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FINITE GROUPS OF THE SAME TYPE AS SUZUKI GROUPS

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Abstract. For a finite group G and a positive integer n, let $G(n)$ be the set of all elements in G such that $x^n = 1$. The groups G and H are said to be of the same (order) type if $|G(n)| = |H(n)|$, for all n. The main aim of this paper is to show that if G is a finite group of the same type as Suzuki groups $\mathrm{Sz}(q)$, where $q = 2^{2m+1} \geq 8$, then G is isomorphic to $\mathrm{Sz}(q)$. This addresses to the well-known J. G. Thompson's problem (1987) for simple groups.

1. Introduction

For a finite group G and a positive integer n, let $G(n)$ consist of all elements x satisfying $x^n = 1$. The *order type* of G is defined to be the function whose value at n is the order of $G(n)$. In 1987, J. G. Thompson [11, Problem 12.37] possed a problem which is related to algebraic number fields:

Is it true that a group is solvable if its type is the same as that of a solvable one?

This problem links to the set $\mathsf{nse}(G)$ of *the number of elements of the same order* in G. Indeed, it turns out that if two groups G and H are of the same type, then $\mathsf{nse}(G) = \mathsf{nse}(H)$ and $|G| = |H|$. Therefore, if a group G has been uniquely determined by its order and $\mathsf{nse}(G)$, then Thompson's problem is true. One may ask this problem for non-solvable groups, in particular, finite simple groups. In this direction, Shao et al [13] studied finite simple groups whose order is divisible by at most four primes. Following this investigation, such problem has been studied for some families of simple groups [1, 2, 3, 13] including Suzuki groups $\mathrm{Sz}(q)$ with q prime [10]. In this paper, we generalize the main result in [10] and prove that

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Theorem 1.1. *Let G be a group with $\mathsf{nse}(G) = \mathsf{nse}(\mathrm{Sz}(q))$ and $|G| = |\mathrm{Sz}(q)|$. Then G is isomorphic to $\mathrm{Sz}(q)$.*

As noted above, as an immediate consequence of Theorem 1.1, we have that

Corollary 1.2. *If G is a finite group of the same type as $\mathrm{Sz}(q)$, then G is isomorphic to $\mathrm{Sz}(q)$.*

In order to prove Theorem 1.1, we use a partition of Suzuki groups $S := \mathrm{Sz}(q)$, where $q = 2^{2m+1} \geq 8$, see Lemma 3.2, that is to say, a set of subgroups H_i of S, for $i = 1, \ldots, s$, such that each nontrivial element of S belongs to exactly one subgroup H_i. We use this information to determine the set $\mathsf{nse}(S)$ in Proposition 3.3. It is also a main tool to show that 2 is an isolated vertex in the prime graph of a group G satisfying hypotheses of Theorem 1.1, see Proposition 4.1. Then we show that G is neither Frobenius, nor 2-Frobenius group. Finally, we obtain a section of G which is isomorphic to S and prove that G is isomorphic to S.

Finally, we give some brief comments on the notation used in this paper. Throughout this article all groups are finite. Our group-theoretic notation is standard, and it is consistent with the notation in [4, 6, 7]. We denote a Sylow p-subgroup of G by G_p. We also use $\mathsf{n}_p(G)$ to denote the number of Sylow p-subgroups of G. For a positive integer n, the set of prime divisors of n is denoted by $\pi(n)$, and if G is a finite group, $\pi(G) := \pi(|G|)$, where $|G|$ is the order of G. We denote the set of elements' orders of G by $\omega(G)$ known as *spectrum* of G. The *prime graph* $\Gamma(G)$ of a finite group G is a graph whose vertex set is $\pi(G)$, and two vertices p and q are adjacent if and only if $pq \in \omega(G)$. Assume further that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \ldots, t(G)$. The positive integers n_i with $\pi(n_i) = \pi_i$ are called order components of G. Clearly, $|G| = n_1 \cdots n_{t(G)}$. In the case where G is of even order, we always assume that $2 \in \pi_1$, and π_1 is said to be the even component of G. In this way, π_i and n_i are called odd components and odd order components of G, respectively. Recall that $\mathsf{nse}(G)$ is the set of the number of elements in G with the same order. In other word, $\mathsf{nse}(G)$ consists of the numbers $\mathsf{m}_i(G)$ of elements of order i in G, for $i \in \omega(G)$. Here, ϕ is the *Euler totient* function.

2. Preliminaries

In this section, we introduce the some known results which will be used in the proof of the main result.

Lemma 2.1. [8, Theorem 9.1.2] *Let G be a finite group, and let n be a positive integer dividing $|G|$. Then n divides $|G(n)|$.*

The proof of the following result is straightforward by Lemma 2.1. Recall that $\mathsf{nse}(G) = \{\mathsf{m}_i(G) \mid i \in \omega(G)\}$.

Lemma 2.2. *Let G be a finite group. Then for every $i \in \omega(G)$, $\phi(i)$ divides $\mathsf{m}_i(G)$, and i divides $\sum_{j|i} \mathsf{m}_j(G)$. Moreover, if $i > 2$, then $\mathsf{m}_i(G)$ is even.*

Lemma 2.3 (Theorem 3 in [15]). *Let G be a finite group of order n. Then the number of elements whose orders are multiples of t is either zero, or a multiple of the greatest divisor of n that is prime to t.*

In what follows, recall that $t(G)$ is the number of connected components of the prime graph $\Gamma(G)$.

Lemma 2.4. [5, Theorem 1] *Let G be a Frobenius group of even order with kernel K and complement H. Then $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$.*

A group G is called 2-Frobenius if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernel K/H and H respectively.

Lemma 2.5. [5, Theorem 2] *Let G be a 2-Frobenius group of even order. Then $t(G) = 2$, $\pi(G/K) \cup \pi(H) = \pi_1$, $\pi(K/H) = \pi_2$, and G/K and K/H are cyclic groups and $|G/K|$ divides $|\mathsf{Aut}(K/H)|$.*

3. Elements of the same order in Suzuki groups

In this section, we determine the set of the number of elements of the same order in Suzuki groups.

Lemma 3.1 ([14])**.** *Let $S = \mathrm{Sz}(q)$ with $q = 2^{2m+1} \geq 8$. Then $\omega(S)$ consists of all factors of 4, $q-1$ and $q \pm \sqrt{2q} + 1$.*

Let G be a group, and let $H_1, \dots, H_t$ be subgroups of G. Then the set $\{H_1, \dots, H_t\}$ forms a partition of G if each non-trivial element of G belongs to exactly one subgroup H_i of G. Lemma 3.2 below introduces a partition of Suzuki groups.

Lemma 3.2. *Let $S = \mathrm{Sz}(q)$ with $q = 2^{2m+1} \geq 8$, and let $\mathbb{F} := \mathsf{GF}(q)$. Then*

- (a) *S possesses cyclic subgroups U_1 and U_2 of orders $q + \sqrt{2q} + 1$ and $q - \sqrt{2q} + 1$, respectively;*
- (b) *if $1 \neq u \in U_i$, for $i = 1, 2$, then ${}_S(u) = U_i$. Moreover, $|\boldsymbol{N}_S(U_i) : U_i| = 4$;*
- (c) *S possesses a cyclic subgroup V of order $q-1$ and $|\boldsymbol{N}_S(V) : V| = 2$;*
- (d) *S possesses a 2-subgroup W of orders q^2 and exponent 4 and $|S : \boldsymbol{N}_S(W)| = q^2 + 1$. Moreover, the elements of W are of the form*

$$w(a,b) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & a\pi & 1 & 0 \\ a^2(a\pi) + ab + b\pi & a(a\pi) + b & a & 1 \end{pmatrix}, \tag{3.1}$$

  where $a, b \in \mathbb{F}$ and $\pi \in \mathsf{Aut}(\mathbb{F})$ maps x to $x^{2^{m+1}}$, for all $x \in \mathbb{F}$.
- (e) *the conjugates of U_1, U_2, V and W form a partition of S.*

Proof. All parts of this result follow from Lemma 3.1 and Theorem 3.10 in [9] except for the facts that $|\mathbf{N}_S(V) : V| = 2$ and $|S : \mathbf{N}_S(W)| = q^2 + 1$ which can be found in the proof of Theorem 3.10 in [9]. □

Proposition 3.3. *Let $S = \mathrm{Sz}(q)$ with $q = 2^{2m+1} \geq 8$. Then the set $\mathsf{nse}(S)$ consists of exactly one the following numbers*

- (a) *1, $(q-1)(q^2+1)$, $q(q-1)(q^2+1)$;*
- (b) *$\phi(i)q^2(q \mp \sqrt{2q} + 1)(q-1)/4$, where $i > 1$ divides $q \pm \sqrt{2q} + 1$;*

(c) $\phi(i)q^2(q^2+1)/2$, *where* $i>1$ *divides* $q-1$.

Proof. Suppose that $i \in \omega(S)$ is an even number. Then by Lemma 3.1, we have that $i = 2$ or $i = 4$ and i divides the order of subgroup W as in Lemma 3.2(d). Then each element of W is of the form $w(a, b)$ as in (3.1). Obviously,

$$w(a, b)w(c, d) = w(a + c, b + d + (a\pi)c).$$

This in particular shows that $w(0, b)$ (with $b \neq 0$) are the only elements of W of order 2. Therefore, the number of involutions in W is $q-1$. Since W is a part of the partition introduced in Lemma 3.2(e), the elements of order 2 of S belong to exactly one of the conjugates of W. Thus by Lemma 3.2(d), there are q^2+1 conjugates of W implying that there are exactly $\mathsf{m}_2(S) = (q-1)(q^2+1)$ involutions in S. It also follows from Lemma 3.2(d) that the number of elements of order 4 in W is q^2-q, and hence applying the partition in Lemma 3.2(e), we conclude that S consists of $\mathsf{m}_4(S) = q(q-1)(q^2+1)$ elements of order 4. This proves part (a).

Suppose now $i \in \omega(S)$ is an odd number. Then, by Lemma 3.1, i divides the order of one the cyclic subgroups U_1, U_2 and V as in Lemma 3.2, say H. Assume $i = np^{\alpha}$ with p odd. Since H is a part of the partition introduced in Lemma 3.2(e), the elements of order i are contained in H and its conjugates. Since also H is cyclic, there are $\phi(i)$ elements of order i in each conjugates of H including H.

Now we consider each possibility of H. If $H = U_t$ of order $q \pm \sqrt{2q}+1$, for $t = 1, 2$, then by Lemma 3.2(b), $|N_S(U_t) : U_t| = 4$, and so $|S : N_S(U_t)| = |S|/4|U_t|$, for $t = 1, 2$. So there are $|S|/4|U_t|$ conjugates of U_t implying that there are exactly $m_i(S) = \phi(i)q^2(q-1)(q \mp \sqrt{2q}+1)/4$ elements of order i in S. This follows part (b). If $H = V$, then Lemma 3.2(c) implies that $|N_S(V) : V| = 2$, and so the same argument as in the previous cases, we conclude that $m_i(S) = \phi(i)|S|/2|V| = \phi(i)q^2(q^2+1)/2$. This follows (c). □

4. Proof of the Main Theorem

In this section, we prove Theorem 1.1. From now on, set $S := \mathrm{Sz}(q)$, where $q = 2^{2m+1} \geq 8$, and recall that G is a finite group with $\mathsf{nse}(G) = \mathsf{nse}(S)$ and $|G| = |S|$. Therefore, by Proposition 3.3, $\mathsf{nse}(S)$ consists of

$$\begin{aligned}
\mathsf{m}_1(S) &= 1;\\
\mathsf{m}_2(S) &= (q-1)(q^2+1);\\
\mathsf{m}_4(S) &= q(q-1)(q^2+1);\\
\mathsf{m}_i(S) &= \phi(i)q^2(q \mp \sqrt{2q}+1)(q-1)/4, \text{ where } i>1 \text{ divides } q \pm \sqrt{2q}+1;\\
\mathsf{m}_i(S) &= \phi(i)q^2(q^2+1)/2, \text{ where } i>1 \text{ divides } q-1.
\end{aligned} \tag{4.1}$$

Proposition 4.1. *The vertex* 2 *is an isolated vertex in* $\Gamma(G)$.

Proof. Assume the contrary. Then there is an odd prime divisor p of $|G|$ such that $2p \in \omega(G)$. Let $f(n)$ be the number of elements of G whose orders are multiples of n. Then by Lemma 2.3, $f(2)$ is a multiple of the greatest divisor of $|G|$ that is prime to 2. Since $(q^2+1)(q-1)$ is the greatest divisor of $|G|$ which is coprime to 2, there exists a positive integer r such that $f(2) = (q^2+1)(q-1)r$ and $(r, 2) = 1$. On the other hand, by Lemma 2.2, it is obvious that $\mathsf{m}_2(G) = \mathsf{m}_2(S)$, and so

$$f(2) = \mathsf{m}_2(G) + \sum_{i>2 \ is \ even} \mathsf{m}_i(G),$$

with m_i as in (4.1). Now applying Proposition 3.3, there is a non-negative integer α such that

$$f(2) = (q^2+1)(q-1) + \alpha q(q^2+1)(q-1) + g(2),$$

where

$$g(2) = \sum_{\substack{i \mid q \pm \sqrt{2q}+1 \\ i \neq 1}} \beta_i \cdot \mathsf{m}_i(S) + \sum_{\substack{i \mid q-1 \\ i \neq 1}} \gamma_i \cdot \mathsf{m}_i(S)$$

for some non-negative integers β_i and γ_i. Since $2p \in \omega(G)$, we have that $\alpha q(q^2+1)(q-1) + g(2) > 0$. Then

$$g(2) = (q^2+1)(q-1)(r-1-\alpha q).$$

We now prove that q^2 divides $g(2)$. It follows from Lemma 3.1 that 2 is an isolated vertex of $\Gamma(S)$. Then a Sylow 2-subgroup of S, say S_2, acts fixed point freely (by conjugation) on the set of elements of order $i \neq 1, 2, 4$ (see Proposition 3.3 and (4.1)). Thus $|S_2|$ divides $\mathsf{m}_i(S)$ with $i \neq 1, 2, 4$. Hence q^2 divides $\mathsf{m}_i(S)$ implying that $g(2)$ is a multiple of q^2.

We now consider the following two cases:

(1) Let $g(2) \neq 0$. Then q^2 divides $r - 1 - \alpha q$, and so $q^2 + \alpha q + 1 \leq r$. This implies that $|G| = q^2(q^2+1)(q-1) < (q^2+1)(q-1)r = f(2)$, which is impossible.

(2) Let $g(2) = 0$. Then $r - 1 - \alpha q = 0$ and $\alpha \neq 0$, and so $\mathsf{m}_{2p}(G) = q(q^2+1)(q-1)$. Therefore

$$f(p) = \sum_{p \mid i} \mathsf{m}_i(G) = \alpha' q(q^2+1)(q-1) + \sum_{\substack{i \mid q \pm \sqrt{2q}+1 \\ i \neq 1}} \beta'_i \cdot \mathsf{m}_i(S) + \sum_{\substack{i \mid q-1 \\ i \neq 1}} \gamma'_i \cdot \mathsf{m}_i(S),$$

where α', β'_i and γ'_i are non-negative integers. Since $\mathsf{m}_{2p} = q(q^2+1)(q-1)$, we have that $\alpha' > 0$. On the other hand, by Lemma 2.3, $f(p) = q^2(q^2+1)(q-1)r'/|G_p|$ with r' a positive integer. Thus

$$\frac{q^2(q^2+1)(q-1)r'}{|G_p|} = \alpha' q(q^2+1)(q-1) + \sum_{\substack{i \mid q \pm \sqrt{2q}+1 \\ i \neq 1}} \beta'_i \cdot \mathsf{m}_i(S) + \sum_{\substack{i \mid q-1 \\ i \neq 1}} \gamma'_i \cdot \mathsf{m}_i(S). \quad (4.2)$$

Since q^2 divides both $q^2(q^2+1)(q-1)r'/|G_p|$ and $\mathsf{m}_i(S)$ in (4.2), it follows that q^2 divides $\alpha' q(q^2+1)(q-1)$. Then $q \mid \alpha'$, and so $|G| = q^2(q^2+1)(q-1) \leq \alpha' q(q^2+1)(q-1) \leq f(p)$, which is impossible. □

Proposition 4.2. *The group G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|\mathsf{Out}(K/H)|$.*

Proof. By Proposition 4.1, the vertex 2 is an isolated vertex in the prime graph $\Gamma(G)$ of G. This implies that the number $t(G)$ of connected components of the prime graph $\Gamma(G)$ is at least two. The assertion follows from [16, Theorem A] provided that G is neither a Frobenius group, nor a 2-Frobenius group.

Let G be a Frobenius group with kernel K and complement H. Then by Lemma 2.4, we must have $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$. By Proposition 4.1, the vertex 2 is an isolated vertex in $\Gamma(G)$. Then either (i) $|K| = q^2$ and $|H| = (q^2+1)(q-1)$, or (ii) $|H| = q^2$ and $|K| = (q^2+1)(q-1)$. Both cases can be ruled out as $|H|$ must divide $|K|-1$.

Let G be a 2-Frobenius group. Then Lemma 2.5 implies that $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernel K/H and H respectively, $\pi(G/K) \cup \pi(H) = \pi_1$, $\pi(K/H) = \pi_2$ and $|G/K|$ divides $|\mathsf{Aut}(K/H)|$. Since 2 is an isolated vertex of $\Gamma(G)$ by Proposition 4.1, $|K/H| = (q^2+1)(q-1)$ and $|G/K|.|H| = q^2$. Since also K is a Frobenius group with kernel H, there is a positive integer α such that $(q^2+1)(q-1)$ divides $2^\alpha - 1$, which is a contradiction. □

4.1. **Proof of Theorem 1.1.**

Proof. Let $S := \mathrm{Sz}(q)$, where $q = 2^{2m+1} \geq 8$. Suppose that G is a finite group with $\mathsf{nse}(G) = \mathsf{nse}(S)$ and $|G| = |S|$. By applying Proposition 4.2, the group G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a non-abelian simple group. Since 3 is a prime divisor of all finite non-abelian simple groups except for Suzuki groups. Moreover, 3 is coprime to $|K/H|$. Then $K/H \cong \mathrm{Sz}(q')$, where $q' = 2^{2m'+1}$. This, in particular, implies that $2^{4m'+2}$ divides 2^{4m+2}, and hence $m' \leq m$. On the other hand, H and G/K are π_1-groups. Then $(q^2+1)(q-1)$ divides $|K/H|$, and so $(q^2+1)(q-1)$ divides $(q'^2+1)(q'-1)$. Since now $m' \leq m$, we must have $m = m'$. Therefore $K/H \cong S$. Now $|G| = |K/H| = |S|$, and hence $G \cong S$. □

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