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GROUPS WITH PERMUTABILITY CONDITIONS FOR SUBGROUPS OF INFINITE RANK

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Abstract. In this paper, the structure of non-periodic generalized radical groups of infinite rank whose subgroups of infinite rank satisfy a suitable permutability condition is investigated.

1. Introduction

A subgroup H of a group G is said to be *permutable* (or quasinormal) if $HK = KH$ for every subgroup K of G. This concept was introduced by Ore [8]. It is clear that every normal subgroup of a group is permutable, but arbitrary permutable subgroups need not to be normal. A group G is called *quasihamiltonian* if every subgroup of G is permutable. The structure of quasihamiltonian groups is well known, we refer to [10] for a detailed account. In particular, it was proved by Iwasawa [6] that in a non-periodic quasihamiltonian group G the set of all elements of finite order T is a subgroup and the factor group G/T is abelian; moreover every subgroup of T is normal in G and, if G is not abelian, then G/T is locally cyclic.

A subgroup H of a group G is called *almost normal* if it has only finitely many conjugates in G or, equivalently, if H is normal in a subgroup of finite index of G, and H is said to be *nearly normal* if it has finite index in its normal closure H^G. In [7], B.H. Neumann has proved that a group has all its subgroups almost normal if and only if its centre has finite index and that every subgroup of a group is nearly normal if and only if its commutator subgroup is finite.

Corresponding properties, where normality is replaced by permutability, have been introduced in [5] and in [4]. More precisely, a subgroup H of a group G is called *almost permutable* if H is permutable in

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a subgroup of finite index of G, while H is called *nearly permutable* if it has finite index in a permutable subgroup of G. In [5], the authors considered groups in which every subgroup is almost permutable, while in [4] groups in which every subgroup is nearly permutable are studied. In those papers, for non-periodic groups, structure theorems of Iwasawa type are obtained for both properties.

In this paper, a property which generalizes almost permutability and nearly permutability is introduced. More precisely, we will say that a subgroup X of a group G is *finite-permutable-finite* if there exist subgroups H and K of G such that $X \leq H \leq K$, the indices $|H : X|$ and $|G : K|$ are finite and H is permutable in K.

A group G is said to have *finite (Prüfer) rank* r if every finitely generated subgroup of G can be generated by at most r elements and r is the least positive integer with such property; if such an r does not exist, we will say that the group G has *infinite rank*. In recent years, many authors have proved that in a (generalized) soluble group of infinite rank the behaviour of subgroups of infinite rank has an influence on the structure of the whole group (for example, see [2] for a survey article on this subject). In particular, in [1] it has been proved that in a (generalized) soluble group of infinite rank Neumann's results hold even restricting the hypothesis on subgroups of infinite rank.

Here, our aim is to investigate the structure of non-periodic groups in which every subgroup is finite-permutable-finite, having in mind Iwasawa's theorem for non-periodic quasihamiltonian groups. Moreover, with the same idea, we will consider non-periodic groups of infinite rank in which every subgroup of infinite rank is finite-permutable-finite.

Most of our notation is standard and can be found in [9].

2. Groups in which every subgroup is finite-permutable-finite

We say that a subgroup X of a group G is *finite-normal-finite* if there exist subgroups H and K of G such that $X \leq H \leq K$, the indices $|H : X|$ and $|G : K|$ are finite and H is normal in K; clearly, every almost normal subgroup and every nearly normal subgroup is finite-normal-finite.

Since every finite-normal-finite subgroup is obviously finite-permutable-finite, first we focus on groups in which every subgroup is finite-normal-finite.

Recall that an element x of a group G is said to be an FC-element of G if x has finitely many conjugates in G or, equivalently, if the centralizer $C_G(x)$ of x has finite index in G. The FC-centre of G is the subgroup of all FC-elements of G and G is called an FC-group if it coincides with its FC-centre.

Lemma 2.1. *Let G be a group and let x be any element of G. The subgroup $\langle x\rangle$ is finite-normal-finite in G if and only if x is an FC-element.*

Proof. If x is an FC-element, then $\langle x\rangle$ is trivially finite-normal-finite. So, assume that $\langle x\rangle$ is finite-normal-finite and let H and K be subgroups of G such that $|H : \langle x\rangle|$ and $|G : K|$ are finite and H is normal in K. Then $\langle x\rangle$ is nearly normal in K and so there exists a positive integer n such that the normal subgroup $(\langle x\rangle^K)^n$ of K is contained in $\langle x\rangle$. As $\langle x\rangle^K/(\langle x\rangle^K)^n$ is finite, we have that

$$K/C_K\left(\langle x\rangle^K/(\langle x\rangle^K)^n\right)$$

is finite and, in particular $N_K(x)$ has finite index in K. As the factor group $N_K(x)/C_K(x)$ is finite, it follows that the index $|K : C_K(x)|$ is finite. Thus, since K has finite index in G, the index $|G : C_G(x)|$ is finite and x is an FC-element of G. □

Corollary 2.2. *Let G be a group whose subgroups are finite-normal-finite. Then every subgroup of G is nearly normal in G.*

Proof. By Lemma 2.1, G is an FC-group. Let X be any subgroup of G and let H and K be subgroups of G such that $|H : X|$ and $|G : K|$ are finite and H is normal in K. Then H is almost normal in the FC-group G and so $|H^G : H|$ is finite ([11], Lemma 7.13). It follows that X has finite index in H^G and so X is nearly normal in G. □

Now we consider groups in which every subgroup is finite-permutable-finite and before proving the main theorem of this section we introduce some preliminary results.

Proposition 2.3. *Let G be a group whose cyclic subgroups are finite-permutable-finite. Then the set of all elements of finite order of G is a subgroup.*

Proof. Let T be the largest periodic normal subgroup of G. Clearly, every cyclic subgroup of G/T is finite-permutable-finite, so that replacing G with G/T it can be assumed without loss of generality that G has no periodic non-trivial normal subgroups. Let x be any element of finite order of G. As $\langle x\rangle$ is finite-permutable-finite, there exist H and K subgroups of G such that $|H : \langle x\rangle|$ and $|G : K|$ are finite and H is permutable in K. Then H is a finite permutable subgroup of K and, by Lemma 6.2.15 of [10], H^K is periodic. Let N be the core of K in G, then H^N is periodic and so $[H, N]$ is a subnormal periodic subgroup of G. Thus $[N, H] = 1$ and, in particular, $N \leq C_G(x)$. Then x is a periodic FC-element of G and, by Dietzmann's Lemma ([9], part 1, p.45), $\langle x\rangle^G$ is finite and, therefore, $x = 1$. Thus G is torsion-free and the proposition is proved. □

Proposition 2.4. *Let G be a group whose cyclic subgroups are finite-permutable-finite. If G contains two elements a and b of infinite order such that $\langle a\rangle \cap \langle b\rangle = \{1\}$, then G is an FC-group.*

Proof. Let x be any element of G. As $\langle x\rangle$ is finite-permutable-finite, there exist H and K subgroups of G such that $|H : \langle x\rangle|$ and $|G : K|$ are finite and H is permutable in K. Let y be any element of K of infinite order and suppose, first, that $\langle x\rangle \cap \langle y\rangle = \{1\}$. Then $H \cap \langle y\rangle = \{1\}$ and hence $H^y = H$ ([10], Lemma 6.2.3). Suppose now that $\langle x\rangle \cap \langle y\rangle \neq \{1\}$, so that in particular x has infinite order and there exists an element z of K of infinite order such that

$$\langle x\rangle \cap \langle z\rangle = \langle y\rangle \cap \langle z\rangle = \{1\}.$$

Thus $H^z = H$. Now, as $\langle y\rangle$ is finite-permutable-finite, there exist M and L subgroups of G such that $|M : \langle y\rangle|$ and $|G : L|$ are finite and M is permutable in L. Let k be a positive integer such that z^k is in L. Then $M^{z^k} = M$ and M is normal in $M\langle yz^k\rangle = M\langle z^k\rangle$, so that yz^k must have infinite order and $M \cap \langle yz^k\rangle = \{1\}$. Since $\langle y\rangle \cap \langle yz^k\rangle = \{1\}$, we have also $\langle x\rangle \cap \langle yz^k\rangle = \{1\}$, so that yz^k normalizes H and $H^y = H$. Therefore H is normalized by any element of infinite order of K and, as K is generated by its elements of infinite order, H is normal in K. Thus $\langle x\rangle$ is finite-normal-finite in G and, hence x is an FC-element of G by Lemma 2.1. Therefore, G is an FC-group. □

Theorem 2.5. *Let G be a non-periodic group whose subgroups are finite-permutable-finite. Then:*

(a) *The set T of all elements of finite order of G is a normal subgroup and the factor group G/T is abelian.*
(b) *Every subgroup of T is finite-normal-finite in G.*
(c) *Either G is an FC-group or the group G/T is locally cyclic.*

Proof. (a) By Proposition 2.3, T is a subgroup of G. In order to prove that G/T is abelian, we may assume that G is torsion-free. Let x be any element of G. As $\langle x\rangle$ is finite-permutable-finite, there exist H and K subgroups of G such that $|H : \langle x\rangle|$ and $|G : K|$ are finite and H is permutable in K. Since H is a cyclic-by-finite torsion-free group, H is cyclic. Put $H = \langle h\rangle$, so that $\langle x\rangle = \langle h^n\rangle$ for some positive integer n. By a contradiction, assume that H is not normal in K, so that there exists an element y of K such that $H^y \neq H$. Let $L = \langle h\rangle\langle y\rangle$. By Lemma 6.2.3 of [10], $M = \langle h\rangle \cap \langle y\rangle$ is a non-trivial subgroup of L contained in $Z(L)$. Therefore L/M is finite so that also the commutator subgroup L' of L is finite, and so L is abelian. This contradiction proves that $\langle h\rangle$ is normal in K and so also $\langle x\rangle$ is normal in K. Therefore all cyclic subgroups of G are almost normal and G is an FC-group. As G is torsion-free, it follows that G is abelian.

(b) Let X be any subgroup of T. As X is finite-permutable-finite, there exist H and K subgroups of G such that $|H : X|$ and $|G : K|$ are finite and H is permutable in K. In particular, H is periodic. Let a be any element of infinite order of K, so that $\langle a\rangle \cap T = \{1\}$. Then $H = H\langle a\rangle \cap T$ is a normal subgroup of $H\langle a\rangle$. It follows that H is normal in K.

(c) This part follows directly from Proposition 2.4. □

Corollary 2.6. *Let G be a non-periodic group whose subgroups are finite-permutable-finite and let T be the torsion subgroup of G. Then every subgroup of T is nearly normal in G.*

Proof. By Theorem 2.5, every subgroup of T is finite-normal-finite in G and so by Corollary 2.2, T' is finite. Without loss of generality, we may replace G with G/T', so that we may assume that T is abelian. Let X be a finite subgroup of T and let H and K be subgroups of G such that $|H : X|$ and $|G : K|$ are finite and H is normal in K. Then H and $|G : N_G(H)|$ are finite, so that H^G is finite ([9], part 1, p. 45). In particular, X^G is finite.

Let H be a periodic subgroup such that $N = N_G(H)$ has finite index in G. If G is an FC-group, then H is nearly normal in G ([11], Lemma 7.13). Suppose that G/T is locally cyclic. Put $G = EN$, where E is finitely generated. If $E \leq T$, $G = N$ and H is normal in G. Suppose that ET is a non-periodic group and, without loss of generality, we may assume that $G = ET$. Then $G = \langle a\rangle \ltimes T$, where a is an element of infinite order of G. Let A and B be subgroups of G such that $|A:\langle a\rangle|$ and $|G:B|$ are finite and A is permutable in B. As $A\cap T$ is finite, $(A\cap T)^G$ is also finite and, replacing G with $G/(A\cap T)^G$, we can assume that $\langle a\rangle$ is permutable in B. Since $B\cap T$ is a subgroup of finite index of T, $T=(T\cap B)X$, where X is a normal finite subgroup of G. Therefore $G=BT=BX$ and $\langle a\rangle X/X$ is permutable in G/X. Since $\langle a\rangle X\cap T=X$, every subgroup of T/X is G-invariant ([3], Lemma 2.5). Thus, HX is normal in G and the index $|HX:H|$ is finite.

Since every subgroup X of T has finite index in a subgroup H which is almost normal in G, it follows that X is nearly normal in G. □

3. **Groups in which every subgroup of infinite rank is finite-permutable-finite**

This section is devoted to the study of the structure of generalized radical groups of infinite rank in which every subgroup of infinite rank is finite-permutable-finite. In the proof of the main theorem, we will need the following lemmas.

Lemma 3.1. *Let G be a periodic group whose subgroups are finite-permutable-finite. Then G is locally finite.*

Proof. Let E be any finitely generated subgroup of G. Since any subgroup of E is finite-permutable-finite, we may assume without loss of generality that G is finitely generated. Let x be any element of G, then there exist H and K subgroups of G such that $|H:\langle x\rangle|$ and $|G:K|$ are finite and H is permutable in K. It follows that H is finite and K is finitely generated and, hence, by Theorem 6.2.18 of [10], H^K is finite. In particular, x has finitely many conjugates in K and so also in G. Therefore, G is an FC-group and hence it is finite. □

Recall that an element x of a group G has finite order modulo a permutable subgroup H of G if the index $|H\langle x\rangle:H|$ is finite; otherwise x is said to have infinite order modulo H.

Lemma 3.2. *Let G be a group and let X be a permutable subgroup of G such that any subgroup of G containing X is finite-permutable-finite. If there exists an element of G having infinite order modulo X, then X is normal in G.*

Proof. Let x and y be elements of G of finite order modulo X and let $L=X\langle x,y\rangle$. The factor group L/X^L has all its subgroups finite-permutable-finite and it is generated by two periodic elements. As a consequence, L/X^L is periodic and so it is also finite by Lemma 3.1. It follows that, since the index $|X^L:X|$ is finite ([10], Theorem 6.2.18), X has finite index in L, so that also the product xy has finite order modulo X. Thus the set T of all elements of G having finite order modulo X is a proper subgroup and G is generated by $G\setminus T$. On the other hand, any element of $G\setminus T$ normalizes X ([10], Lemma 6.2.3) and the lemma is proved. □

Here, we highlight the following argument, as it will be frequently used in the next proofs. It is known that in generalized soluble groups of infinite rank the existence of abelian subgroups of infinite rank plays a crucial role.

Let G be a group of infinite rank whose subgroups of infinite rank are finite-permutable-finite. Let $A = A_1 \times A_2$ be an abelian subgroup of G, where both factors A_1 and A_2 have infinite rank. By hypothesis, A_1 and A_2 are finite-permutable-finite, so there exist subgroups H_i and K_i of G such that the indices $|H_i : A_i|$, $|G : K_i|$ are finite and H_i is permutable in K_i, for $i = 1, 2$. It follows that $N = (K_1 \cap K_2)_G$ is a normal subgroup of finite index of G, $H_i \cap N$ is a permutable subgroup of N and $|H_i \cap N : A_i \cap N|$ is finite, for $i = 1, 2$. Then, replacing A with $\bar{A} = (A_1 \cap N) \times (A_2 \cap N)$, we will always assume that H_1 and H_2 are both permutable in the same normal subgroup N of finite index of G. If G is a group of infinite rank whose subgroups of infinite rank are finite-normal-finite, then the same argument can be used, simply replacing permutability with normality.

Next proposition shows that, restricting the hypotheses to the subgroups of infinite rank, it is possible to obtain a result similar to Corollary 2.2.

Proposition 3.3. *Let G be a generalized radical group of infinite rank in which all subgroups of infinite rank are finite-normal-finite. Then every subgroup of G is nearly normal.*

Proof. Let $A = A_1 \times A_2$ be an abelian subgroup of G, with A_1 and A_2 of infinite rank. Then there exists a normal subgroup N of finite index of G such that A_i has finite index in a N-invariant subgroup H_i of N, for $i = 1, 2$. Every subgroup of N/H_i is finite-normal-finite and so, by Corollary 2.2, $N'H_i/H_i$ is finite, for $i = 1, 2$. Since $H_1 \cap H_2$ is finite, N' is finite and, replacing G with G/N', we may assume that N is abelian. Therefore, N contains a direct product $Y_1 \times Y_2$ of G-invariant subgroups of infinite rank Y_1 and Y_2 ([1], Lemma 6) and it follows that $G'Y_i/Y_i$ is finite, for $i = 1, 2$. Hence, also G' is finite and every subgroup of G is nearly normal. □

For the convenience of the reader, we state as a Lemma the results of [5] and [4] concerning non-periodic groups in which either every subgroup is almost permutable or every subgroup is nearly permutable.

Lemma 3.4 ([5],[4])**.** *Let G be a non-periodic group in which every subgroup is almost permutable (resp. nearly permutable). Then the set of all elements of finite order T of G is a normal subgroup of G and the factor group G/T is abelian; moreover, every subgroup of T is almost normal (resp. nearly normal) in G and either G is an FC-group or G/T is locally cyclic.*

Theorem 3.5. *Let G be a non-periodic generalized radical group of infinite rank whose subgroups of infinite rank are finite-permutable-finite. Then:*

(a) *The set T of all elements of finite order of G is a normal subgroup of G and the factor group G/T is abelian.*

(b) *Every subgroup of T is finite-normal-finite in G and either G is an FC-group or G/T is locally cyclic.*

Proof. (a) Suppose that G has no non-trivial periodic normal subgroups. Let x be an element of infinite order of G and let $A = A_1 \times A_2$ be an abelian subgroup of G such that A_1 and A_2 have both infinite rank and $A \cap \langle x \rangle = \{1\}$. Then there exists a normal subgroup N of finite index of G such that A_i has finite index in a subgroup H_i which is a permutable subgroup of N, for $i = 1, 2$. Put $\langle y \rangle = \langle x \rangle \cap N$, then $\langle y \rangle \cap H_1 = \langle y \rangle \cap H_2 = \{1\}$ and so, by Lemma 3.2, H_1 and H_2 are normal subgroups of N. Therefore every subgroup of N/H_i is finite-permutable-finite and hence, by Theorem 2.5, $N'H_i/H_i$ is periodic, for $i = 1, 2$. Since $H_1 \cap H_2$ is finite, N' is periodic and $N' = \{1\}$. Thus N contains two G-invariant subgroups Y_1 and Y_2 of infinite rank such that $Y_1 \cap Y_2 = \{1\}$ ([1], Lemma 6). Since G embeds in the direct product $G/Y_1 \times G/Y_2$, it follows by Theorem 2.5 that G' is periodic. Thus $G' = \{1\}$ and G is a torsion-free abelian group.

Now, let T be the largest normal periodic subgroup of G. If T has finite rank, then G/T has infinite rank and so, by the previous argument, G/T is a torsion-free abelian group. On the other hand, if T has infinite rank, then G/T is a non-periodic group whose subgroups are finite-permutable-finite and then, by Theorem 2.5, G/T is a torsion-free abelian group. In both cases, T coincides with the set of all periodic elements of G and G/T is abelian.

(b) Assume that T has finite rank, so that G has infinite torsion-free rank. Let $A = A_1 \times A_2$ be an abelian subgroup of G, with A_1 and A_2 of infinite rank, such that $A \cap T = \{1\}$. Then there exists a normal subgroup N of finite index of G such that A_i has finite index in a subgroup H_i which is a permutable subgroup of N, for $i = 1, 2$. Let x be an element of A_2; as $|H_1 : A_1|$ is finite, $H_1 \cap \langle x \rangle = \{1\}$. Therefore, by Lemma 3.2, H_1 is normal in N. Similarly, H_2 is a normal subgroup of N. Thus, N/H_i is a non-periodic group with infinite torsion-free rank in which every subgroup is finite-permutable-finite and, by Theorem 2.5, N/H_i is an FC-group, for $i = 1, 2$. It follows that N is finite-by-FC and so it is an FC-group. Since $G/Z(N)$ is a periodic group, $Z(N)$ is a G-invariant subgroup with infinite torsion-free rank. Let X be a subgroup of T, so $Z(N)$ contains a torsion-free subgroup $Y = Y_1 \times Y_2$, with Y_1 and Y_2 G-invariant of infinite rank, such that $X \cap Y = \{1\}$ ([1], Lemma 6). Thus, G/Y_i is an FC-group and XY_i is finite-normal-finite in G, for $i = 1, 2$. It follows that G is an FC-group and $X = XY_1 \cap XY_2$ is finite-normal-finite in G.

Assume that T has infinite rank and let X be any subgroup of T of infinite rank. Then there exist subgroups H and K such that $|H : X|$ and $|G : K|$ are finite and H is permutable in K. In particular, H is periodic. Let a be an element of infinite order of K, then $H = H\langle a \rangle \cap T$ is a normal subgroup of $H\langle a \rangle$. It follows that H is normal in K and X is finite-normal-finite in G. By Proposition 3.3, T' is finite. Without loss of generality, we may replace G with G/T' and assume that T is abelian. Let X be a subgroup of T of finite rank and let $A = A_1 \times A_2$ be an abelian subgroup of T, with A_1 and A_2 of infinite rank, such that $A \cap X = \{1\}$. Then XA_i is finite-normal-finite in G, for $i = 1, 2$, and $X = XA_1 \cap XA_2$ is finite-normal-finite in G. Let N be a normal subgroup of finite index of G such that A_i has finite index in a normal subgroup H_i of N, for $i = 1, 2$.

Suppose that G/T is not locally cyclic. Then, by Theorem 2.5, N/H_i is an FC-group, for $i = 1, 2$, and so N is an FC-group. It follows that G is FC-by-finite and, in particular, G satisfies locally the maximal condition on subgroups.

In order to prove that G is an FC-group, we first show that any subgroup of T is nearly normal in G. It is enough to show that if a periodic subgroup is almost normal in G, then it is also nearly normal in G. So, let $H \leq T$ be an almost normal subgroup of G. Then, $N = N_G(H)$ has finite index in G and $G = EN$, where E is finitely generated. Put $L = ET$, so $H^G = H^L$. Since G is FC-by-finite, the FC-centre F of L has finite index in L. Moreover, as every subgroup of T is finite-normal-finite in G, $T \leq F$ and $F = (F \cap E)T$, where $F \cap E$ is finitely generated. As $F/Z(F)$ is locally finite, it follows that the G-invariant abelian subgroup $A = TZ(F)$ has finite index in F and hence also in L. Put $L = AM$, where M is finitely generated, and $A \cap M$ is a normal, finitely generated abelian subgroup of L. Then, $L/(A \cap M)$ is a group of infinite rank whose subgroups of infinite rank are finite-permutable-finite and consequently its periodic subgroups are finite-normal-finite. Hence $M^L/(A \cap M)$ is finite and M^L is finitely generated. Replacing M with M^L, we may assume that M is normal in L. Therefore, $[H, M]$ is a finitely generated subgroup of L' and, by (a), L' is contained in T. Therefore $[H, M]$ is finite and H has finite index in $H[H, M] = H^M = H^L$.

Let $A = A_1 \times A_2$ be an abelian subgroup of T, with A_1 and A_2 of infinite rank. Then G/A_i^G is an FC-group and, since $|A_i^G : A_i|$ is finite for $i = 1, 2$, G is finite-by-FC and G is an FC-group. □

Now we will apply Theorem 3.5 to the study of groups of infinite rank in which either every subgroup of infinite rank is almost permutable or every subgroup of infinite rank is nearly permutable.

Corollary 3.6. *Let G be a non-periodic generalized radical group of infinite rank whose subgroups of infinite rank are almost permutable. Then*

(a) *The set T of all elements of finite order of G is a normal subgroup of G and the factor group G/T is abelian.*

(b) *Every subgroup of T is almost normal in G.*

(c) *Either G is an FC-group or G/T is locally cyclic.*

Proof. (a) and (c) follow directly from Theorem 3.5.

(b) If T has finite rank and $X \leq T$, take Y_1 and Y_2 as in the proof of (b) in Theorem 3.5. Then, by Lemma 3.4, XY_i is almost normal in G and also $X = XY_1 \cap XY_2$ is almost normal in G.

If T has infinite rank, let X be a subgroup of T of infinite rank and let K be a subgroup of finite index of G such that X is permutable in K. Then the same argument used in the proof of Theorem 3.5 shows that X is normal in K. Therefore, every subgroup of infinite rank of T is almost normal in G and $T/Z(T)$ is finite by Theorem A of [1]. Now, let X be any subgroup of finite rank of T and let $A = A_1 \times A_2$ be a subgroup of $Z(T)$ with A_1 and A_2 of infinite rank and $X \cap A = \{1\}$. Then $X = XA_1 \cap XA_2$ is almost normal in G. □

Corollary 3.7. *Let G be a non-periodic generalized radical group of infinite rank whose subgroups of infinite rank are nearly permutable. Then*

(a) *The set T of all elements of finite order of G is a normal subgroup of G and the factor group G/T is abelian.*

(b) *Every subgroup of T is nearly normal in G.*

(c) *Either G is an FC-group or G/T is locally cyclic.*

Proof. (a) and (c) follow directly from Theorem 3.5.

(b) If T has finite rank and $X \leq T$, take Y_1 and Y_2 as in the proof of (b) in Theorem 3.5. Then, by Lemma 3.4, XY_i is nearly normal in G and also $X = XY_1 \cap XY_2$ is nearly normal in G.

If T has infinite rank, let X be a subgroup of T of infinite rank and let K be a permutable subgroup of G such that X has finite index in K. Then the same argument used in the proof of Theorem 3.5 shows that K is normal in G. Therefore, every subgroup of infinite rank of T is nearly normal in G and T' is finite by Theorem B of [1]. We can now replace G with G/T' and assume that T is abelian. Thus, if X is any subgroup of finite rank of T and $A = A_1 \times A_2$ is a subgroup of T, with A_1 and A_2 of infinite rank and $X \cap A = \{1\}$, then $X = XA_1 \cap XA_2$ is nearly normal in G. □

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