www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. x No. x (201x), pp. xx-xx.

www.ui.ac.ir

THE MASCHKE PROPERTY FOR THE SYLOW p-SUBGROUPS OF THE SYMMETRIC GROUP S_{p^n}

D. J. GREEN, L. HÉTHELYI AND E. HORVÁTH*

Communicated by Attila Maróti

ABSTRACT. In this paper we prove that the Maschke property holds for coprime actions on some important classes of p-groups like: metacyclic p-groups, p-groups of p-rank two for $p > 3$ and some weaker property holds in the case of regular p-groups. The main focus will be the case of coprime actions on the iterated wreath product P_n of cyclic groups of order p, i.e. on Sylow p-subgroups of the symmetric groups S_{p^n}, where we also prove that a stronger form of the Maschke property holds. These results contribute to a future possible classification of all p-groups with the Maschke property. We apply these results to describe which normal partition subgroups of P_n have a complement. In the end we also describe abelian subgroups of P_n of largest size.

1. Introduction

Coprime actions have been extensively studied for many decades, see e.g. [12, Ch. 5]. Even nowadays it is a very popular research subject. Many authors deal with various aspects of this topic, see e.g. [9], [10], [11], [16], [21], [23], [29]. Maschke's well known result, if one considers vector spaces over finite fields, also gives examples of coprime action: if V is a finite dimensional vector space over a finite field of characteristic p and G is a finite group whose order is relatively prime to p such that G is acting on V then for every G-invariant subspace $W_1 \leq V$ there is a G-invariant complementary subspace W_2 in V, i.e. $V = W_1 \oplus W_2$.

Several authors have considered generalizations of Maschke's Theorem in the context of groups acting on groups. For example, the case of coprime action on an elementary abelian p-group is found in [18]; and in [1] Berkovich studied the case of coprime action on an abelian group, see also [2, §6].

MSC(2010): Primary: 20D45; Secondary: 20B35, 20D20.

Keywords: Maschke's Theorem, coprime action, Sylow p-subgroup of symmetric group, iterated wreath product, uniserial action.

Received: 23 October 2016, Accepted: 14 August 2017.

*Corresponding author.

In [8, Thm. 11.4], as a consequence of Gaschütz's Theorem, a general form of Maschke's theorem is proved for a group G acting on an abelian group A. As a special case one gets that if $(|G|, |A|) = 1$ and $A = A_1 \times A_2$, where A_1 is a G-invariant subgroup and A_2 is a subgroup of A, then there exists a G-invariant subgroup A_2^* of A such that $A = A_1 \times A_2^*$. In [15, Thm. 10.16] another generalization of Maschke's theorem is proved. As a special case one gets: Let G be a finite group acting on a group $V = V_1 \times V_2$ via automorphisms, where V_1 is abelian and G-invariant and $(|G|, |V_1|) = 1$. Then there exists a G-invariant normal subgroup $V_2^* \lhd V$ such that $V = V_1 \times V_2^*$.

Here we study a different generalization of the property in Maschke's theorem for coprime actions on finite groups.

Throughout this paper, p denotes a prime number and n a natural number. Moreover, π denotes a set of prime numbers, and π' is defined to be its complement in the set of all prime numbers.

Definition 1.1 (Maschke property). *A π-group V has the* Maschke property *if for every π'-group G acting on V the following property holds: if N is a G-invariant normal subgroup of V which has a complement in V, then it also has a G-invariant complement.*

For the sake of brevity we will often say "V is Maschke" instead of "the π-group V has the Maschke property", provided that the choice of π is clear from the context.

We call it a property rather than a theorem because it does not hold for all groups: one counterexample is an action of the cyclic group C_3 on the 2-group $Q_8 * C_4$, a central product (Example 2.4). As we mentioned above, all abelian groups are Maschke. We will prove in Section 2 that all metacyclic p-groups and p-groups of p-rank two if $p > 3$ are also Maschke (Proposition 2.2). Our main result is:

Theorem 1.2. *The Sylow p-subgroups of the symmetric group S_{p^n} have the Maschke property.*

We shall prove Theorem 1.2 in Section 11 by reducing it to the case of coprime action in S_{p^n}, where we will prove the following result:

Theorem 1.3. *Let P_n be a Sylow p-subgroup of S_{p^n}. Then there is a Hall p'-subgroup H of $N_{S_{p^n}}(P_n)$ which has the following property:*

> *If $N \trianglelefteq P_n$ is a normal subgroup which has a complement in P_n, then N has an H-invariant complement in P_n.*

The proof of Theorem 1.3 occupies much of the paper and is completed in Section 11.

Remark 1.4. *As N is not required to be H-invariant, Theorem 1.3 is a strengthening of the Maschke property. For $p^n = 3^3$, Example 12.4 constructs a complemented normal subgroup which does indeed fail to be H-invariant.*

This strengthening is false in the original context of Maschke's Theorem. For example, the dihedral group D_8 has an irreducible ordinary representation in degree two. Every one-dimensional subspace of the representation space has a complement, but by irreducibility there is no invariant complement.

In order to show that Theorem 1.2 follows from Theorem 1.3, we need some known properties of the Sylow p-subgroups of S_{p^n}. However these properties may never have been written down in the following uniform manner for all values of p:

Proposition 1.5 (various authors)**.** *Let P_n be a Sylow p-subgroup of S_{p^n}. Then:*

(1) $C_{S_{p^n}}(P_n) = Z(P_n)$ *and* $N_{S_{p^n}}(P_n)/P_n \cong C_{p-1}^n$.

(2) $\mathrm{Aut}(P_n)$ *has a normal Sylow p-subgroup, with factor group* C_{p-1}^n.

So for $p = 2$, $\mathrm{Aut}(P_n)$ is a 2-group and P_n is self-normalizing in S_{p^n}.

We give a proof of Proposition 1.5 in Section 6.

Remark 1.6. *Part (1) for odd primes was proved by Cárdenas and Lluis* [4]*. Both* [7] *and* [2, Corollary A.13.3] *say that P. Hall proved the case $p = 2$ in 1956. A modern treatment may be found in* [2, Appendix 13]*.*

Turning to (2), Bodnarchuk described the full structure of $\mathrm{Aut}(P_n)$ *for odd primes* [3]*, whereas we have not yet located a proof for $p = 2$. For recent developments on automorphisms of wreath products, see e.g.* [20]*,* [25]*.*

Structure of the paper. In Section 2 we prove the Maschke property for some basic types of p-groups. Section 3 recalls the identification of P_n as an iterated wreath product, introduces the generators σ_i and recalls Weir's subgroup A^{n-1}. Then in Section 4 we recall Weir's filtration T_j and his notion of depth before proving the useful Proposition 4.6, generalising results of Weir an Dmitruk. Uniserial action is the topic of Section 5, where we generalise another result of Weir to prove Proposition 5.1. These preparations then allow us to prove Proposition 1.5 in Section 6, where (in Lemma 6.1) we also construct the Hall subgroup for Theorem 1.3. The next four sections assemble the necessary tools for the proof of Theorem 1.3: see the introduction to Section 7 for the strategy. This allows us to prove both theorems in the short Section 11. The paper ends with an extensive selection of examples, and the application of our results to Weir's partition subgroups. In an appendix we briefly consider the largest abelian subgroups of P_n.

2. The Maschke property for some basic types of p-groups

The following lemma will be used to prove the metacyclic and rank two cases of Proposition 2.2 below. It shows that for regular p-groups a weakar form of the Maschke property holds.

Lemma 2.1. *Let G act coprimely on the regular p-group V. If $N \trianglelefteq V$ is a G-invariant normal subgroup which has a cyclic complement in V, then N has a G-invariant complement in V.*

Proof. Let L be a cyclic complement of N in V, and let $|L| = p^\ell$. Set $V_1 := \Omega_\ell(V) = \langle g \in V \mid g^{p^\ell} = 1\rangle$, which is characteristic and hence G-invariant. Then $L \leq V_1$, and L is a complement in V_1 to $N_1 := N \cap V_1$. Any G-invariant complement to N_1 in V_1 will be a complement to N in V, too.

As N_1 has a cyclic complement, [13, Prop 5.2] says that there is a G-invariant cyclic subgroup $C \leq V_1$ with $N_1 C = V_1$. We want $N_1 \cap C = 1$. Now, $|C : C \cap N_1| = |V_1 : N_1| = |L| = p^\ell$, so if

$N_1 \cap C \neq 1$ then the cyclic group $C \leq V_1$ has order $> p^\ell$. But as V is regular and $V_1 = \Omega_\ell(V)$, [14, 10.5 Hauptsatz p. 324] says that $V_1 = \{g \in V \mid g^{p^\ell} = 1\}$. So $C \cap N_1 = 1$ and C is the desired G-invariant complement. □

Proposition 2.2. *Let V be a finite group. If*

(1) *V is a metacyclic p-group; or*

(2) *V is a p-group of p-rank two, for $p > 3$*

then V has the Maschke property.

Remark 2.3. Bettina Wilkens has shown us an argument that our result (1) can be improved to arbitrary finite metacyclic groups (not necessarily p-groups).

Proof. Suppose that G acts coprimely on V, and that the G-invariant normal subgroup $N \trianglelefteq V$ has the complement L in V. Assume $N \neq 1$.

V metacyclic, $p = 2$: By [22, Lemma 1], if G acts nontrivially and V is nonabelian then $V = Q_8$. But then only $N = 1$ and $N = V$ have complements.

V metacyclic, p odd: If $[V, N] = 1$ then $V' = L'$. Hence V/L' is abelian, and there is $L' \leq W \trianglelefteq V$ with W/L' a G-invariant complement to $NL'/L' \cong N$. So W is a G-invariant complement to N.

So we assume $[V, N] \neq 1$. Let $K \trianglelefteq V$ be cyclic with V/K cyclic, so $V' \leq K$ and V is a regular p-group by [14, III.10.2 Satz p. 322]. Since N is normal and $V' \leq K$ we have $K \cap N \neq 1$. As K is cyclic and $N \cap L = 1$, it follows that $L \cap K = 1$. Therefore $L \cong LK/K \leq V/K$ is cyclic, and the result follows by Lemma 2.1.

V has p-rank two, $p \geq 5$: Set $F = N \cap \Omega_1(Z(V))$; from $N \neq 1$ it follows that $F \neq 1$. If $E \leq L$ is elementary abelian then EF is elementary abelian, too. As $E \cap N = 1$ if follows that EF has rank larger than that of E. It follows that L has p-rank one. So L is cyclic, by [14, III.8.2 Satz p. 310].

By Lemma 2.1 it suffices to show that V is regular. By a theorem of Blackburn [14, III.12.4 Satz p. 343], V satisfies one of three conditions. In Blackburn's Case (1), V is metacyclic. In Case (2), $V' = \langle Z^{p^{n-3}} \rangle$ is cyclic; and in Case (3), V has nilpotency class $3 < p$. So V is regular in cases (2) and (3) by a) and c) of [14, III.10.2 Satz p. 322]. □

First we give counterexamples of groups with p-rank two for $p = 2, 3$. The counterexample for $p = 3$ is also of maximal class.

Example 2.4. For $p = 2$ let V be the central product $V = Q_8 * C_4$. That is, $V = \langle i, j, k, x \rangle$ with x central, $x^2 = -1$ and $|V| = 16$. Observe that V has 2-rank two.

There is an automorphism ϕ of order 3 which acts on the set $\{i, j, k, x\}$ as the 3-cycle $(i\,j\,k)$. So $G = \langle \phi \rangle \cong C_3$ acts coprimely on V, and $N = Q_8 = \langle i, j, k \rangle$ is a G-invariant normal subgroup which has a complement: each of the six involutions $\pm ix$, $\pm jx$, $\pm kx$ generates a complement. But ϕ acts on this set of six complements as a permutation of type 3^2, and there are no other complements. So $Q_8 * C_4$ does not have the Maschke property.

Example 2.5. Let V be the semidirect product $V = (\mathbb{Z}/9\mathbb{Z})^2 \rtimes C_3$, where the action of $C_3 = \langle x \rangle$ on $(\mathbb{Z}/9\mathbb{Z})^2$ is as follows:

$$ {}^x v = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{for} \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. $$

Observe that V has order 3^5; it has 3-rank two; and it is of maximal class.

Now, $V' = \{v \in (\mathbb{Z}/9\mathbb{Z})^2 \mid v_1 \in 3\mathbb{Z}/9\mathbb{Z}\}$, which has order 3^3. Consequently, $N := \langle V', x \rangle$ is a normal subgroup of order 3^4. Moreover,

$$ v + {}^x v + {}^{x^2} v = 0 \quad \text{for every } v \in (\mathbb{Z}/9\mathbb{Z})^2, $$

and so (v, x) has order 3 for every $v \in (\mathbb{Z}/9\mathbb{Z})^2$. So $C_v := \langle (v, x) \rangle$ is cyclic of order 3 for every $v \in (\mathbb{Z}/9\mathbb{Z})^2$; and C_v is a complement of N in V for every $v \in (\mathbb{Z}/9\mathbb{Z})^2 \setminus V'$. As every $v \in (\mathbb{Z}/9\mathbb{Z})^2 \setminus V'$ has order 9, it follows that every complement of N in V is a C_v with $v \in (\mathbb{Z}/9\mathbb{Z})^2 \setminus V'$.

By construction of V, $\alpha(v, x) := (-v, x)$ defines an automorphism of V, of order 2. Then N is α-invariant, and $\alpha(C_v) = C_{-v}$. As $C_v \neq C_{-v}$ for $0 \neq v \in (\mathbb{Z}/9\mathbb{Z})^2$, it follows that N has no $\langle \alpha \rangle$-invariant complement in V. So V does not have the Maschke property.

3. The iterated wreath product

Recall that if $S \leq \mathrm{Sym}(X)$ and $G \leq \mathrm{Sym}(Y)$ are permutation groups acting on finite sets X and Y, then there is a wreath product group

$$ G \wr S = G^{|X|} \rtimes S \leq \mathrm{Sym}(Y \times X) $$

with S-action given by $({}^\sigma \underline{g})_x = g_{\sigma^{-1}(x)}$ for $\sigma \in S$, $\underline{g} \in G^{|X|}$ and $x \in X$. By [14, I.15.4 Hilfssatz, p. 96] we have associativity: $G \wr (S \wr T) \cong (G \wr S) \wr T$.

Let p be a prime number. The cyclic group C_p embeds in the symmetric group S_p as the subgroup generated by a p-cycle, and so the n-fold iterated wreath product

$$ P_n := \underbrace{C_p \wr C_p \wr \cdots \wr C_p}_{n \text{ copies of } C_p} $$

embeds in S_{p^n}. Kaloujnine (in [17]; see also [14, III.15.3 Satz, p. 378]) proved that P_n is a Sylow p-subgroup of S_{p^n}.

We shall treat S_n as the group of permutations of $\{0, 1, \ldots, n-1\}$ rather than of $\{1, 2, \ldots, n\}$. Using the p-adic representation $a = \sum_{i=0}^{n-1} b_i p^{n-1-i}$ we can identify $a \in \{0, 1, \ldots, p^n - 1\}$ with $(b_0, b_1, \ldots, b_{n-1}) \in \mathbb{F}_p^n$. In particular, we may identify S_{p^n} with the symmetric group $\mathrm{Sym}(\mathbb{F}_p^n)$.

Remark 3.1. *The n-fold iterated wreath product of C_p can also be considered as the automorphism group of the complete p-ary rooted tree of height n. In the literature these are extensively studied, see e.g.* [24], [27], [28], [31], [26]. *These results have significant role in the representation theory of iterated wreath products, as well as they have applications in signal processing, fast Fourier transforms, molecular symmetry, automata theory etc.*

Here we take a different approach. We consider groups represented on iterated wreath products. In some respects iterated wreath products of C_p behave similar to vector spaces, since they have the Maschke property.

Lemma 3.2. *Denote by σ the p-cycle $\sigma = (0\ 1\ 2\ \cdots\ p-1) \in \mathrm{Sym}(\mathbb{F}_p)$. For $0 \leq i \leq n-1$ define $\sigma_i \in \mathrm{Sym}(\mathbb{F}_p^n)$ as follows:*

$$\sigma_i(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) = \begin{cases} (\lambda_0, \ldots, \lambda_i, \ldots, \lambda_{n-1}) & \exists\, j < i \,:\, \lambda_j \neq 0 \\ (\lambda_0, \ldots, \sigma(\lambda_i), \ldots, \lambda_{n-1}) & \forall\, j < i \,:\, \lambda_j = 0 \end{cases}.$$

Then $\langle \sigma_0, \ldots, \sigma_{n-1} \rangle$ is a copy of P_n in $\mathrm{Sym}(\mathbb{F}_p^n)$. It acts transitively.

Proof. More generally, for $G \leq \mathrm{Sym}(Y)$ and $S \leq \mathrm{Sym}(X)$, the group $G \wr S = G^X \rtimes S$ is the following subgroup of $\mathrm{Sym}(X \times Y)$: The action of $\pi \in S$ on $X \times Y$ is $(x_0, y) \mapsto (\pi(x_0), y)$, and the action of $(g_x)_{x \in X} \in G^X$ is $(x_0, y) \mapsto (x_0, g_{x_0}(y))$. This is indeed an action of $G \wr S$, since

$$\begin{aligned} \pi(g_x)_{x \in X}(x_0, y) &= \pi(x_0, g_{x_0}(y)) = (\pi(x_0), g_{x_0}(y)) \\ &= (g_{\pi^{-1}(x)})_{x \in X}(\pi(x_0), y) = (g_{\pi^{-1}(x)})_{x \in X}\pi(x_0, y)\,. \end{aligned}$$

For $g \in G$ and $x \in X$ define $\delta_x(g) \in G^X$ by $(\delta_x g)_{x'} = \begin{cases} g & x = x' \\ \mathrm{Id} & \text{otherwise} \end{cases}$. Then ${}^{\pi}\delta_x(g) = \delta_{\pi(x)}(g)$. So as $(g_x)_{x \in X} = \prod_{x \in X} \delta_x(g_x)$, we see: If S is transitive and $x_0 \in X$ then G^X is the normal closure of $\mathrm{Im}(\delta_{x_0})$, and $G \wr S$ is generated by S and $\mathrm{Im}(\delta_{x_0})$. We apply this to $P_n = C_p \wr P_{n-1}$ and use induction over n. Note in particular that σ_{n-1} is $\delta_{x_0}(\sigma)$ for $x_0 = (0, \ldots, 0) \in \mathbb{F}_p^{n-1}$.

Transitive: More generally, if G and S are transitive, then so is $G \wr S$. □

Example 3.3. Consider $P_3 = \langle \sigma_0, \sigma_1, \sigma_2 \rangle$ for $p = 3$. Then for example $15 = 1{\cdot}9 + 2{\cdot}3 + 0{\cdot}1 \in \{0, 1, \ldots, 26\}$ corresponds to $(1, 2, 0) \in \mathbb{F}_3^3$. Hence

$$\begin{aligned} \sigma_0 &= (0\ 9\ 18)(1\ 10\ 19)(2\ 11\ 20)(3\ 12\ 21)(4\ 13\ 22)(5\ 14\ 23)(6\ 15\ 24)\,\cdot \\ &\qquad (7\ 16\ 25)(8\ 17\ 26) \\ \sigma_1 &= (0\ 3\ 6)(1\ 4\ 7)(2\ 5\ 8) \\ \sigma_2 &= (0\ 1\ 2)\,. \end{aligned}$$

Lemma 3.4. *All p^{n-1} conjugates of σ_{n-1} in P_n commute with each other.*

Proof. P_{n-1} has degree p^{n-1}, and in the isomorphism $P_n \cong C_p \wr P_{n-1}$ the P_{n-1} is generated by $\sigma_0, \ldots, \sigma_{n-2}$, and the C_p by σ_{n-1}. □

Remark 3.5. *For $x \in P_n$, observe that ${}^x\sigma_{n-1} \in P_n$ moves $(\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{F}_p^n$ if and only if x sends $(0, \ldots, 0) \in \mathbb{F}_p^n$ to $(\lambda_0, \ldots, \lambda_{n-2}, \mu)$ for some $\mu \in \mathbb{F}_p$. So ${}^x\sigma_{n-1}$ is a p-cycle on those p points whose first $n-1$ coordinates coincide with those of $x(0, \ldots, 0)$. One such value of x is $x = \sigma_0^{\lambda_0}\sigma_1^{\lambda_1} \cdots \sigma_{n-2}^{\lambda_{n-2}}$; this lies in P_{n-1}, viewed as a subgroup of P_n via the isomorphism $P_n \cong C_p \wr P_{n-1}$.*

Notation 3.6. Following Weir we write A^{n-1} for the base group $C_p^{p^{n-1}}$ of $P_n = C_p \wr P_{n-1} = C_p^{p^{n-1}} \rtimes P_{n-1}$. Then A^{n-1} is elementary abelian, and normal in P_n. Also, A^{n-1} is the normal closure of $\langle \sigma_{n-1} \rangle$ in P_n.

Remark 3.7. *For odd p, Weir [30, Thm 6] shows that A^{n-1} is the unique maximal abelian normal subgroup of P_n, and hence characteristic. But if $p = 2$ then A^{n-1} is not characteristic: for example, $P_2 \cong D_8$ and A^1 is one of the two rank two elementary abelian subgroups in D_8; but these two elementary abelians are conjugate in D_{16}.*

Example 3.8. For $p = 3$, $A^2 \leq P_3$ is elementary abelian of rank 9 with basis

$$\begin{array}{lll} \sigma_2 = (0\ 1\ 2) & {}^{\sigma_1}\sigma_2 = (3\ 4\ 5) & {}^{\sigma_1^2}\sigma_2 = (6\ 7\ 8) \\ {}^{\sigma_0}\sigma_2 = (9\ 10\ 11) & {}^{\sigma_0\sigma_1}\sigma_2 = (12\ 13\ 14) & {}^{\sigma_0\sigma_1^2}\sigma_2 = (15\ 16\ 17) \\ {}^{\sigma_0^2}\sigma_2 = (18\ 19\ 20) & {}^{\sigma_0^2\sigma_1}\sigma_2 = (21\ 22\ 23) & {}^{\sigma_0^2\sigma_1^2}\sigma_2 = (24\ 25\ 26)\,. \end{array}$$

Corollary 3.9. *Both A^{n-1} and P_n are self-centralizing[1] in S_{p^n}.*

Proof. By Remark 3.5, A^{n-1} is generated by a set X of p-cycles whose supports are disjoint and cover $\mathbb{F}_p^n$. Suppose that $\pi \in S_{p^n}$ centralizes A^{n-1}, and pick $\sigma \in X$; then $[\pi, \sigma] = 1$. Since σ is a p-cycle and C_p is self-centralizing in S_p, it follows that π has the form $\pi = \pi' \cdot \sigma^r$, where the supports of π' and σ are disjoint. As the supports of the $\sigma \in X$ cover $\mathbb{F}_p^n$, it follows that $\pi \in A^{n-1}$. So A^{n-1} is self-centralizing in S_{p^n}, and the result for P_n follows. ☐

4. Weir's filtration T_j and depth

Notation 4.1. Associativity implies that $P_n \cong P_{n-j} \wr P_j$ for all $0 \leq j \leq n$. Weir [30] writes T_j for the base group of this wreath product, so $T_j \cong P_{n-j}^{p^j}$.

Hence $P_n = P_j T_j$, $T_{n-1} = A^{n-1}$ and $P_n = T_0 \geq T_1 \geq \cdots \geq T_{n-1} \geq T_n = 1$. Also, T_{j-1}/T_j is the subgroup A^{j-1} of $P_j \cong P_n/T_j$. For odd p this means that each T_j is characteristic in P_n, as A^{n-1} is characteristic.

Example 4.2. If $p^n = 3^3$ then $T_0 = P_3$; $T_3 = 1$; $T_2 = A^2$, which we described in Example 6.2; and T_1/T_2 is elementary abelian of rank 3, generated by the cosets of the three $\langle \sigma_0 \rangle$-conjugates of σ_1.

Notation 4.3. Weir defines the *depth* j of a subgroup $S \leq P_n$ to be the largest i such that $S \leq T_i$. That is, T_j is the smallest group in the series $P_n = T_0 > T_1 > \cdots > T_n = 1$ which contains S.

Lemma 4.4. *Let $N \trianglelefteq P_n$. If $N \cap T_{j+1} \lneq N \cap T_j$ then $N \cap T_{k+1} \lneq N \cap T_k$ for all $j \leq k \leq n-1$.*

Proof. If $g \in N \cap (T_j \setminus T_{j+1})$ then $gT_{j+1} \neq 1$ in the elementary abelians A^j of $P_n/T_{j+1} \cong P_{j+1}$. Replacing g by a conjugate, we may assume that σ_j lies in the support[2] of gT_{j+1}. Then $[g, \sigma_{j+1}] \in N \cap (T_{j+1} \setminus T_{j+2})$, since σ_{j+1} commutes with all nontrivial P_j-conjugates of σ_j. ☐

[1] We say that H is self-centralizing in G if $C_G(H) \leq H$.

[2] A^j is an $\mathbb{F}_p$-vector space, with basis the P_j-conjugates of σ_j.

Remark 4.5. *The following result is used in the proofs of Theorem 1.3 and Proposition 5.1. Special cases of this result have been proved before: The case where p is odd and N is a partition subgroup is due to Weir* [30, Thm 4]*, and the case where $p = 2$ and N is characteristic is due to Dmitruk* [6, Thm 5a].

Proposition 4.6. *If $N \trianglelefteq P_n$ has depth j, then $[T_j, T_j] \leq N$.*

Proof. T_n and T_{n-1} are abelian. If $j < n-1$, then $N \cap T_{j+1}$ has depth $j+1$ by Lemma 4.4. So by downward induction on j we may assume that $[T_{j+1}, T_{j+1}] \leq N$.

T_j is generated by T_{j+1} and the P_j-conjugates of σ_j. So by the formulae[3] for the commutators $[x, yz]$ and $[xy, z]$ it suffices to show that $[x, y] \in N$ if each of x, y is either an element of T_{j+1} or a P_j-conjugate of σ_j. As these conjugates commute with each other, we need only consider the case of $[\sigma_j, y]$, with $y \in T_{j+1}$.

If $[\sigma_j, y] = 1$ then we are done, hence we may assume that σ_j, y lie in the same factor $F \cong P_{n-j}$ of the base group of $P_{n-j} \wr P_j$. As in the proof of Lemma 4.4 there is some $g \in N$ such that σ_j occurs in the support of $gT_{j+1} \in A^j$. That is, some power g^r has component $\sigma_j z$ in F, with $z \in T_{j+1}$. Hence $[\sigma_j, y] = [g^r z^{-1}, y]$. Using the commutator formulae again we have $[\sigma_j, y] \in N$. □

Corollary 4.7. (see [5, Thm 4.4.1]) *Let p be an arbitrary prime. If $B \trianglelefteq P_n$ is an abelian normal subgroup, then $B \leq T_{n-2}$.*

Proof. If not, then $T'_{n-3} \leq B$ by Proposition 4.6. But T_{n-3} is a direct product of copies of P_3, and P'_3 is nonabelian as $[[\sigma_0, \sigma_1], [\sigma_0, \sigma_2]] \neq 1$. □

5. Uniserial action

In this section we first recall the definition of uniserial action and then prove the following result (due to Weir for $p \neq 2$), which we require for the proof of Proposition 1.5 (2). Recall that P_n is the n-fold iterated wreath product $C_p \wr C_p \wr \cdots \wr C_p$, and so $P_n \cong C_p \wr P_{n-1}$.

Proposition 5.1. *The Sylow p-subgroup P_n of S_{p^n} has a characteristic abelian subgroup B with the following properties:*

1. $P_n/B \cong P_{n-1}$
2. *The action of $P_{n-1} \cong P_n/B$ on B is uniserial.*

Remark 5.2. *Weir* [30] *proved this for $p \neq 2$; for B he used the subgroup which he calls A^{n-1} (see Notation 3.6), and which Huppert constructs in* [14, III.15.4 Satz a), p. 380]. *For $p = 2$, Huppert constructs our B in* [14, III.15.4 Satz b), p. 381]. *He only remarks that it is abelian, normal and not contained in A^{n-1}; Covello shows that it is characteristic* [5, Thm 4.4.6]. *For $p^n = 2^3$, our B is the group $\mathfrak{H}_7$ which Dmitruk constructs in* [6, p. 124].

Definition 5.3. *Let P, M be finite p-groups, with M abelian and P acting on M. Recall from* [19, §4.1] *that the action is called* uniserial *if the following equivalent conditions hold:*

[3] See e.g. [12, Lemma 2.2.4, p. 20].

(1) *$[P, N]$ has index p in N for every P-invariant subgroup $1 \neq N \leq M$.*
(2) *$M_\ell \neq 0$, where $\ell = \log_p(|M|)$, $M_1 = M$ and $M_{r+1} = [P, M_r]$.*

Recall further that if the action is uniserial, then

(1) *$M = M_1 > M_2 > \cdots > M_\ell > M_{\ell+1} = 0$.*
(2) *$N \leq M$ is P-invariant if and only if N is one of the M_r.*
(3) *The set of P-invariant subgroups of M is linearly ordered by inclusion.*

One calls ℓ the length *of M.*

Remark 5.4. *It follows that $C_M(P) = M_\ell$.*

Lemma 5.5. *Let P, M be 2-groups, with P acting on M. Then the natural action of $Q = P \wr C_2$ on $M^2 = M \oplus M$ has the following properties:*

(1) *If $[P, M]$ has index 2 in M, then $M^2 > [Q, M^2] > [Q, Q, M^2] = [P, M]^2$.*
(2) *If the action of P on M is uniserial, then so is the action of Q on M^2.*

Proof. Here we write $[a, b] = aba^{-1}b^{-1}$ and $[a, b, c] = [a, [b, c]]$.
(1): Since the action of Q on M^2 is nilpotent and $[P, M]^2$ has index 4 in M^2, it suffices to show that $[P, M]^2 \leq [Q, Q, M^2]$. We have $Q = P^2 \rtimes \langle \sigma \rangle$, where σ transposes the two copies of P^2. Then for $g \in P$ and $x \in M$ we have

$$[(g, 1), \sigma, (1, x)] = [(g, 1), (x, x^{-1})] = ([g, x], 1).$$

Hence $[P, M] \times 1$ lies in $[Q, Q, M^2]$. Similarly, $1 \times [P, M] \leq [Q, Q, M^2]$.
(2): Set $\ell = \log_2 |M|$. Define M_r, N_r by $M_1 = M$, $N_1 = M^2$, $M_{r+1} = [P, M_r]$ and $N_{r+1} = [Q, N_r]$. Then $M_\ell \neq 0$, and we need $N_{2\ell} \neq 0$. As M is uniserial, $M_{r+1} = [P_{n-1}, M_r]$ has index 2 in M_r for $r \leq \ell$. So by induction on r we have $N_{2r-1} = M_r^2$ for $r \leq \ell + 1$, since if $r \leq \ell$ and $N_{2r-1} = M_r^2$ then

$$N_{2r+1} = [Q, Q, M_r^2] = [P, M_r]^2 = M_{r+1}^2$$

by (1). In particular $N_{2\ell-1} = M_\ell^2 \neq 0$. Another application of (1) shows that $N_{2\ell} = [Q, M_\ell^2] > [Q, Q, M_\ell^2]$, hence $N_{2\ell} \neq 0$. So M^2 is uniserial. □

Lemma 5.6 (Weir)**.** *The action of P_n on A^n is uniserial of length p^n.*

Proof. By [30, Theorem 2] we need only consider the case $p = 2$. For $n = 1$ this is immediate; and for $n \geq 2$ it follows from Lemma 5.5 (2) by induction on n, as the action of P_n on A^n is the induced action of $P_{n-1} \wr C_2$ on $(A^{n-1})^2$. □

Proof of Proposition 5.1. For odd p we may take $B = A^{n-1}$ by Theorems 2 and 6 of Weir's paper [30], so from now on we take $p = 2$. Corollary 4.7 tells us that if $N \trianglelefteq P_n$ is abelian then $N \leq T_{n-2}$. Now T_{n-2} is the base group of $P_2 \wr P_{n-2}$, and $P_2 \cong D_8$: so $T_{n-2} \cong (D_8)^{2^{n-2}}$; and $P_n \cong T_{n-2} \rtimes P_{n-2}$, where P_{n-2} acts by permuting the copies of D_8 transitively.

Since N is abelian, its projection onto each D_8 must be abelian, too; and since P_{n-2} acts transitively, each projection must be the same abelian subgroup of D_8. But D_8 has only one abelian subgroup of exponent four. So if N has exponent four then it is contained in $B = (C_4)^{2^{n-2}}$. Since B is normal in P_n it follows that B is the unique largest abelian normal subgroup of exponent four, and hence characteristic in P_n.

We have $P_n/B \cong (D_8/C_4)^{2^{n-2}} \rtimes P_{n-2} \cong C_2^{2^{n-2}} \rtimes P_{n-2} \cong P_{n-1}$. Writing B^{n-1} for B, we see that $P_1 \cong D_8/C_4$ acts uniserially on $B^1 \cong C_4$; and that $B^{n-1} \cong (B^{n-2})^2$ in $P_n \cong P_{n-1} \wr C_2$. So the action of P_{n-1} on B^{n-1} is uniserial by Lemma 5.5 (2). □

6. The Hall subgroup and Proposition 1.5

In this section we construct the Hall subgroup for Theorem 1.3 and prove Proposition 1.5. We include a proof of Proposition 1.5 for two reasons: the $p = 2$ case of part (2) may not be in the literature; and the proof of Theorem 1.3 necessitates our constructing explicit generators for a Hall p'-subgroup of $N_{S_{p^n}}(P_n)$.

The group of units $\mathbb{F}_p^\times$ is cyclic of order $p-1$: let r be a generator. Define $\eta \in \mathrm{Sym}(\mathbb{F}_p)$ by $\eta(x) = rx$. Then η is a $(p-1)$-cycle, with $\eta(0) = 0$. Since the σ of Lemma 3.2 is given by $\sigma(x) = x + 1$, we have ${}^{\eta}\sigma = \sigma^r$.

Lemma 6.1. *For $0 \leq i \leq n-1$ define $\eta_i \in \mathrm{Sym}(\mathbb{F}_p^n)$ by*

$$\eta_i(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) = (\lambda_0, \ldots, \eta(\lambda_i), \ldots, \lambda_{n-1})$$

and set $H = \langle \eta_0, \ldots, \eta_{n-1} \rangle$. Then $H \cong (C_{p-1})^n$, and $H \leq N_{S_{p^n}}(P_n)$.

Corollary 6.5 below shows that H is a Hall p'-subgroup of $N_{S_{p^n}}(P_n)$.

Proof. $H \cong (C_{p-1})^n$ is clear. And ${}^{\eta_j}\sigma_i = \begin{cases} \sigma_i^r & j = i \\ \sigma_i & j \neq i \end{cases}$, since $\eta(0) = 0$. □

Example 6.2. For $p^n = 3^3$ we have $r = 2$ and

$$\eta_0 = (9\ 18)(10\ 19)(11\ 20)(12\ 21)(13\ 22)(14\ 23)(15\ 24)(16\ 25)(17\ 26)$$
$$\eta_1 = (3\ 6)(4\ 7)(5\ 8)(6\ 9) \cdot (12\ 15)(13\ 16)(14\ 17) \cdot (21\ 24)(22\ 25)(23\ 26)$$
$$\eta_2 = (1\ 2)(4\ 5)(7\ 8)(10\ 11)(13\ 14)(16\ 17)(19\ 20)(22\ 23)(25\ 26)\,.$$

We are now in a position to prove Proposition 1.5.

Remark 6.3. *Suppose that G is a finite group that has a normal Sylow p-subgroup Q with abelian factor group G/Q. Observe that G is solvable, so by a theorem of P. Hall* [12, Thm 6.4.1, p. 231]*, G has a Hall p'-subgroup H, and every p'-subgroup of G is conjugate to a subgroup of H. Observe that H is isomorphic to G/Q.*

Remark 6.4. *The following observation follows from the fact that every submodule of $\mathbb{Z}^n$ is free of rank $\leq n$: A finite abelian group G is isomorphic to a subgroup of $(C_m)^n$ if and only if the exponent of G divides m, and G has a generating set of size $\leq n$.*

Proof of Proposition 1.5. We show that $\mathrm{Aut}(P_n)$ has a normal Sylow p-subgroup Q, and an abelian Hall p'-subgroup A with exponent dividing $p-1$ and at most n generators; the result follows by Corollary 3.9 and Lemma 6.1.

It is well known that $\mathrm{Aut}(C_p) \cong C_{p-1}$, see e.g. [12, Thm 1.3.10, p. 12]. That deals with the case $n = 1$, so now take $n \geq 2$.

Step 1: The subgroups B_i and the map ϕ.

Proposition 5.1 says that P_n has a characteristic abelian subgroup B such that $P_n/B \cong P_{n-1}$ acts uniserially on B. Define B_i inductively for $i \geq 0$ by $B_0 = B$ and $B_{i+1} = [P_n, B_i]$. Then each B_i is characteristic in P_n; $B_i \leq B_{i-1}$; and the factor group B_{i-1}/B_i is cyclic of order p for all $i \leq p^{n-1}$, and $B_{p^{n-1}} = 1$.

As each term is characteristic in P_n, the normal series $P_n > B = B_0 > B_1 > \cdots > B_{p^{n-1}} = 1$ induces

$$\phi\colon \mathrm{Aut}(P_n) \to \mathrm{Aut}(P_n/B) \times \prod_{i=1}^{p^{n-1}} \mathrm{Aut}(B_{i-1}/B_i)\,,$$

Step 2: $\mathrm{Aut}(P_n)$ has a normal Sylow p-subgroup Q, and an abelian Hall p'-subgroup A of exponent dividing $p-1$.

The kernel of ϕ is a p-group by [12, Cor 5.3.3, p. 179]. Since $P_n/B \cong P_{n-1}$ and $\mathrm{Aut}(B_{i-1}/B_i) \cong \mathrm{Aut}(C_p) \cong C_{p-1}$, our ϕ is a map

$$\phi\colon \mathrm{Aut}(P_n) \to \mathrm{Aut}(P_{n-1}) \times (C_{p-1})^{p^{n-1}}\,.$$

By induction, $\mathrm{Aut}(P_{n-1})$ has a normal Sylow p-subgroup whose factor group is abelian of exponent dividing $p-1$. Now apply Remark 6.3.

Step 3: The kernel K of $A \hookrightarrow \mathrm{Aut}(P_n) \to \mathrm{Aut}(P_n/B)$ is cyclic.

Suppose that $\alpha \in K$ acts trivially on B_{i-1}/B_i for some i. From $B_i = [P_n, B_{i-1}]$ it follows that B_i/B_{i+1} is generated by elements of the form $[g,x]B_{i+1}$, with $x \in B_{i-1}$ and $g \in P_n$. Then $\alpha([g,x]) = [\alpha(g), \alpha(x)]$. Since α acts trivially on B_{i-1}/B_i, we have $\alpha(x) = xy$ for some $y \in B_i$; and since $\alpha \in K$ we have $\alpha(g) = gz$ for some $z \in B$. So $\alpha([g,x]) = [gz, xy] = [g, xy] \in [g,x] \cdot [P_n, B_i] = [g,x]B_{i+1}$. So α acts trivially on B_i/B_{i+1}, too. Hence: if $\alpha \in K$ acts trivially on B_0/B_1 then it acts trivially on each B_{i-1}/B_i, meaning that $\alpha \in \ker(\phi) \subseteq Q$. But $A \cap Q = 1$, so K acts faithfully on $B_0/B_1 \cong C_p$ and is therefore cyclic.

Step 4: A has at most n generators.

K is cyclic, and A/K is isomorphic to a p'-subgroup of $\mathrm{Aut}(P_{n-1})$. By induction and Remark 6.3, A/K is isomorphic to a subgroup of C_{p-1}^{n-1} and has at most $n-1$ generators. So A has at most n generators. □

Corollary 6.5. *The group H constructed in Lemma 6.1 is a Hall p'-subgroup of $N_{S_{p^n}}(P_n)$, and its image in $\mathrm{Aut}(P_n)$ is a Hall p'-subgroup of $\mathrm{Aut}(P_n)$.*

Proof. By Proposition 1.5 it has the correct order. □

7. Direct summands of M^n for uniserial M

In this and the next three sections we assemble the necessary tools for the proof of Theorem 1.3 in Section 11. If $N \trianglelefteq P_n$ has depth j, then it contains $K := [T_j, T_j]$ by Proposition 4.6, and N/K is an $\mathbb{F}_p P_j$-submodule of T_j/K. Now, T_j/K is a direct sum of $n - j$ copies of the uniserial module A^j; and if N has a complement in P_n, then N/K is a direct summand of T_j/K. So in this section we suppose that M is any uniserial module; we characterise which submodules of M^n have complements, and show that if N has a complement then it has one of the form M_Z. In Sections 8 and 9 we apply this general theory in the case $M^n = T_j/K$. In particular we establish a necessary condition on N (Lemma 9.3, which builds on Lemmas 8.3 and 9.2), without which N cannot have a complement, even if N/K does. Finally in Section 10 we construct certain permutations ρ_i and use them to show that if N/K has a complement and N satisfies the necessary condition of Lemma 9.3, then the complement M_Z of N/K lifts to an H-invariant complement of N.

In this section we take P to be a finite p-group.

Lemma 7.1. *Let $M \neq 0$ be a uniserial $\mathbb{F}_p P$-module. Then there is some $a_0 \in \mathbb{F}_p P$ with $a_0 M = C_M(P)$ and $a_0[P, M] = 0$.*

Proof. Let $I = \{a \in \mathbb{F}_p P \mid aM = 0\}$, the annihilator of M in $\mathbb{F}_p P$. Observe that I is a two-sided ideal in $\mathbb{F}_p P$, and proper since $M \neq 0$. Hence the quotient ring $R = \mathbb{F}_p P/I$ has order p^d for some $d \geq 1$. Now, the p-group $P \times P$ acts on R via $(x, y) \cdot (r + I) = xry^{-1} + I$; and so the number of length one orbits has to be divisible by p. As $0 + I$ is one such orbit, it follows that $a_0 + I$ is fixed by $P \times P$ for some $a_0 \notin I$. Then for all $g, h \in P$ and all $x \in M$ we have $ga_0 hx = a_0 x$. Hence $a_0 M$ is a submodule of $C_M(P)$; and since $[P, M]$ is generated by elements of the form $(h - 1)x$, it follows that $a_0[P, M] = 0$. Moreover, since $a_0 \notin I$ we have $a_0 M \neq 0$. But since M is uniserial it follows that $C_M(P)$ is simple, so $a_0 M = C_M(P)$. □

Lemma 7.2. *Let P be a p-group and M a length ℓ uniserial $\mathbb{F}_p P$-module. Let $N_v \subseteq M^n$ be the cyclic submodule generated by $v = (v_1, \dots, v_n) \in M^n$. Then the following statements are equivalent:*

1. *As an $\mathbb{F}_p P$-module, N_v is uniserial of length ℓ.*
2. $\dim_{\mathbb{F}_p}(N_v) = \ell$ *and* $\dim_{\mathbb{F}_p}(C_{N_v}(P)) = 1$.
3. *There is some $i \in \{1, \dots, n\}$ with the following properties:*
   - (a) *If $a \in \mathbb{F}_p P$ satisfies $av_i = 0$, then $av = 0$.*
   - (b) *$v_i \in M$ lies outside $[P, M]$.*

Example 12.1 shows that we cannot dispense with condition (3a).

Proof. (1) ⇒ (2): Follows from Remark 5.4.
(2) ⇒ (3): Pick $0 \neq w \in C_{N_v}(P)$, then $w_i \neq 0$ for some $i \in \{1, \dots, n\}$. Now consider the $\mathbb{F}_p P$-module map $\phi \colon N_v \to M$, $u \mapsto u_i$. If $u \in \ker(\phi)$, then the submodule $U \subseteq N_v$ generated by u satisfies $x_i = 0$ for all $x \in U$. If $U \neq 0$ then $C_U(P) \neq 0$ and therefore $U \cap C_{N_v}(P) \neq 0$. So there is $0 \neq w' \in U \cap C_{N_v}(P)$.

As $w_i \neq 0$ and $w' \subseteq U \subseteq \ker(\phi)$, we see that w, w' are linearly independent, a contradiction. Hence ϕ is injective.

Since $av_i = \phi(av)$, this proves (3a). Also, since ϕ is injective, it is surjective for dimension reasons. So $v_i = \phi(v)$ generates M, since v generates N_v. This shows (3b), since $[P, M]$ is a proper submodule.
(3) $\Rightarrow$ (1): Conversely, (3a) means that ϕ is injective, and since M is uniserial, (3b) means that ϕ is surjective. So N_v is isomorphic to M. □

Lemma 7.3. *Let $M \neq 0$ be a length ℓ uniserial $\mathbb{F}_p P$-module; $v_1, \ldots, v_r \in M^n$ elements satisfying the equivalent conditions of Lemma 7.2; and $N = \sum_{i=1}^r N_{v_i}$ the $\mathbb{F}_p P$-submodule they generate. Then the following statements are equivalent:*

1. *The sum $N = \sum_{i=1}^r N_{v_i}$ is direct.*
2. *The images of $v_1, \ldots, v_r$ in $M^n/[P, M^n]$ are linearly independent.*
3. *If w_i is a basis element of $C_{N_{v_i}}(P)$, then $w_1, \ldots, w_r$ are linearly independent.*

Proof. (2) $\Leftrightarrow$ (3): Let $a_0 \in \mathbb{F}_p P$ be as in Lemma 7.1. Then the map $w \mapsto a_0 w$ induces an isomorphism $M^n/[P, M^n] \to C_{M^n}(P)$. Up to multiplication by an invertible scalar we then have $w_i = a_0 v_i$, hence (2) $\Leftrightarrow$ (3).
(3) $\Rightarrow$ (1): If $\sum_i u_i = 0$ with $u_i \in N_{v_i}$ and not all $u_i = 0$, then by nilpotence we get a linear dependence between the w_i.
(1) $\Rightarrow$ (3): If $\sum_i N_{v_i}$ is direct, then $\sum_i C_{N_{v_i}}(P)$ is direct, too. □

Lemma 7.4. *Let $M \neq 0$ be a length ℓ uniserial $\mathbb{F}_p P$-module. For an $\mathbb{F}_p P$-submodule N of M^n, the following four statements are equivalent:*

1. *N is a direct summand of M^n.*
2. *N has a generating set $v_1, \ldots, v_r$ satisfying the equivalent conditions of Lemma 7.3.*
3. *The $\mathbb{F}_p$-vector spaces $(N + [P, M^n])/[P, M^n]$ and $C_N(P)$ have the same dimension.*
4. *M_Z is a complement of N in M^n for some $Z \subseteq \{1, 2, \ldots, n\}$. Here, $M_Z = \{(u_1, \ldots, u_n) \in M^n \mid u_i = 0 \text{ for all } i \notin Z\}$.*

If these equivalent conditions hold then:

5. *For any complement L of N in M^n, the normal subgroup N of $M^n \rtimes P$ has complement $L \rtimes P$.*
6. *With r as in (2), we have $\dim C_N(P) = r$ in (3) and $|Z| = n - r$ in (4).*

Proof. (1) $\Rightarrow$ (2): M^n is a direct sum of n copies of the length ℓ uniserial module M. By Krull-Schmidt, N is also a direct sum of length ℓ uniserial modules; and uniserial modules are cyclic.
(2) $\Rightarrow$ (3) and first part of (6): As $N = \bigoplus_{i=1}^r N_{v_i}$ we have $\dim C_N(P) = r$ since $\dim C_{N_{v_i}}(P) = 1$, and $\dim N/[P, N] = r$ since $\dim N_{v_i}/[P, N_{v_i}] = 1$.
(3) $\Rightarrow$ (4) and second part of (6): $C_N(P)$ is a subspace of $C_{M^n}(P)$. Pick $0 \neq w \in C_M(P)$, and define $w_i \in C_{M^n}(P)$ by $w_i = (0, \ldots, 0, w, 0, \ldots, 0) \in M^n$, with w in the ith position. Then $w_1, \ldots, w_n$ is a basis of $C_{M^n}(P)$, so by the exchange lemma there is $Z \subseteq \{1, \ldots, n\}$ such that the subspace $W_Z \subseteq C_{M^n}(P)$ on the w_i with $i \in Z$ is a complement to $C_N(P)$. In particular, $|Z| = n - \dim C_N(P)$.

Since M_Z has socle W_Z and N has socle $C_N(P)$, it follows that the sum $M_Z + N$ is direct. Now suppose that $x \in M_Z$, $y \in N$ and $x + y \in [P, M^n]$. With a_0 as in Lemma 7.1 we have $a_0 x + a_0 y = 0$. Since $M_Z + N$ is direct, it follows that $a_0 x = a_0 y = 0$, and hence $x, y \in [P, M^n]$. Therefore

$$\dim \frac{(M_Z \oplus N) + [P, M^n]}{[P, M^n]} = \dim \frac{M_Z + [P, M^n]}{[P, M^n]} + \dim \frac{N + [P, M^n]}{[P, M^n]}$$
$$= (n - \dim C_N(P)) + \dim C_N(P) = n\,.$$

Hence $(M_Z \oplus N) + [P, M^n] = M^n$, so as $[P, \_]$ is nilpotent we conclude that $M_Z \oplus N = M^n$.
Finally, (4) ⇒ (1) and (5) are clear. □

8. Uniserial modules and P_j

Remark 8.1. *By Lemma 5.6 the natural action of P_n on $M = (\mathbb{F}_p)^{p^n}$ is uniserial of length p^n. Observe that the socle $C_M(P_n)$ is the diagonal subgroup*

$$\Delta(\mathbb{F}_p) = \{\underline{v} \in (\mathbb{F}_p)^{p^n} \mid v_i = v_j \text{ for all } i, j\}\,,$$

and that

$$[P_n, M] = \{\underline{v} \in (\mathbb{F}_p)^{p^n} \mid \sum_i v_i = 0\}\,.$$

Lemma 8.2. *Let $0 \le j \le n$. Set $K := [T_j, T_j]$. For any $U \le P_n$ write $\bar{U} = UK/K$. Then $K \le T_j$, and*

1. *The quotient module $\bar{T}_j = T_j/K$ is an $\mathbb{F}_p P_j$-module.*
2. *$\bar{T}_j$ is the direct sum of $n - j$ length p^j uniserial modules isomorphic to A^j; these summands are generated by $\sigma_j K, \sigma_{j+1} K, \dots, \sigma_{n-1} K$.*
3. *$C_{\bar{T}_j}(P_j)$ is the $\mathbb{F}_p$-vector space with basis $\Delta(\sigma_j)K, \dots, \Delta(\sigma_{n-1})K$.*
4. *If L is a direct summand of the $\mathbb{F}_p P_j$-module $\bar{T}_j$, then there is some $Z \subseteq \{\sigma_j, \dots, \sigma_{n-1}\}$ such that $\bar{T}_j = L \oplus M_Z$, where $M_Z \subseteq \bar{T}_j$ is the submodule generated by $\{\sigma_i K \mid \sigma_i \in Z\}$.*

Proof. (1): In the factorization $P_n = P_j T_j$, note that $P_j = \langle \sigma_0, \dots, \sigma_{j-1} \rangle$, and that T_j is the normal closure of $\langle \sigma_j, \dots, \sigma_{n-1} \rangle$ under the action of P_j. Since each σ_i has order p, the abelianization of T_j is elementary abelian.
(2): The submodule of $\bar{T}_j$ generated by $\sigma_i K$ has basis consisting of the p^j conjugates of $\sigma_i K$ under the action of P_j. So $\bar{T}_j$ is the direct sum of these submodules, and each is isomorphich to A^j. As we recalled in Remark 8.1, Weir showed that A^j is uniserial of length p^j.
(3): $C_{\bar{T}_j}(P_j)$ is the diagonal subgroup by Remark 8.1.
(4): Lemma 7.4, specifically (1) ⇔ (4). □

For the next lemma we suppose that L is a direct summand of the $\mathbb{F}_p P_j$-module $\bar{T}_j$. From Lemma 8.2 we know that $\bar{T}_j = L \oplus M_Z$ for some $Z \subseteq \{\sigma_j, \dots, \sigma_{n-1}\}$; and that $C_{\bar{T}_j}(P_j)$ has basis $\Delta(\sigma_j)K, \dots, \Delta(\sigma_{n-1})K$.

Lemma 8.3. *Under these circumstances we have:*

1. *If $L \not\subseteq \bar{T}_{j+1} + [P_j, \bar{T}_j]$ then we may choose Z such that $\sigma_j \notin Z$.*
2. *If $L + [P_j, \bar{T}_j] = \bar{T}_{j+1} + [P_j, \bar{T}_j]$ then $Z = \{\sigma_j\}$.*

(3) *If* $L+[P_j,\bar{T}_j] \subsetneq \bar{T}_{j+1}+[P_j,\bar{T}_j]$ *then for every complement* D *of* L *there are* $x, y \in C_D(P_j)$ *such that* $\Delta(\sigma_j)K$ *lies in the support of* x*, whereas the support of* $y \neq 0$ *does not contain* $\Delta(\sigma_j)K$.

Proof. Let D be a complement of L. By Lemma 7.1 there is some $a_0 \in \mathbb{F}_p P_j$ such that $a_0 \bar{T}_j = C_{T_j}(P_j)$, and hence

$$C_{\bar{T}_j}(P_j) = C_L(P_j) \oplus C_D(P_j)\,.$$

Moreover, multiplication by a_0 induces an isomorphism $\dfrac{\bar{T}_j}{[P_j,\bar{T}_j]} \xrightarrow[\cong]{\mu} C_{\bar{T}_j}(P_j)$ which restricts to isomorphisms

$$\frac{L+[P_j,\bar{T}_j]}{[P_j,\bar{T}_j]} \xrightarrow{\cong} C_L(P_j) \quad \text{and} \quad \frac{D+[P_j,\bar{T}_j]}{[P_j,\bar{T}_j]} \xrightarrow{\cong} C_D(P_j)\,.$$

(1): Recall from the proof of Lemma 7.4 that Z is chosen using the exchange lemma: Z is any subset of $\{\sigma_j,\ldots,\sigma_{n-1}\}$ such that $\{\Delta(\sigma_i)K \mid \sigma_i \in Z\}$ is the basis of a complement to $C_L(P_j)$. Now, $\dfrac{\bar{T}_{j+1}+[P_j,\bar{T}_j]}{[P_j,\bar{T}_j]}$ is precisely the preimage under μ of the subspace spanned by $\Delta(\sigma_{j+1})K,\ldots,\Delta(\sigma_{n-1})K$. So by assumption there is some $x \in L$ such that a_0x has $\Delta(\sigma_j)K$ in its support. Beginning the exchange lemma with x, we can ensure that $\sigma_j \notin Z$.
(2): $C_L(P_j)$ has basis $\Delta(\sigma_{j+1})K,\ldots,\Delta(\sigma_{n-1})K$, hence $Z=\{\sigma_j\}$.
(3): $C_L(P_j)$ is a proper subspace of the span of $\Delta(\sigma_{j+1})K,\ldots,\Delta(\sigma_{n-1})K$. The result follows, as $C_D(P_j)$ is a complement in $C_{\bar{T}_j}(P_j)$. □

9. Complements and uniserial modules

We recall some notation from Lemma 8.2: So $K=[T_j,T_j]$, and $\bar{U}=UK/K$.

Lemma 9.1. *Suppose that* $N \trianglelefteq P_n$ *has a complement* C*. Set* $D=T_j \cap C$*, where* j *is the depth of* N*. Then*

(1) D *is elementary abelian, and* $\bar{D} \cong D$.
(2) $\bar{D}$ *is an* $\mathbb{F}_pP_j$ *module, and* $\bar{T}_j = \bar{N} \oplus \bar{D}$.

Proof. (1): Since $D \leq T_j$, $D \cap N = 1$ and $K \leq N$, the map from T_j to its abelianization $\bar{T}_j = T_j/K$ is injective on D. By Lemma 8.2, the abelianization is elementary abelian.
(2): D is a group-theoretic complement of N in T_j, but it is conceivable that it is not normalized by P_j. However, if $a \in P_j$ then $a=cn$ with $c \in C$ and $n \in N$, so for $d \in D$ we have ${}^a d = {}^c(d \cdot d^{-1}ndn^{-1}) \in {}^c(DK) = DK$. Hence ${}^a\bar{D} = \bar{D}$. □

Lemma 9.2. *Let* Q *be a* p*-group and* $P = Q \wr C_p$*, so* $P = B \rtimes C_p$ *with* $B=Q^p$*. Suppose that* $x \in P \setminus B$ *and* $y \in B \cap C_P(x)$*. If* $y^p=1$ *then* $y \in P'$.

Proof. We have $x=(q_0,\ldots,q_{p-1})\sigma$, with σ a p-cycle. Rearranging the factors in $B=Q^p$ if necessary, we may assume that $\sigma = (0\ 1\ 2\ \ldots\ p-2\ p-1)$, so $\sigma(i)=i+1$ modulo p.
Special case: Q abelian: Since y lies in B it has the form $y=(q'_0,\ldots,q'_{p-1})$. As Q is abelian we have ${}^x y = {}^\sigma y = (q'_{p-1},q'_0,q'_1,\ldots,q'_{p-2})$; and so since $y \in C_P(x)$ it follows that $y=(q,q,\ldots,q)$ for some $q \in Q$. As $y^p=1$ we have $q^p=1$. Now set $z=(1,q,q^2,\ldots,q^{p-1}) \in Q^p$, then ${}^\sigma z = (q,q^2,\ldots,q^{p-1},1) = (q,q^2,\ldots,q^p) = yz$. So $y = {}^\sigma z \cdot z^{-1} \in P'$.

General case: We have $B' = (Q')^p$, so $P/B' \cong (Q/Q') \wr C_p$ and $yB' \in (P/B')'$ by the special case. Hence $y \in P'$. □

Lemma 9.3. *Suppose that $N \trianglelefteq P_n$ has depth j. If N has a complement in P_n, then $\bar{N} + [P_j, \bar{T}_j]$ is not a proper subgroup of $\bar{T}_{j+1} + [P_j, \bar{T}_j]$.*

Proof. Call the complement C, and set $D = C \cap T_j$. Lemma 9.1 says that the $\mathbb{F}_p P_j$-module $\bar{D}$ is a complement of $\bar{N}$ in $\bar{T}_j$. If $\bar{N} + [P_j, \bar{T}_j]$ is a proper subgroup of $\bar{T}_{j+1} + [P_j, \bar{T}_j]$ then Lemma 8.3 says that there are $x, y \in D$ such that the support of $xK \in C_{\bar{D}}(P_j)$ contains $\Delta(\sigma_j)K$, and yK is a nontrivial element of $C_{\bar{T}_{j+1}}(P_j)$.
Lemma 9.1 says that $\langle x, y \rangle$ is elementary abelian. On the other hand, $x, y \in T_j \cong (P_{n-j})^{p^j}$. Let $x_i, y_i \in P_{n-j}$ be the images of x, y in the ith of these p^j factors, then $\langle x_i, y_i \rangle$ is elementary abelian, too. Moreover, our choice of x means that each x_i lies outside the base subgroup $(P_{n-j-1})^p$ of $P_{n-j} = P_{n-j-1} \wr C_p$; but y_i does lie in this base group, since $y \in T_{j+1}$. So $y_i \in P'_{n-j} \leq K \leq N$ by Lemma 9.2 and Proposition 4.6. As y is the product of the y_i, we have $y \in N \cap D = 1$: a contradiction. □

Lemma 9.4. *Suppose that $N \trianglelefteq P_n$ has depth j. Then*

(1) *If $K \leq L \leq T_j$, then $LP'_n/P'_n \cong \dfrac{\bar{L} + [P_j, \bar{T}_j]}{[P_j, \bar{T}_j]}$.*
(2) *$NP'_n/P'_n = T_{j+1}P'_n/P'_n$ if and only if $\bar{N} + [P_j, \bar{T}_j] = \bar{T}_{j+1} + [P_j, \bar{T}_j]$.*
(3) *NP'_n/P'_n is a proper subgroup of $T_{j+1}P'_n/P'_n$ if and only if $\bar{N} + [P_j, \bar{T}_j]$ is a proper subgroup of $\bar{T}_{j+1} + [P_j, \bar{T}_j]$.*

Proof. By Proposition 4.6, $K = [T_j, T_j]$, satisfies $K \leq N \cap T_{j+1}$. Since $P_n = T_j.P_j$ we have $P'_n = K.[P_j, T_j].P'_j$ and therefore

$$P'_n \cap T_j = K.[P_j, T_j] \,.$$

(1): $L \cap P'_n = L \cap (P'_n \cap T_j)$, and so $LP'_n/P'_n \cong L(P'_n \cap T_j)/(P'_n \cap T_j)$. The result follows, since $L(P'_n \cap T_j) = LK[P_j, T_j] = L[P_j, T_j]$.
(2) and (3): Follow by applying (1) to the cases $L = N$ and $L = T_{j+1}$. □

10. The permutations ρ_i

Notation 10.1. Now suppose that $1 \leq i \leq n-1$. Observe that $\langle \sigma_{i-1}, \sigma_i \rangle \cong P_2$, whose centre is cyclic of order p and generated by $\prod_{s=0}^{p-1} {}^{\sigma_{i-1}^s}\sigma_i$. Set

$$\rho_i = \prod_{s=1}^{p-1} {}^{\sigma_{i-1}^s}\sigma_i \,,$$

so $\sigma_i \rho_i$ generates the centre of $\langle \sigma_{i-1}, \sigma_i \rangle$.

Lemma 10.2. (1) *$\rho_i^p = 1$.*
(2) *$\rho_i[T_j, T_j] = \sigma_i^{-1}[T_j, T_j] \; \forall \, i > j$.*
(3) *Every P_j-conjugate of ρ_i commutes with every P_j-conjugate of ρ_k, for all $i, k > j$.*
(4) *${}^{\eta_k}\rho_k = \rho_k^r$, and ${}^{\eta_k}\rho_i = \rho_i$ for all $i \neq k$, where r is as defined at the beginning of Section 6.*

Example 10.3. If $p^n = 3^3$ then

$$\rho_1 = (9\ 12\ 15)(10\ 13\ 16)(11\ 14\ 17)(18\ 21\ 24)(19\ 22\ 25)(20\ 23\ 26)$$
$$\rho_2 = (3\ 4\ 5)(6\ 7\ 8)\,.$$

Proof. (1): The σ_{i-1}-conjugates of σ_i commute with each other, and each has exponent p.
(2): $\sigma_i^p = 1$, and for $i > j$ have ${}^{\sigma_{i-1}^k}\sigma_i \in \sigma_i[T_j, T_j]$.
(3): Let $i > j$ and $\pi \in P_j$. Then ${}^{\pi}\rho_i$ only alters $(\lambda_0, \ldots, \lambda_{n-1})$ if $\lambda_{i-1} \neq 0$; $\lambda_k = 0$ for $j \leq k < i-1$; and $(\lambda_0, \ldots, \lambda_{j-1}) = \pi(0, \ldots, 0)$. If these conditions hold, then the value of λ_i is increased by 1. Any two such permutations commute with each other.
(4): By inspection. □

Proposition 10.4. *Suppose that $N \trianglelefteq P_n$. Set $j := \operatorname{depth}(N)$. (See Section 4.) Let H be as in Theorem 1.3. Then $K := T_j' \leq N$, and the following statements are equivalent:*

1. *N has a complement in P_n.*
2. *N has an H-invariant complement in P_n.*
3. *N/K is a direct summand of the $\mathbb{F}_p P_j$-submodule T_j/K, and NP_n'/P_n' is not a proper subgroup of $T_{j+1}P_n'/P_n'$.*

Remark 10.5. *As T_j/K is a direct sum of copies of the length p^j uniserial module A^j, one may use the equivalent conditions of Lemma 7.4 in order to determine whether N/K is a direct summand.*

Proof. Proposition 4.6 tells us that $K := [T_j, T_j] \leq N$. The implication (2) ⇒ (1) is clear. As in Lemma 8.2 we write $\bar{U} = UK/K$.
(1) ⇒ (3): Lemma 9.1 says that $\bar{N}$ is a direct summand of $\bar{T}_j$, and Lemma 9.3 says that $\bar{N} + [P_j, \bar{T}_j]$ is not a proper subgroup of $\bar{T}_{j+1} + [P_j, \bar{T}_j]$. So NP_n'/P_n' is not a proper subgroup of $T_{j+1}P_n'/P_n'$ by Lemma 9.4.
(3) ⇒ (2): By Lemma 9.4, $\bar{N} + [P_j, \bar{T}_j]$ is a not proper subgroup of $\bar{T}_{j+1} + [P_j, \bar{T}_j]$. So we are in one of the first two cases of Lemma 8.3.

Suppose case (1) applies. Let $D \leq T_j$ be the subgroup generated by all P_j-conjugates of the ρ_i for which $\sigma_i \in Z$. Lemma 10.2 says that D is elementary abelian, and by construction it is normalized by P_j. Moreover, the formula for ${}^{\eta_j}\sigma_i$ in the proof of Lemma 6.1 shows that P_j is H-invariant. So from Lemma 10.2 (4) we conclude that D and $C := D \rtimes P_j$ are H-invariant.

From $\sigma_j \notin Z$ and Lemma 10.2 (2) it follows that $\bar{D}$ is M_Z, which is a complement of $\bar{N}$ in $\bar{T}_j$. Moreover, D and M_Z are elementary abelian of the same rank, so $D \cap K = 1$ and $D \cap N \cong \bar{D} \cap \bar{N} = 1$. Hence $C \cap N = (T_j \cap C) \cap N = D \cap N = 1$, and C is a complement of N in P_n.

Now suppose case (2) applies. Let $D \leq T_j$ be the subgroup generated by all P_j-conjugates of σ_j. Then D is elementary abelian, $D \cap K = 1$, and $\bar{D}$ is a complement to $\bar{N}$ in $\bar{T}_j$. Hence $C = D \rtimes P_j$ is a complement to N in P_n. □

11. Proofs of the theorems

Proof of Theorem 1.3. If $p = 2$ then $H = 1$ by Proposition 1.5. For odd p, Corollary 6.5 says that the subgroup $H \leq S_{p^n}$ of Lemma 6.1 is a Hall p'-subgroup of the normalizer of $P_n = \langle \sigma_0, \dots, \sigma_{n-1} \rangle$. (See Lemma 3.2 for the definition of σ_i $i = 0, \dots, n-1$.) The result for this H follows from Proposition 10.4. □

Proof of Theorem 1.2. Since $C_{S_{p^n}}(P_n) = Z(P_n)$, the Hall p'-subgroup H of Theorem 1.3 embeds in $\mathrm{Aut}(P_n)$. Proposition 1.5 says that H is also a Hall p'-subgroup of $\mathrm{Aut}(P_n)$, and so $\mathrm{Aut}(P_n)$ is solvable. By Remark 6.3, every p'-subgroup of $\mathrm{Aut}(P_n)$ is conjugate to a subgroup of H. The result follows by Theorem 1.3. □

12. Examples

Example 12.1. This example concerns Lemma 7.2: it demonstrates that (3a) does not follow from (3b). For $p^n = 3^2$ let M be the length 9 uniserial P_2-module $M = (\mathbb{F}_3)^9$. Consider $v = (v_1, v_2) \in M^2$ given by

$$v_1 = (1, 0, 0, 0, 0, 0, 0, 0, 0) \qquad v_2 = (0, 0, 0, -1, 1, 0, 0, 0, 0)\,.$$

Note that $v_1 \notin [P_2, M]$ and $v_2 \in [P_2, M]$: so (3b) is satisfied. Setting

$$a = \sigma_1 - \mathrm{Id} = (0\ 1\ 2) - \mathrm{Id} \in \mathbb{F}_3 P_2 \qquad b = {}^{\sigma_0}a = (3\ 4\ 5) - \mathrm{Id} \in \mathbb{F}_3 P_2$$

we see that v fails to satisfy (3a), since

$$av_1 = (-1, 1, 0, 0, 0, 0, 0, 0, 0) \qquad av_2 = \underline{0}$$
$$bv_1 = \underline{0} \qquad bv_2 = (0, 0, 0, 1, 1, 1, 0, 0, 0)\,.$$

Since $0 \neq av \in M \oplus 0$ and $0 \neq bv \in 0 \oplus M$, it follows that $\mathrm{soc}(M^2) \subseteq M_v$. So (2) is also violated, as $C_{M^2}(P_2) = \mathrm{soc}(M^2)$; and since $\mathrm{soc}(M^2)$ has three dimension one submodules, M_v is not uniserial either, i.e. (1) is violated, too.

The remaining examples concern normal subgroups of P_n.

Remark 12.2. *We need a method for determining whether N has a complement, and constructing an H-invariant complement C if there is one.*
The proof of (3) ⇒ (2) in Proposition 10.4 can readily be adapted for this purpose. First one checks whether $\bar{N}$ is a direct summand of $\bar{T}_j$, possibly using the equivalent conditions of Lemma 7.4. If $\bar{N}$ is a direct summand, then there are three possibilities:

(1) *If $\bar{N}$ has a complement of the form M_Z with $Z \subseteq \{\sigma_{j+1}, \dots, \sigma_{n-1}\}$ then we take $C = \langle X \rangle$ for $X = \{\sigma_0, \dots, \sigma_{j-1}\} \cup \{\rho_i \mid \sigma_i \in Z\}$. This is the case $\sigma_j \notin Z$ and $N \nleq T_{j+1}P_n'$.*
(2) *If $NP_n'/P_n' = T_{j+1}P_n'/P_n'$ then $\bar{N}$ has complement M_Z for $Z = \{\sigma_j\}$. We take $C = \langle \sigma_0, \dots, \sigma_j \rangle$.*
(3) *If $NP_n'/P_n' \lneq T_{j+1}P_n'/P_n'$ then N has no complement in P_n.*

Sometimes it may be better to begin by comparing NP_n'/P_n' and $T_{j+1}P_n'/P_n'$.

Example 12.3. On why ρ_i replaces σ_i in Case (1) of Remark 12.2.

For $p^n = 3^3$ let N be the normal closure of $\langle \sigma_0 \rangle$ in P_3. Then $j = 0$, so $K = P_3'$ and $\bar{T}_0$ is the $\mathbb{F}_3$-vector space with basis $\sigma_0 K$, $\sigma_1 K$, $\sigma_2 K$. As $\bar{N}$ has basis $\sigma_0 K$, it has complement M_Z for $Z = \{\sigma_1, \sigma_2\}$. Case (1) of Remark 12.2 says that $C = \langle \rho_1, \rho_2 \rangle$ is an H-invariant complement of N in P_3.

Since this complement is elementary abelian, it has order 3^2. By contrast, $\langle \sigma_1, \sigma_2 \rangle \cong P_2$ has order 3^4. Hence $\langle \sigma_1, \sigma_2 \rangle \cong P_2$ is not a complement of N. In particular, $[\sigma_1, \sigma_2] \in \langle \sigma_1, \sigma_2 \rangle \cap N$, since $P_3' = K \leq N$.

Example 12.4. This example features Case (2) of Remark 12.2. More significantly, it demonstrates that the normal subgroup N need not be H-invariant.

For $p^n = 3^3$ let N be the normal closure of $\langle \gamma \sigma_2 \rangle$ in P_3, where $\gamma = \sigma_1 \cdot {}^{\sigma_0}\sigma_1 \cdot {}^{\sigma_0^2}\sigma_1$ is the product of the three $\langle \sigma_0 \rangle$-conjugates of σ_1.

Then N has depth $j = 1$, and $NP_3'/P_3' = \langle \sigma_2 \rangle P_3'/P_3' = T_2 P_3'/P_3'$. We are in Case (2) of Remark 12.2 – proivded that $\bar{N}$ does have a complement.

$\bar{T}_1$ is a direct sum of two copies of the uniserial $\mathbb{F}_3 P_1$-module A^1: one on $\sigma_1 K$, and the other on $\sigma_2 K$. Moreover, $\bar{N}$ is generated by $v = \gamma K + \sigma_2 K$. But γK lies in the socle of the summand on $\sigma_1 K$, hence v satisfies Lemma 7.2 (3) with $i = 2$. So v is a generating set for $\bar{N}$ satisfying the conditions of Lemma 7.3, meaning that $\bar{N}$ is a direct summand of $\bar{T}_1$ by Lemma 7.4. We conclude that N does have an H-invariant complement in P_3.

By Case (2), one H-invariant complement is $C = \langle \sigma_0, \sigma_1 \rangle$.

Observe that if $\pi_1 = \sigma_2, \pi_2, \ldots, \pi_9$ are the nine P_3-conjugates of σ_2 (see Example 3.8), then

$$N = \left\{ \gamma^{\sum_{i=1}^{9} e_i} \prod_{i=1}^{9} \pi_i^{e_i} \,\middle|\, e_1, \ldots, e_9 \in \mathbb{Z} \right\} .$$

So as η_2 fixes σ_0, σ_1 and inverts σ_2, we have

$${}^{\eta_2}N = \left\{ \gamma^{-\sum_{i=1}^{9} e_i} \prod_{i=1}^{9} \pi_i^{e_i} \,\middle|\, e_1, \ldots, e_9 \in \mathbb{Z} \right\} \neq N .$$

So the normal subgroups N and ${}^{\eta_2}N$ of P_3 fail to be H-invariant – and yet each of them has a H-invariant complement in C.

Example 12.5. This example features Case (3) of Remark 12.2.

As in Example 12.4 we let N be the normal closure of $\langle \gamma \sigma_2 \rangle$, but this time we take $p^n = 3^4$, and so N is the normal closure in P_4. Once more, the depth is $j = 1$ and $\bar{N}$ is uniserial of length 3 and hence a direct summand of $\bar{T}_1$. However this time $NP_4'/P_4' = \langle \sigma_2 \rangle P_4'/P_4'$ is a proper subgroup of $T_2 P_4'/P_4' = \langle \sigma_2, \sigma_3 \rangle P_4'/P_4'$. So N does not have a complement in P_4, even though $\bar{N}$ is a direct summand of $\bar{T}_1$.

Example 12.6. The H-invariant complement need not be unique. Also, the distinction between cases (1) and (2) of Remark 12.2 is slightly arbitrary: for $N \not\leq T_{j+1} P_n'$ there may be Z with $\bar{T}_j = \bar{N} \oplus M_Z$ and $\sigma_j \in Z$.

For $p^n = 3^2$ let $N \trianglelefteq P_2$ be the normal closure of $\langle \sigma_0\sigma_1 \rangle$. This has depth $j = 0$, so $\bar{T}_0 = P_2/P_2'$ is the $\mathbb{F}_3$-vector space with basis $\sigma_0 K, \sigma_1 K$, and $\bar{N}$ is the subspace spanned by $\sigma_0 K + \sigma_1 K$. So we may take $Z = \{\sigma_1\}$, obtaining the H-invariant complement $\langle \rho_1 \rangle$; or we may take $Z = \langle \sigma_0 \rangle$, obtaining the H-invariant complement $\langle \sigma_0 \rangle$. Observe that $\langle \sigma_1 \rangle$ is a third H-invariant complement.

Example 12.7. In this example, $\bar{N}$ is not a direct summand of $\bar{T}_j$.

For $p^n = 3^4$ we let N be the normal closure of $\langle \beta \rangle$ in P_4, for $\delta = \sigma_2 \cdot {}^{\sigma_0}(\sigma_3^{-1} \cdot {}^{\sigma_1}\sigma_3)$. So $j = 2$ and $\bar{T}_2$ is the direct sum of two copies of A^2, which is uniserial of length 9; and one can verify that δK corresponds to the element v of Example 12.1. So $\bar{N}$ is not a direct summand of $\bar{T}_2$.

13. Partition subgroups

Remark 13.1. *Following Weir* [30, p. 537] *we define A_i^{n-1} inductively for $i \geq 0$ by $A_0^{n-1} = A^{n-1}$ and $A_{i+1}^{n-1} = [P_n, A_i^{n-1}]$. Then each A_i^{n-1} is normal in P_n, whence $A_i^{n-1} \leq A_{i-1}^{n-1}$. By* [30, Theorem 2]*, the factor group A_{i-1}^{n-1}/A_i^{n-1} is cyclic of order p for all $i \leq \log_p(|A^{n-1}|) = p^{n-1}$, and $A_{p^{n-1}}^{n-1} = 1$.*

Since $P_n = A^{n-1} \rtimes P_{n-1}$, one may view A_i^j as a subgroup of P_n for all $0 \leq j \leq n-1$. Since A_i^{n-1} is normal in P_n, it follows that every product of the form $Q = A_{i_0}^0 A_{i_1}^1 \cdots A_{i_{n-1}}^{n-1}$ is a subgroup of P_n. Weir calls the subgroups of this form partition subgroups [30, p. 538].

Observe that the depth of the partition subgroup $A_{i_0}^0 A_{i_1}^1 \cdots A_{i_{n-1}}^{n-1}$ is the smallest j such that $i_j < p^j$. Weir [30, Theorem 4] *shows that a depth j partition subgroup is normal in P_n if and only if $i_k \leq p^j$ for all $k \geq j$.*

Lemma 13.2. *T_j' is the partition subgroup $A_{p^j}^{j+1} \cdots A_{p^j}^{n-1}$.*

Proof. Suppose that $k \leq s \leq n-1$. Weir [30, Lemma 2] shows that if $x \in T_k \setminus T_{k+1}$ then the smallest normal subgroup of P_{s+1} containing $[A^s, x]$ is $A_{p^k}^s$. This and the fact that A^j is abelian imply the result. □

Proposition 13.3. *Suppose that $N = A_{i_j}^j A_{i_{j+1}}^{j+1} \cdots A_{i_{n-1}}^{n-1}$ is a depth j normal partition subgroup of P_n. Then the following three statements are equivalent:*

(1) *N has a complement in P_n.*
(2) *N has an H-invariant complement in P_n.*
(3) *$i_j = 0$ and $i_k \in \{0, p^j\}$ for all $j \leq k \leq n-1$.*

Proof. (1) and (2) are equivalent by Proposition 10.4, and it suffices to show that (3) is equivalent to $\bar{N} = N/T_j'$ being a direct summand of the $\mathbb{F}_p P_j$-module $\bar{T}_j$, and NP_n'/P_n' not being a proper subgroup of $T_{j+1}P_n'/P_n'$.

Since $A_{p^j}^j = 1$, the $\mathbb{F}_p P_j$-module $\bar{N} = N/T_j'$ is the direct sum

$$\bar{N} = \bigoplus_{k=j}^{n-1} A_{i_k}^k / A_{p^j}^k \,.$$

Now, $A_{i_k}^k / A_{p^j}^k$ is uniserial of length $p^j - i_k$, whereas $\bar{T}_j$ is a direct sum of several copies of a length p^j uniserial module. By Lemmas 7.4 and 7.3 it follows that $\bar{N}$ is a direct summand of $\bar{T}_j$ if and only if

$i_k \in \{0, p^j\}$ for all $j \leq k \leq n-1$. Finally, if $i_j = 0$ then NP_n'/P_n' is not subgroup of $T_{j+1}P_n'/P_n'$, let alone a proper subgroup; whereas $i_j = p^j$ would mean that N has depth $j+1$, a contradiction. □

Lemma 13.4. *If $N \trianglelefteq P_n$ has a complement then there is a partition subgroup $Q \trianglelefteq P_n$ such that N and Q have a common H-invariant complement.*

Proof. From the proof of Proposition 13.3 one sees that partition subgroups with complements always fall into Case (1) of Remark 12.2, with $Z = \{\sigma_k \mid i_k = p^j\}$. Conversely, every $Z \subseteq \{\sigma_{j+1}, \ldots, \sigma_{n-1}\}$ occurs in this way. That leaves Case (2): $\langle \sigma_0, \ldots, \sigma_j \rangle$ is a complement of the partition subgroup $T_{j+1} = A_0^{j+1} \cdots A_0^{n-1}$, which has depth $j+1$ rather than j. □

Appendix A. Abelian subgroups of largest size

Proposition A.1. *Let p be an arbitrary prime, $n \geq 2$ and P_n a Sylow p-subgroup of S_{p^n}. Set*

$$d = \max\{|A| \mid A \leq P_n,\ A \text{ abelian}\}, \quad \text{and}$$
$$\mathcal{M} = \{A \leq P_n \mid A \text{ abelian and } |A| = d\}.$$

Then

1. *$d = p^{p^{n-1}}$, even in the case $n = 1$.*
2. *If p is odd then $\mathcal{M} = \{A^{n-1}\}$.*
3. *If $p = 2$ then $|M| = 3^{2^{n-2}}$, and every $A \in \mathcal{M}$ lies in $T_{n-2} \cong (D_8)^{2^{n-2}}$.*
4. (see [5, Thm 4.4.6]) *If $p = 2$ then $\{C \in \mathcal{M} \mid C \trianglelefteq P_n\} = \{A^{n-1}, B, W\}$, where $B \cong (C_4)^{2^{n-2}}$ is the characteristic subgroup of Proposition 5.1 and W is conjugate to A^{n-1} under the action of the outer automorphism group. Moreover, B is the only exponent four homocyclic group in $\mathcal{M}$.*

Proof. (1): For $n = 1$ we have $d = p$, since $P_1 \cong C_p$, so assume $n \geq 2$. Then $P_n \cong P_{n-1} \wr C_p = Q \rtimes C_p$ for $Q = P_{n-1}^p$; and $d \geq p^{p^{n-1}}$ since A^{n-1} is abelian.

Now suppose $n = 2$. If $C \leq P_2$ is abelian with $C \nleq Q$ then C contains some $x \in P_2 \setminus Q$. As $Q = A^1$ is abelian, conjugation by x acts on $Q = (C_p)^p$ by permuting the p factors cyclically. Hence $C_Q(x)$ is the diagonal subgroup of $(C_p)^p$, which is cyclic of order p. Since $C \cap Q \leq C_Q(x)$ we have

$$|C| = p\,|C \cap Q| \leq p^2 \leq p^p = \left|A^1\right|.$$

So if p is odd, then $|C| < \left|A^1\right|$, whence $d = p^p$ and $\mathcal{M} = \{A^1\}$. If $p = 2$, then $P_2 \cong D_8$, so $d = 2^2$ and $\mathcal{M}$ consists of the three maximal subgroups of D_8.

Now suppose $n > 2$. Again let $C \leq P_n$ be abelian with $C \nleq Q$. Set $D = C \cap Q$. From $|P_n : Q| = p$ it follows that $|C : D| = p$. Now, $D \leq Q = P_{n-1}^p$, so we may consider the projection D_i onto the ith factor P_{n-1}. Then each D_i is abelian, and $D \leq \bar{D} = \prod_{i=1}^p D_i$.

Pick $x \in C \setminus D$; then x normalizes D, hence conjugation by x permutes the D_i transitively, and so $|D_i| = |D_1|$ for all i. Moreover $D \cap D_1 = 1$: for conjugation by x fixes D pointwise, but it also maps every $1 \neq y \in D_1$ into one of the other factors D_i. Hence $\left|\bar{D} : D\right| \geq |D_1|$ and so $|D| \leq |D_1|^{p-1}$.

By induction we have $|D_1| \leq p^{p^{n-2}}$. Hence

$$|C| = p\,|D| \leq p\,|D_1|^{p-1} \leq p^{p^{n-1}-p^{n-2}+1} < p^{p^{n-1}} = \left|A^{n-1}\right| .$$

(2), (3): The proof of (1) deals with the case $n = 2$, so assume $n \geq 3$. Let $C \in \mathcal{M}$. Then $C \leq Q$ by the proof of (1), so as above we have $C \leq \bar{D} = \prod_{i=1}^{p} D_i$, with D_i the projection of C onto the ith factor of Q. As $\bar{D}$ is abelian we have $C = \bar{D}$ by maximality, and again by maximality, D_i lies in the $\mathcal{M}$ for P_{n-1}. These two cases follow by induction.
(4): T_{n-2} is the direct product of 2^{n-2} copies of D_8, each of which has three order four subgroups; two being elementary abelian, and one cyclic of order four. Proposition 5.1 shows that B is characteristic. As $P_n \cong D_8 \wr P_{n-2}$ and P_{n-2} acts by permuting the factors D_8 transitively, a normal subgroup must have the same projection onto every copy D_8: hence there are only three normal subgroup. Finally, let $\alpha \in \mathrm{Aut}(D_8)$ be the automorphism which interchanges the two elementary abelian subgroups of rank two: α can be constructed as an inner automorphism in D_{16}. Then the automorphism $\alpha \wr \mathrm{Id}$ of P_n interchanges the two elementary abelian normal subgroups in $\mathcal{M}$. □

Corollary A.2. *P_n has p-rank p^{n-1} for all primes p.* □

Acknowledgments

We thank the referee for his helpful comments. Héthelyi and Horváth received support from the Hungarian National Science Foundation Research Grant No. 77476, and from the National Research, Development and Innovation Office - NKFIH Grant No. 115288 and 115799.

References

[1] Y. Berkovich, Some consequences of Maschke's theorem, *Algebra Colloq.*, **5** no. 2 (1998) 143–158.

[2] Y. Berkovich, *Groups of prime power order*,. V 1, de Gruyter Expositions in Mathematics, **46** Walter de Gruyter GmbH & Co. KG, Berlin, 2008.

[3] Yu. V. Bodnarchuk, Structure of the group of automorphisms of a Sylow p-subgroup of the symmetric group S_{p^n} ($p \neq 2$), *Ukrainian Math. J.*, **36** no. 6 (1984) 512–516.

[4] H. Cárdenas and E. Lluis, El normalizador del p-grupo de Sylow del grupo simetrico S_{p^n} [The normalizer of the Sylow p-group of the symmetric group S_{p^n}], *Bol. Soc. Mat. Mexicana*, **9** (1964) 1–6.

[5] S. Covello,*Minimal parabolic subgroups in the symmetric groups*, M. Phil. Thesis, University of Birmingham, 1998.

[6] Yu. V. Dmitruk, Structure of Sylow two-subgroups of the symmetric group of degree 2^n, *Ukrainian Math. J.*, **30** no. 2 (1978) 117–124.

[7] Yu. V. Dmitruk and V. I. Sushchanskii, Structure of Sylow 2-subgroups of the alternating groups and normalizers of Sylow subgroups in the symmetric and alternating groups, *Ukrainian Math. J.*, **33** no. 3 (1981) 235–241.

[8] K. Doerk and T. Hawkes, *Finite soluble groups*, De Gruyter Expositions in Mathematics **4**, Walter de Gruyter, Berlin, New York, 1992.

[9] S. Dolfi, Large orbits in coprime actions of solvable groups, *Trans. Amer. Math. Soc.*, **360** (2008) 135–152.

[10] P. Flavell, The fixed points of coprime action, *Arch. Math.*, **75** no. 3 (2000) 173–177.

[11] D. Gluck, Coprime actions with all orbit sizes small, *Proc. AMS*, **143** no. 6 (2015) 2371–2377.

[12] D. Gorenstein, *Finite groups*, Chelsea Publ. Co., New York, 1980.

[13] L. Héthelyi and E. Horváth, Galois actions on blocks and classes of finite groups, *J. Algebra*, **320** (2008) 660–679.

[14] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin, Heidelberg, New York 1967.

[15] I. M. Isaacs, *Finite group theory*, Graduate Studies in Mathematics, **92**,American Mathematical Society, Providences, Rhode Island, 2008.

[16] I. M. Isaacs, M. L. Lewis and G. Navarro, Invariant characters and coprime actions on finite nilpotent groups, *Arch. Math.*, **74** no. 6 (2000) 401–403.

[17] L. Kaloujnine, La structure des p-groupes de Sylow des groupes symétriques finis, *Ann. Sci. École Norm. Sup. (3),* **65** (1948) 239–276.

[18] H. Kurzweil and B. Stellmacher, *The theory of finite groups*, Springer Universitext, 2004.

[19] C. R. Leedham-Green and S. McKay, *The structure of groups of prime power order*, London Mathematical Society Monographs, New Series, **27** Oxford University Press, Oxford, 2002.

[20] Zhengxing Li, Coleman automorphisms of permutational wreath products, *Comm. Alg.*, **44** no. 9 (2016) 3933–3938.

[21] G. Malle, G. Navarro and B. Späth, Invariant blocks under coprime actions, *Doc. Math.*, **20** (2015) 491–506.

[22] V. D. Mazurov, Finite groups with metacyclic Sylow 2-subgroups, *Sib. Math. J.*, **8** no. 5 (1967) 733–745.

[23] A. Moreto and L. Sanus, Coprime actions and degrees of primitive inducers of invariant characters, *Bull. Austral. Math. Soc.*, **64** (2001) 315–320.

[24] R. C. Orellana, M. E. Orrison and D. N. Rockmore, Rooted trees and iterated wreath products, *Adv. in Appl. Math.*, **33** (2004) 531–547.

[25] J. M. Riedl, Automorphisms of regular wreath product p-groups, *Int. J. Math. Math. Sci.*, (2009), doi:10.1155/2009/245617.

[26] D. N. Rockmore, Fast Fourier transforms and wreath products, *Appl. Comput. Harmon. Anal.*, **2** no. 3 (1995) 279–292.

[27] M. Seong and A. Wu, Generalized iterated wreath products of cyclic groups and rooted tree correspondence, arXiv:1409.0603v1[math.RT] 2, 2014.

[28] S. Sidki, *Regular trees and their automorphisms*, Monografias de Matematica, **56** , IMPA, Rio de Janeiro, 1998.

[29] R. Waldecker, A theorem about coprime action, *J. Algebra*, **320** (2008) 2027–2030.

[30] A. J. Weir, The Sylow subgroups of the symmetric groups, *Proc. Amer. Math. Soc.*, **6** (1955) 534–541.

[31] A. Woryna, The automaton realizations of iterated wreath products of cyclic groups, *Comm. Alg.*, **42** no. 3 (2014) 1354–1361.

D. J. Green

Institut für Mathematik, Friedrich-Schiller-Universität Jena, P.O.Box 07737, Jena, Germany

`Email: david.green@uni-jena.de`

L. Héthelyi

Department of Algebra, Budapest University of Technology and Economics, P.O.Box 1111, Budapest, Hungary

`Email: fobaba@t-online.hu`

E. Horváth

Department of Algebra, Budapest University of Technology and Economics, P.O.Box 1111, Budapest, Hungary

`Email: he@math.bme.hu`