DIFFERENCE BASES IN DIHEDRAL GROUPS

TARAS BANAKH AND VOLODYMYR GAVRYLKIV

Communicated by Attila Maroti

Abstract. A subset $B$ of a group $G$ is called a difference basis of $G$ if each element $g \in G$ can be written as the difference $g = ab^{-1}$ of some elements $a, b \in B$. The smallest cardinality $|B|$ of a difference basis $B \subseteq G$ is called the difference size of $G$ and is denoted by $\Delta[G]$. The fraction $\mathcal{D}[G] := \frac{\Delta[G]}{\sqrt{|G|}}$ is called the difference characteristic of $G$. We prove that for every $n \in \mathbb{N}$ the dihedral group $D_{2n}$ of order $2n$ has the difference characteristic $\sqrt{2} \leq \mathcal{D}[D_{2n}] \leq \frac{48}{\sqrt{890}} \approx 1.983$. Moreover, if $n \geq 2 \cdot 10^{15}$, then $\mathcal{D}[D_{2n}] < \frac{4}{7\sqrt{2}} \approx 1.633$. Also we calculate the difference sizes and characteristics of all dihedral groups of cardinality $\leq 80$.

1. Introduction

A subset $B$ of a group $G$ is called a difference basis for a subset $A \subseteq G$ if each element $a \in A$ can be written as $a = xy^{-1}$ for some $x, y \in B$. The smallest cardinality of a difference basis for $A$ is called the difference size of $A$ and is denoted by $\Delta[A]$. For example, the set $\{0, 1, 4, 6\}$ is a difference basis for the interval $A = \{-6, 6\} \cap \mathbb{Z}$ witnessing that $\Delta[A] \leq 4$.

The definition of a difference basis $B$ for a set $A$ in a group $G$ implies that $|A| \leq |B|^2$ and gives a lower bound $\sqrt{|A|} \leq \Delta[A]$. The fraction

$$\mathcal{D}[A] := \frac{\Delta[A]}{\sqrt{|A|}} \geq 1$$

is called the difference characteristic of $A$.

For a real number $x$ we put

$$\lfloor x \rfloor = \min \{ n \in \mathbb{Z} : n \geq x \} \text{ and } \lceil x \rceil = \max \{ n \in \mathbb{Z} : n \leq x \}.$$
The following proposition is proved in [3, 1.1].

**Proposition 1.** Let $G$ be a finite group. Then

1. $1 + \sqrt{4|G|^3} \leq |G| \leq \left\lfloor \frac{|G|}{2} \right\rfloor$,
2. $\Delta(G) \leq \Delta(H) \cdot \Delta(G/H)$ and $\partial(G) \leq \partial(H) \cdot \partial(G/H)$ for any normal subgroup $H \subseteq G$;
3. $\Delta[G] \leq |H| + |G/H| - 1$ for any subgroup $H \subseteq G$.

In [10] Kozma and Lev proved (using the classification of finite simple groups) that each finite group $G$ has difference characteristic $\partial(G) \leq \frac{4}{\sqrt{3}} \approx 2.3094$.

In this paper we shall evaluate the difference characteristics of dihedral groups and prove that each dihedral group $D_{2n}$ has $\partial[D_{2n}] \leq \frac{48}{\sqrt{586}} \approx 1.983$. Moreover, if $n \geq 2 \cdot 10^{15}$, then $\partial[D_{2n}] < \frac{4}{\sqrt{6}} \approx 1.633$.

We recall that the dihedral group $D_{2n}$ is the isometry group of a regular $n$-gon. The dihedral group $D_{2n}$ contains a normal cyclic subgroup of index 2. A standard model of a cyclic group of order $n$ is the multiplicative group $C_n = \{z \in C : z^n = 1\}$ of $n$-th roots of 1. The group $C_n$ is isomorphic to the additive group of the ring $\mathbb{Z}/n\mathbb{Z}$.

Difference bases have applications in the study of structure of superextensions of groups, see [1, 3].

A subset $B$ of a group $G$ is called a basis of $G$ if each element $g \in G$ can be written as $g = ab$ for some $a, b \in B$. Bases in dihedral groups were studied in [7].

**Theorem 2.** For any numbers $n, m \in \mathbb{N}$ the dihedral group $D_{2nm}$ has the difference size

$$2\sqrt{nm} \leq \Delta[D_{2nm}] \leq \Delta[D_{2n}] \cdot \Delta[C_m]$$

and the difference characteristic $\sqrt{2} \leq \partial[D_{2nm}] \leq \partial[D_{2n}] \cdot \partial[C_m]$.

**Proof.** It is well-known that the dihedral group $D_{2nm}$ contains a normal cyclic subgroup of order $nm$, which can be identified with the cyclic group $C_{nm}$. The subgroup $C_m \subseteq C_{nm}$ is normal in $D_{2nm}$ and the quotient group $D_{2nm}/C_m$ is isomorphic to $D_{2n}$. Applying Proposition 1(2), we obtain the upper bounds $\Delta[D_{2n}] \leq \Delta[D_{2nm}/C_m] \cdot \Delta[C_m] = \Delta[D_{2n}] \cdot \Delta[C_m]$ and $\partial[D_{2nm}] \leq \partial[D_{2n}]$.

Next, we prove the lower bound $2\sqrt{nm} \leq \Delta[D_{2nm}]$. Fix any element $s \in D_{2nm} \setminus C_{nm}$ and observe that $s = s^{-1}$ and $sx^{-1}s^{-1} = x^{-1}$ for all $x \in C_{nm}$. Fix a difference basis $D \subseteq D_{2nm}$ of cardinality $|D| = \Delta[D_{2nm}]$ and write $D$ as the union $D = A \cup sB$ for some sets $A, B \subseteq C_{nm} \subseteq D_{2nm}$. We claim that $AB^{-1} = C_{nm}$. Indeed, for any $x \in C_{nm}$ we get $xs \in sC_{nm} \cap (A \cup sB)(A \cup sB)^{-1} = AB^{-1}s^{-1} \cup sBA^{-1}$ and hence

$$x \in AB^{-1}s^{-1} \cup sBA^{-1}s^{-1} = AB^{-1} \cup BA^{-1} = AB^{-1}.$$

So, $C_{nm} = AB^{-1}$ and hence $nm \leq |A| \cdot |B|$. Then $\Delta[D_{2nm}] = |A| + |B| \geq \min\{l + k : l, k \in \mathbb{N}, lk \geq nm\} \geq 2\sqrt{nm}$ and $\partial[D_{2nm}] = \frac{\Delta[D_{2nm}]}{\sqrt{2nm}} \geq \frac{2\sqrt{nm}}{\sqrt{2nm}} = \sqrt{2}$. \hfill $\square$

**Corollary 3.** For any number $n \in \mathbb{N}$ the dihedral group $D_{2n}$ has the difference size

$$2\sqrt{n} \leq \Delta[D_{2n}] \leq 2 \cdot \Delta[C_n]$$

and the difference characteristic $\sqrt{2} \leq \partial[D_{2n}] \leq \sqrt{2} \cdot \partial[C_n]$. 

The difference sizes of finite cyclic groups were evaluated in [2] with the help of the difference sizes of the order-intervals $[1, n] \cap \mathbb{Z}$ in the additive group $\mathbb{Z}$ of integer numbers. For a natural number $n \in \mathbb{N}$ by $\Delta[n]$ we shall denote the difference size of the order-interval $[1, n] \cap \mathbb{Z}$ and by $\delta[n] := \frac{\Delta[n]}{\sqrt{n}}$ its difference characteristic. The asymptotics of the sequence $(\delta[n])_{n=1}^{\infty}$ was studied by Rédei and Rényi [11], Leech [9] and Golay [8] who eventually proved that

$$\sqrt{2 + \frac{4}{3\pi}} < \sqrt{2 + \max_{0 < \varphi < 2\pi} \frac{2\sin(\varphi)}{\varphi + \pi}} \leq \lim_{n \to \infty} \delta[n] = \inf_{n \in \mathbb{N}} \delta[n] \leq \delta[6166] = \frac{128}{\sqrt{6166}} < \delta[6] = \sqrt{\frac{8}{3}}.$$

In [2] the difference sizes of the order-intervals $[1, n] \cap \mathbb{Z}$ were applied to give upper bounds for the difference sizes of finite cyclic groups.

**Proposition 4.** For every $n \in \mathbb{N}$ the cyclic group $C_n$ has difference size $\Delta[C_n] \leq \Delta[\lceil \frac{n-1}{2} \rceil]$, which implies that

$$\limsup_{n \to \infty} \delta[C_n] \leq \frac{1}{\sqrt{2}} \inf_{n \in \mathbb{N}} \delta[n] \leq \frac{64}{\sqrt{3083}} < \frac{2}{\sqrt{3}}.$$

The following upper bound for the difference sizes of cyclic groups were proved in [2].

**Theorem 5.** For any $n \in \mathbb{N}$ the cyclic group $C_n$ has the difference characteristic:

1. $\delta[C_n] \leq \delta[C_4] = \frac{3}{2}$;
2. $\delta[C_n] \leq \delta[C_2] = \delta[C_8] = \sqrt{2}$ if $n \neq 4$;
3. $\delta[C_n] \leq \frac{12}{\sqrt{73}} < \sqrt{2}$ if $n \geq 9$;
4. $\delta[C_n] \leq \frac{24}{\sqrt{293}} < \frac{12}{\sqrt{73}}$ if $n \geq 9$ and $n \neq 292$;
5. $\delta[C_n] < \frac{2}{\sqrt{3}}$ if $n \geq 2 \cdot 10^{15}$.

For some special numbers $n$ we have more precise upper bounds for $\Delta[C_n]$. A number $q$ is called a prime power if $q = p^k$ for some prime number $p$ and some $k \in \mathbb{N}$.

The following theorem was derived in [2] from the classical results of Singer [13], Bose, Chowla [4], [5] and Rusza [12].

**Theorem 6.** Let $p$ be a prime number and $q$ be a prime power. Then

1. $\Delta[C_{q^2 + q + 1}] = q + 1$;
2. $\Delta[C_{q^2 - 1}] \leq q - 1 + \Delta[C_{q - 1}] \leq q - 1 + \frac{3}{2}\sqrt{q - 1}$;
3. $\Delta[C_{p^2 - p}] \leq p - 3 + \Delta[C_p] + \Delta[C_{p - 1}] \leq p - 3 + \frac{3}{2}\sqrt{p + \sqrt{p - 1}}$.

The following Table 1 of difference sizes and characteristics of cyclic groups $C_n$ for $n \leq 100$ is taken from [2].
Using Theorem 6(1), we shall prove that for infinitely many numbers $n$ the lower and upper bounds given in Theorem 2 uniquely determine the difference size $\Delta[D_{2n}]$ of $D_{2n}$.

**Theorem 7.** If $n = 1 + q + q^2$ for some prime power $q$, then

$$
\Delta[D_{2n}] = 2 \cdot \Delta[C_n] = \lceil 2\sqrt{n} \rceil = \left\lceil \sqrt{2|D_{2n}|} \right\rceil = 2 + 2q.
$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\Delta[C_n]$</th>
<th>$\bar{d}[C_n]$</th>
<th>$n$</th>
<th>$\Delta[C_n]$</th>
<th>$\bar{d}[C_n]$</th>
<th>$n$</th>
<th>$\Delta[C_n]$</th>
<th>$\bar{d}[C_n]$</th>
<th>$n$</th>
<th>$\Delta[C_n]$</th>
<th>$\bar{d}[C_n]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>26</td>
<td>6</td>
<td>1.1766...</td>
<td>51</td>
<td>8</td>
<td>1.1202...</td>
<td>76</td>
<td>10</td>
<td>1.1470...</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1.4142...</td>
<td>27</td>
<td>6</td>
<td>1.1547...</td>
<td>52</td>
<td>9</td>
<td>1.2480...</td>
<td>77</td>
<td>10</td>
<td>1.1396...</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1.1547...</td>
<td>28</td>
<td>6</td>
<td>1.1338...</td>
<td>53</td>
<td>9</td>
<td>1.2362...</td>
<td>78</td>
<td>10</td>
<td>1.1322...</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>1.5</td>
<td>29</td>
<td>7</td>
<td>1.2998...</td>
<td>54</td>
<td>9</td>
<td>1.2247...</td>
<td>79</td>
<td>10</td>
<td>1.1250...</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>1.3416...</td>
<td>30</td>
<td>7</td>
<td>1.2780...</td>
<td>55</td>
<td>9</td>
<td>1.2135...</td>
<td>80</td>
<td>11</td>
<td>1.2298...</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>1.2247...</td>
<td>31</td>
<td>6</td>
<td>1.0776...</td>
<td>56</td>
<td>9</td>
<td>1.2026...</td>
<td>81</td>
<td>11</td>
<td>1.2222...</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>1.1338...</td>
<td>32</td>
<td>7</td>
<td>1.2374...</td>
<td>57</td>
<td>8</td>
<td>1.0596...</td>
<td>82</td>
<td>11</td>
<td>1.2147...</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>1.4142...</td>
<td>33</td>
<td>7</td>
<td>1.2185...</td>
<td>58</td>
<td>9</td>
<td>1.1817...</td>
<td>83</td>
<td>11</td>
<td>1.2074...</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>1.3333...</td>
<td>34</td>
<td>7</td>
<td>1.2004...</td>
<td>59</td>
<td>9</td>
<td>1.1717...</td>
<td>84</td>
<td>11</td>
<td>1.2001...</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>1.2649...</td>
<td>35</td>
<td>7</td>
<td>1.1832...</td>
<td>60</td>
<td>9</td>
<td>1.1618...</td>
<td>85</td>
<td>11</td>
<td>1.1931...</td>
</tr>
<tr>
<td>11</td>
<td>4</td>
<td>1.2060...</td>
<td>36</td>
<td>7</td>
<td>1.1666...</td>
<td>61</td>
<td>9</td>
<td>1.1523...</td>
<td>86</td>
<td>11</td>
<td>1.1861...</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>1.1547...</td>
<td>37</td>
<td>7</td>
<td>1.1507...</td>
<td>62</td>
<td>9</td>
<td>1.1430...</td>
<td>87</td>
<td>11</td>
<td>1.1793...</td>
</tr>
<tr>
<td>13</td>
<td>4</td>
<td>1.1094...</td>
<td>38</td>
<td>8</td>
<td>1.2977...</td>
<td>63</td>
<td>9</td>
<td>1.1338...</td>
<td>88</td>
<td>11</td>
<td>1.1726...</td>
</tr>
<tr>
<td>14</td>
<td>5</td>
<td>1.3363...</td>
<td>39</td>
<td>7</td>
<td>1.1208...</td>
<td>64</td>
<td>9</td>
<td>1.125</td>
<td>89</td>
<td>11</td>
<td>1.1659...</td>
</tr>
<tr>
<td>15</td>
<td>5</td>
<td>1.2909...</td>
<td>40</td>
<td>8</td>
<td>1.2649...</td>
<td>65</td>
<td>9</td>
<td>1.1163...</td>
<td>90</td>
<td>11</td>
<td>1.1595...</td>
</tr>
<tr>
<td>16</td>
<td>5</td>
<td>1.25</td>
<td>41</td>
<td>8</td>
<td>1.2493...</td>
<td>66</td>
<td>10</td>
<td>1.2309...</td>
<td>91</td>
<td>10</td>
<td>1.0482...</td>
</tr>
<tr>
<td>17</td>
<td>5</td>
<td>1.2126...</td>
<td>42</td>
<td>8</td>
<td>1.2344...</td>
<td>67</td>
<td>10</td>
<td>1.2216...</td>
<td>92</td>
<td>11</td>
<td>1.1468...</td>
</tr>
<tr>
<td>18</td>
<td>5</td>
<td>1.1785...</td>
<td>43</td>
<td>8</td>
<td>1.2199...</td>
<td>68</td>
<td>10</td>
<td>1.2126...</td>
<td>93</td>
<td>12</td>
<td>1.2443...</td>
</tr>
<tr>
<td>19</td>
<td>5</td>
<td>1.1470...</td>
<td>44</td>
<td>8</td>
<td>1.2060...</td>
<td>69</td>
<td>10</td>
<td>1.2038...</td>
<td>94</td>
<td>12</td>
<td>1.2377...</td>
</tr>
<tr>
<td>20</td>
<td>6</td>
<td>1.3416...</td>
<td>45</td>
<td>8</td>
<td>1.1925...</td>
<td>70</td>
<td>10</td>
<td>1.1952...</td>
<td>95</td>
<td>12</td>
<td>1.2311...</td>
</tr>
<tr>
<td>21</td>
<td>5</td>
<td>1.0910...</td>
<td>46</td>
<td>8</td>
<td>1.1795...</td>
<td>71</td>
<td>10</td>
<td>1.1867...</td>
<td>96</td>
<td>12</td>
<td>1.2247...</td>
</tr>
<tr>
<td>22</td>
<td>6</td>
<td>1.2792...</td>
<td>47</td>
<td>8</td>
<td>1.1669...</td>
<td>72</td>
<td>10</td>
<td>1.1785...</td>
<td>97</td>
<td>12</td>
<td>1.2184...</td>
</tr>
<tr>
<td>23</td>
<td>6</td>
<td>1.2510...</td>
<td>48</td>
<td>8</td>
<td>1.1547...</td>
<td>73</td>
<td>9</td>
<td>1.0533...</td>
<td>98</td>
<td>12</td>
<td>1.2121...</td>
</tr>
<tr>
<td>24</td>
<td>6</td>
<td>1.2247...</td>
<td>49</td>
<td>8</td>
<td>1.1428...</td>
<td>74</td>
<td>10</td>
<td>1.1624...</td>
<td>99</td>
<td>12</td>
<td>1.2060...</td>
</tr>
<tr>
<td>25</td>
<td>6</td>
<td>1.2</td>
<td>50</td>
<td>8</td>
<td>1.1313...</td>
<td>75</td>
<td>10</td>
<td>1.1547...</td>
<td>100</td>
<td>12</td>
<td>1.2</td>
</tr>
</tbody>
</table>
Moreover, if \( q \) to it suffices to check that \( (2 + 2q) - 2\sqrt{q^2 + q + 1} < 1 \), which is equivalent to \( \sqrt{q^2 + q + 1} > q + \frac{1}{2} \) and to \( q^2 + q + 1 > q^2 + q + \frac{1}{4} \).

A bit weaker result holds also for the dihedral groups \( D_{8(q^2+q+1)} \).

**Proposition 8.** If \( n = 1 + q + q^2 \) for some prime power \( q \), then

\[
4q + 3 \leq \Delta[D_{8n}] \leq 4q + 4.
\]

**Proof.** By Theorem 6(1), \( \Delta[C_n] = 1 + q \). Since

\[
2\sqrt{q^2 + q + 1} = 2\sqrt{n} \leq \Delta[D_{2n}] \leq \Delta[D_2] \cdot \Delta[C_n] = 2 \cdot \Delta[C_n] = 2 + 2q,
\]

it suffices to check that \( (2 + 2q) - 2\sqrt{q^2 + q + 1} < 1 \), which is equivalent to \( \sqrt{q^2 + q + 1} > q + \frac{1}{2} \) and to \( q^2 + q + 1 > q^2 + q + \frac{1}{4} \). \( \square \)

In Table 2 we present the results of computer calculation of the difference sizes and characteristics of dihedral groups of order \( \leq 80 \). In this table \( lb[D_{2n}] := \lceil \sqrt{4n} \rceil \) is the lower bound given in Theorem 2.

With the boldface font we denote the numbers \( 2n \in \{14, 26, 42, 62\} \), equal to \( 2(q^2 + q + 1) \) for a prime power \( q \). For these numbers we know that \( \Delta[D_{2n}] = lb[D_{2n}] = 2q + 2 \). For \( q = 2 \) and \( n = q^2 + q + 1 = 7 \) the table shows that \( \Delta[D_{56}] = \Delta[D_{8n}] = 11 = 4q + 3 \), which means that the lower bound \( 4q + 3 \) in Proposition 8 is attained.

**Theorem 9.** For any number \( n \in \mathbb{N} \) the dihedral group \( D_{2n} \) has the difference characteristic

\[
\sqrt{2} \leq \overline{d}[D_{2n}] \leq \frac{48}{\sqrt{586}} \approx 1.983.
\]

Moreover, if \( n \geq 2 \cdot 10^{15} \), then \( \overline{d}[D_{2n}] < \frac{4}{\sqrt{6}} \approx 1.633. \)

**Proof.** By Corollary 3, \( \sqrt{2} \leq \overline{d}[D_{2n}] \leq \sqrt{2} \cdot \overline{d}[C_n] \). If \( n \geq 9 \) and \( n \neq 292 \), then \( \overline{d}[C_n] \leq \frac{24}{\sqrt{293}} \) by Theorem 5(4), and hence \( \overline{d}[D_{2n}] \leq \sqrt{2} \cdot \overline{d}[C_n] \leq \sqrt{2} \cdot \frac{24}{\sqrt{293}} = \frac{48}{\sqrt{586}} \). If \( n = 292 \), then known values \( \overline{d}[C_{73}] = \frac{9}{\sqrt{73}} \) (given in Table 1), \( \overline{d}[D_8] = \frac{4}{\sqrt{8}} = \sqrt{2} \) (given in Table 2) and Theorem 2 yield the upper bound

\[
\overline{d}[D_{292}] = \overline{d}[D_{8 \cdot 73}] \leq \overline{d}[D_8] \cdot \overline{d}[C_{73}] = \sqrt{2} \cdot \frac{9}{\sqrt{73}} < \frac{48}{\sqrt{586}}.
\]

Analyzing the data from Table 2, one can check that \( \overline{d}[D_{2n}] \leq \frac{48}{\sqrt{586}} \approx 1.983 \) for all \( n \leq 8 \).

If \( n \geq 2 \cdot 10^{15} \), then \( \overline{d}[C_n] < \frac{2}{\sqrt{3}} \) by Theorem 5(5), and hence

\[
\overline{d}[D_{2n}] \leq \sqrt{2} \cdot \overline{d}[C_n] < \frac{4}{\sqrt{6}}.
\]

\( \square \)

**Question 10.** Is \( \sup_{n \in \mathbb{N}} \overline{d}[D_{2n}] = \overline{d}[D_{22}] = \frac{8}{\sqrt{22}} \approx 1.7056 \)?

To answer Question 10 affirmatively, it suffices to check that \( \overline{d}[D_{2n}] \leq \frac{8}{\sqrt{22}} \) for all \( n < 1212464. \)
Table 2. Difference sizes and characteristics of dihedral groups $D_{2n}$ for $2n \leq 80$.

<table>
<thead>
<tr>
<th>$2n$</th>
<th>$lb[D_{2n}]$</th>
<th>$\Delta[D_{2n}]$</th>
<th>$2\Delta[C_n]$</th>
<th>$\delta[D_{2n}]$</th>
<th>$2n$</th>
<th>$lb[D_{2n}]$</th>
<th>$\Delta[D_{2n}]$</th>
<th>$2\Delta[C_n]$</th>
<th>$\delta[D_{2n}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1.4142...</td>
<td>42</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>1.5430...</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>1.5</td>
<td>44</td>
<td>10</td>
<td>10</td>
<td>12</td>
<td>1.5075...</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>1.6329...</td>
<td>46</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>1.6218...</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>1.4142...</td>
<td>48</td>
<td>10</td>
<td>10</td>
<td>12</td>
<td>1.4433...</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>1.5811...</td>
<td>50</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>1.5556...</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>1.4433...</td>
<td>52</td>
<td>11</td>
<td>11</td>
<td>12</td>
<td>1.5254...</td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>1.6035...</td>
<td>54</td>
<td>11</td>
<td>12</td>
<td>12</td>
<td>1.6329...</td>
</tr>
<tr>
<td>16</td>
<td>6</td>
<td>6</td>
<td>8</td>
<td>1.5</td>
<td>56</td>
<td>11</td>
<td>11</td>
<td>12</td>
<td>1.4699...</td>
</tr>
<tr>
<td>18</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>1.6499...</td>
<td>58</td>
<td>11</td>
<td>12</td>
<td>14</td>
<td>1.5756...</td>
</tr>
<tr>
<td>20</td>
<td>7</td>
<td>7</td>
<td>8</td>
<td>1.5652...</td>
<td>60</td>
<td>11</td>
<td>12</td>
<td>14</td>
<td>1.5491...</td>
</tr>
<tr>
<td>22</td>
<td>7</td>
<td>8</td>
<td>8</td>
<td>1.7056...</td>
<td>62</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>1.5240...</td>
</tr>
<tr>
<td>24</td>
<td>7</td>
<td>7</td>
<td>8</td>
<td>1.4288...</td>
<td>64</td>
<td>12</td>
<td>12</td>
<td>14</td>
<td>1.5</td>
</tr>
<tr>
<td>26</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>1.5689...</td>
<td>66</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>1.6001...</td>
</tr>
<tr>
<td>28</td>
<td>8</td>
<td>8</td>
<td>10</td>
<td>1.5118...</td>
<td>68</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>1.5764...</td>
</tr>
<tr>
<td>30</td>
<td>8</td>
<td>8</td>
<td>10</td>
<td>1.4605...</td>
<td>70</td>
<td>12</td>
<td>12</td>
<td>14</td>
<td>1.4342...</td>
</tr>
<tr>
<td>32</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>1.5909...</td>
<td>72</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>1.5320...</td>
</tr>
<tr>
<td>34</td>
<td>9</td>
<td>9</td>
<td>10</td>
<td>1.5434...</td>
<td>74</td>
<td>13</td>
<td>14</td>
<td>14</td>
<td>1.6274...</td>
</tr>
<tr>
<td>36</td>
<td>9</td>
<td>9</td>
<td>10</td>
<td>1.5</td>
<td>76</td>
<td>13</td>
<td>14</td>
<td>16</td>
<td>1.6059...</td>
</tr>
<tr>
<td>38</td>
<td>9</td>
<td>10</td>
<td>10</td>
<td>1.6222...</td>
<td>78</td>
<td>13</td>
<td>14</td>
<td>14</td>
<td>1.5851...</td>
</tr>
<tr>
<td>40</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>1.4230...</td>
<td>80</td>
<td>13</td>
<td>14</td>
<td>16</td>
<td>1.5652...</td>
</tr>
</tbody>
</table>

Proposition 11. The inequality $\delta[D_{2n}] \leq \sqrt{2} \cdot \delta[C_n] \leq \frac{8}{\sqrt{22}}$ holds for all $n \geq 1212464$.

Proof. It suffices to prove that $\delta[C_n] \leq \frac{4}{\sqrt{11}}$ for all $n \geq 1212464$. To derive a contradiction, assume that $\delta[C_n] > \frac{4}{\sqrt{11}}$ for some $n \geq 1212464$. Let $(q_k)_{k=1}^{\infty}$ be an increasing enumeration of prime powers. Let $k \in \mathbb{N}$ be the unique number such that $12q_k^2 + 14q_k + 15 < n \leq 12q_{k+1}^2 + 14q_{k+1} + 15$. By Corollary 4.9 of [2], $\Delta[C_n] \leq 4(q_{k+1} + 1)$. The inequality $\delta[C_n] > \frac{4}{\sqrt{11}}$ implies

$4(q_{k+1} + 1) \geq \Delta[C_n] > \frac{4}{\sqrt{11}} \sqrt{n} \geq \frac{4}{\sqrt{11}} \sqrt{12q_k^2 + 14q_k + 16}$.

By Theorem 1.9 of [6], if $q_k \geq 3275$, then $q_{k+1} \leq q_k + \frac{q_k}{2 \ln^2(q_k)}$. On the other hand, using WolframAlpha computational knowledge engine it can be shown that the inequality $1 + x + \frac{x}{2 \ln^2(x)} \leq \frac{1}{\sqrt{11}} \sqrt{12x^2 + 14x + 16}$ holds for all $x \geq 43$. This implies that $q_k < 3275$. 


Analyzing the table\(^1\) of (maximal gaps between) primes, it can be shown that \(11(q_{k+1} + 1)^2 \leq 12q_k^2 + 14q_k + 16\) if \(q_k \geq 331\). So, \(q_k \leq 317, q_{k+1} \leq 331\) and \(11 \cdot (q_{k+1} + 1)^2 = 11 \cdot 332^2 = 1\,212\,464 \leq n\), which contradicts \(4(q_{k+1} + 1) > \frac{4}{\sqrt{11}} \sqrt{n}\). □

**References**


**Taras Banakh**

Ivan Franko National University of Lviv (Ukraine), and

Institute of Mathematics, Jan Kochanowski University in Kielce (Poland)

Email: t.o.banakh@gmail.com

**Volodymyr Gavrylkiv**

Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine

Email: vgavrylkiv@gmail.com

---

\(^1\)See https://primes.utm.edu/notes/GapsTable.html and https://primes.utm.edu/lists/small/1000.txt