient $G'' = L''/[L'', L]$. As we have known, $G''_n = G'' \cap G_n$. We are interested in $G'' \cap G_5$.

The second derived ideal of L can be expressed as

$$[L_2, L_2] \oplus [L_3, L_2] \oplus ([L_4, L_2] + [L_3, L_3]) \oplus \cdots .$$

Hence, we have

$$\begin{aligned} [L'', L] &= [[L_2, L_2] \oplus [L_3, L_2] \oplus \cdots, L_1 \oplus L_2 \oplus \cdots] \\ &= [L_2, L_2, L_1] \oplus [L_3, L_2, L_1] \oplus \cdots . \end{aligned}$$

For degree 5, we have

$$\begin{aligned} G''_5 &= G_5 \cap G'' \\ &\cong (L_5/(L_5 \cap [L'', L]) \cap L''/[L'', L]) \\ &\cong (L_5 \cap L'')/(L_5 \cap [L'', L]). \end{aligned}$$

Since L'' has only the subspace $[L_3, L_2]$ and $[L'', L]$ has only the subspace $[L_2, L_2, L_1]$ for degree 5, we have $L_5 \cap L'' = [L_3, L_2]$ and $L_5 \cap [L'', L] = [L_2, L_2, L_1]$. Hence,

$$G''_5 \cong [L_3, L_2]/[L_2, L_2, L_1].$$

As a result, we obtain

$$\dim(G''_5) = \dim[L_3, L_2] - \dim[L_2, L_2, L_1]$$

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or

$$\dim[L_2, L_2, L_1] = \dim[L_3, L_2] - \dim(G''_5).$$

This completes the proof of the lemma. □

Now we are ready for our main result. The following theorem was obtained by the author and R. Stöhr in [6]. Here, we give an alternative proof.

Theorem 3.2. *Let L be the free Lie algebra of rank r over a field F. If $r \geq 5$, then the dimension of $[L_2, L_2, L_1]$ over a field of characteristic 2 is strictly less than the dimension of $[L_2, L_2, L_1]$ over a field of characteristic other than 2.*

Proof. Suppose that F is the field of characteristic other than 2. According to Theorem 2.1., we have

$$\dim(G''_q) = \binom{k}{2} - m.$$

If q is multilinear, namely, $m = k$,

$$\dim(G''_q) = \binom{k}{2} - k = \frac{1}{2}k(k-1) - k = \binom{k-1}{2} - 1.$$

Suppose that $\mathrm{char}F = 2$. According to Theorem 2.2., if q is multilinear, then

$$\dim(G''_q) = \binom{k-1}{2}.$$

If at least one of the parts of q is greater than 1, then

$$\dim(G''_q) = \binom{k}{2} - m.$$

We show the formulae of dimensions for G''_q in the following diagram:

	Char$F$=2	Char$F \neq 2$
q multilinear	$\binom{k\text{-}1}{2}$	$\binom{k\text{-}1}{2}$-1
q non-multilinear	$\binom{k}{2}$-m	$\binom{k}{2}$-m

By this diagram, it is easy to see that for q multilinear composition of 5, the dimension of G''_q over a field of characteristic 2 is more by 1 than the dimension of G''_q over a field of characteristic other than 2. Therefore, since the dimension of G''_5 is the sum of the dimensions of the fine homogeneous components G''_q. The dimension of G''_5 over a field of characteristic 2 is greater than the dimension of G''_q over a field of characteristic other than 2. By Lemma 3.1., we have

$$\dim[L_2, L_2, L_1] = \dim[L_3, L_2] - \dim(G''_5).$$

Therefore, it is clear that the dimension of $[L_2, L_2, L_1]$ over a field of characteristic 2 is strictly less than the dimension of $[L_2, L_2, L_1]$ over a field of characteristic other than 2. □

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