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ON THE RELATIONSHIPS BETWEEN THE FACTORS OF THE UPPER AND LOWER CENTRAL SERIES IN SOME NON-PERIODIC GROUPS

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Abstract. This paper deals with the mutual relationships between the factor group $G/\zeta(G)$ (respectively $G/\zeta_k(G)$) and G' (respectively $\gamma_{k+1}(G)$ and $G^{\mathfrak{N}}$). It is proved that if $G/\zeta(G)$ (respectively $G/\zeta_k(G)$) has finite 0-rank, then G' (respectively $\gamma_{k+1}(G)$ and $G^{\mathfrak{N}}$) also have finite 0-rank. Furthermore, bounds for the 0-ranks of $G', \gamma_{k+1}(G)$ and $G^{\mathfrak{N}}$ are obtained.

1. Introduction

Let G be a group and let $\zeta(G)$ denote the centre of G. Starting from the centre we can define the upper central series

$$1 = \zeta_0(G) \leq \zeta_1(G) \leq \cdots \zeta_\alpha(G) \leq \zeta_{\alpha+1}(G) \leq \cdots \zeta_\eta(G)$$

of G in the usual way by setting $\zeta_1(G) = \zeta(G), \zeta_{\alpha+1}(G)/\zeta_\alpha(G) = \zeta(G/\zeta_\alpha(G))$, for all ordinals α, $\zeta_\rho(G) = \bigcup_{\alpha<\rho} \zeta_\alpha(G)$, for all limit ordinals ρ and $\zeta(G/\zeta_\eta(G))$ is trivial.

Likewise we let $\gamma_2(G) = G'$, the derived subgroup of G and define the lower central series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \gamma_\alpha(G) \geq \gamma_{\alpha+1}(G) \geq \cdots \gamma_\delta(G)$$

of G by setting $\gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G]$, for all ordinals α, $\gamma_\rho(G) = \bigcap_{\alpha<\rho} \gamma_\alpha(G)$, for all limit ordinals ρ and $\gamma_\delta(G) = [\gamma_\delta(G), G]$.

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It is well-known that $G = \zeta_n(G)$, for some natural number n if and only if $\gamma_{n+1}(G) = 1$ and this prompts the general question:

> For which classes of groups $\mathfrak{X}$ does the fact that $G/\zeta_n(G) \in \mathfrak{X}$ imply that $\gamma_{n+1}(G) \in \mathfrak{X}$?

The classes $\mathfrak{X}$ with this property are called Baer classes in [14]. The motivation for this name lies in the fact that *if $G/\zeta_n(G)$ is finite, then $\gamma_{n+1}(G)$ is also finite*, a result which became known as Baer's Theorem and which R. Baer noted can be obtained from Zusatz zum Endlichkeitssatz in his paper [2]. A particular case of this theorem is the result that *if $G/\zeta(G)$ is finite, then $G' = \gamma_2(G)$ is also finite*, a theorem which is traditionally called Schur's Theorem. This tradition was initiated by P. Hall who attributes the result to the famous algebraist I. Schur, without giving a reference. This tradition has permeated the literature, which often cites [21] as the source of this result. Although Schur proved many important results,this was not one of them. In [21] Schur introduced a new concept, now called the Schur multiplier or Schur multiplicator, which he defined only for finite groups. The result we know as Schur's Theorem appeared in the literature for the first time in a paper of B. H. Neumann [18], a student of Schur. However at the end of this paper Neumann writes that he received a letter from Baer, informing him that Neumann's result is a consequence of a more general result proved by Baer in [1]. In fact in [1, Theorem 3] it was proved that if the normal subgroup H of the group G has finite index then $(G' \cap H)/[H, G]$ is also finite. One year after the publication of [18], Baer [2] gave another proof of this result. It therefore appears that the result mentioned above should be called the Baer-Neumann Theorem. This theorem and Baer's Theorem serve as a starting point for numerous generalizations and we refer the reader to the papers [20, 4, 8, 12, 16, 5, 13] and also the survey [7]. In the current article we also obtain generalizations of these results associated with the class of groups of finite 0-rank.

Let G be a group which has an ascending series whose factors are either infinite cyclic or periodic. If the number of infinite cyclic factors is finite then the group G is said to have *finite* 0*-rank*. The 0-rank of the group G is the number of infinite cyclic factor groups in the series and is denoted by $\mathbf{r}_0(G)$, an invariant of the group. Thus if the number of infinite cyclic factors is exactly r then $\mathbf{r}_0(G) = r$. If no such integer r exists then we shall say that G has infinite 0-rank. If G has no such ascending series the 0-rank is undefined.

We consider groups G whose central factor group $G/\zeta(G)$ has finite 0-rank and the question then arises as to whether or not G' has finite 0-rank. In general the answer is in the negative since in [16] an example is constructed of a group H with the property that $H = H'$, $\zeta(H)$ is free abelian of infinite 0-rank and $H/\zeta(H)$ is periodic (and hence has finite 0-rank). Thus a positive answer to this question is only possible under additional hypotheses and one common limitation is to restrict attention to the class of generalized radical groups, where a group G is generalized radical if G has an ascending series whose factors are locally nilpotent or locally finite

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The main result of Section 2 is as follows:

Theorem A. *Let G be a locally generalized radical group and suppose that C is a subgroup of $\zeta(G)$ such that G/C has finite 0-rank r. Then G' has finite 0-rank at most $r(5r^2+5r-1)/2$.*

This result is the 0-rank version of the well-known fact that if G is a group such that $G/\zeta(G)$ is polycyclic-by-finite, then G' is polycyclic-by-finite also. It should also be compared with [4, Theorem A] and [16].

In Section 3 we discuss the generalizations of Baer's Theorem in the 0-rank case. There are two main results in this case. First we prove the following theorem, analogous to [5, Theorem A], for which the bound is rather complicated.

Theorem B. *Let G be a locally generalized radical group and suppose that, for some natural number k, $G/\zeta_k(G)$ has finite 0-rank r. Then $\gamma_{k+1}(G)$ has finite 0-rank and there is a function ν such that $\boldsymbol{r}_0(\gamma_{k+1}(G)) \leq \nu(r,k)$.*

In Section 3, we also prove the following result which shows that the bound for the nilpotent residual is much easier to determine. The proof of this result is more technical and requires knowledge of the Z-decomposition of an abelian normal subgroup, first introduced by D. I. Zaitsev [22], details of which can be found in [15] and [12].

Theorem C. *Let G be a locally generalized radical group and suppose that for some natural number k, $G/\zeta_k(G)$ has finite 0-rank r. Then the nilpotent residual $G^{\mathfrak{N}}$ of G has finite 0-rank and $\boldsymbol{r}_0(G^{\mathfrak{N}}) \leq r(5r^2+5r+1)/2$.*

It is worth noting that $\mathbf{r}_0(G^{\mathfrak{N}})$ is bounded by a function of $\mathbf{r}_0(G/\zeta_k(G))$ only. We would also like to note that if G is a locally generalized radical group, with maximal normal torsion subgroup $\mathbf{Tor}\,(G)$, then $G/\mathbf{Tor}\,(G)$ has finite special rank, a result which appeared in [6]. In Theorem 3.6 of the current paper, a simpler bound for the special rank of $G/\mathbf{Tor}\,(G)$ is obtained, namely $G/\mathbf{Tor}\,(G)$ has special rank at most $5r^2(r+1)/2$, where $r = \mathbf{r}_0(G)$.

2. Groups whose central factor group has finite 0-rank

We begin with the following elementary lemma whose proof we omit.

Lemma 2.1. *Let G be a group. Suppose that H is a subgroup of G and L is a normal subgroup of G.*

(i) *If G has finite 0-rank, then H has finite 0-rank and $\boldsymbol{r}_0(H) \leq \boldsymbol{r}_0(G)$;*
(ii) *G has finite 0-rank if and only if both L and G/L have finite 0-rank. In this case $\boldsymbol{r}_0(G) = \boldsymbol{r}_0(L) + \boldsymbol{r}_0(G/L)$.*

Next we prove a result whose proof is similar to that of [4, Corollary 3.8].

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Lemma 2.2. *Let G be a group and let A be a normal abelian subgroup of G. Suppose that G satisfies the following conditions:*

(i) $G/C_G(A) = \langle x_1C_G(A), \dots, x_mC_G(A)\rangle$, *where* $x_1, \dots, x_m \in G$;

(ii) $A \cap \zeta(G)$ *contains a subgroup* C *such that* A/C *has finite* 0*-rank* r.

Then $[A, G]$ *has finite* 0*-rank at most* rm.

Proof. The map $\phi_i : A \longrightarrow [A, x_i]$ defined by $\phi_i(a) = [a, x_i]$ is an endomorphism of A with kernel $C_A(x_i)$ so $A/C_A(x_i) \cong [A, x_i]$. Since $C \leq C_A(x_i)$ $\mathbf{r}_0(A/C_A(x_i)) \leq r$ and hence $\mathbf{r}_0([A, x_i]) \leq r$. However, $[A, G] = [A, x_1] \cdots [A, x_m]$ which therefore has 0-rank at most rm. □

The next corollary is modeled on [4, Corollary 3.9]. We recall that if G is a group, then G has finite special rank r if every finitely generated subgroup of G is at most r-generator and r is the least natural number with this property.

Corollary 2.3. *Let G be a group and let A be a normal abelian subgroup of G such that $G/C_G(A)$ has finite special rank m. If $A \cap \zeta(G)$ contains a subgroup C such that A/C has finite* 0*-rank r, then $[A, G]$ has finite* 0*-rank at most rm.*

Proof. Let

$$\mathcal{L} = \{F \leq G | C_G(A) \leq F \text{ and } F/C_G(A) \text{ is finitely generated}\}$$

denote the local system of subgroups in which $F/C_G(A)$ is at most m-generator. Since $[A, G] = \bigcup_{F\in\mathcal{L}}[A, F]$, Lemma 2.2 implies that each of the subgroups $[A, F]$ has 0-rank at most rm. Hence $[A, G]$ has 0-rank at most rm also, by [6, Proposition 2]. □

We next prove a special case of Theorem A.

Lemma 2.4. *Let G be a group and let C be a subgroup of $\zeta(G)$ such that G/C is abelian of finite* 0*-rank r. Then G' has finite* 0*-rank at most $r(r-1)/2$.*

Proof. Notice that G is a nilpotent group of class at most 2. Let $H/C = \langle x_1C, \dots, x_rC\rangle$ be a torsion-free abelian subgroup of G/C such that G/H is periodic. Let $g, h \in G$ and observe that there are natural numbers m, n such that $g^m, h^n \in H$. Then $[g^m, h^n] = [g, h]^{mn} \in H'$, so it follows that G'/H' is periodic. Hence $\mathbf{r}_0(G') = \mathbf{r}_0(H')$ and it suffices to show that $\mathbf{r}_0(H') \leq r(r-1)/2$. However if $x, y \in H$, then there are integers $s_1, \dots, s_r, t_1, \dots, t_r$ and elements $c, d \in C$ such that

$$\begin{aligned}[x, y] &= [x_1^{s_1} \cdots x_r^{s_r} c, x_1^{t_1} \cdots x_r^{t_r} d] \\ &= [x_1, x_2]^{u_{12}}[x_1, x_3]^{u_{13}} \cdots [x_i, x_j]^{u_{i,j}} \cdots [x_{r-1}, x_r]^{u_{r-1,r}}\end{aligned}$$

for certain integers $u_{i,j}$. Thus H' can be generated by the elements of the form $[x_i, x_j]$, where $i < j$ so that H' is at most an $r(r-1)/2$-generator abelian group. Hence $\mathbf{r}_0(H') \leq r(r-1)/2$, as required. □

Corollary 2.5. *Let G be a group containing normal subgroups A, B such that the following conditions hold:*

(i) $A \leq B$, $A \leq \zeta(G)$ *and* $B/A \leq \zeta(G/A)$*;*

(ii) B/A *has finite* 0*-rank* r *and* G/B *has finite special rank* k*.*

Then $[B, G]$ *has finite* 0*-rank at most* $r(r + 2k - 1)/2$*.*

Proof. Applying Lemma 2.4 to B we see that $\mathbf{r}_0(B') \leq r(r-1)/2$. Of course B/B' is abelian and $(G/B')/C_{G/B'}(B/B')$ has finite special rank at most k. Applying Corollary 2.3 to G/B' we deduce that $[B/B', G/B'] = [B, G]/B'$ has finite 0-rank at most rk. Finally, by Lemma 2.1, we have

$$\mathbf{r}_0([B, G]) = \mathbf{r}_0(B') + \mathbf{r}_0([B, G]/B') \leq r(r-1)/2 + rk = r(r + 2k - 1)/2,$$

as required. □

Corollary 2.6. *Let G be a group containing normal subgroups A, B such that the following conditions hold:*

(i) $A \leq B$, $A \leq \zeta(G)$ *and* $B \leq \zeta_m(G)$*, for some natural number* m*;*

(ii) B/A *has finite* 0*-rank* r *and* G/B *has finite special rank* k*.*

Then $[B, G]$ *has finite* 0*-rank at most* $r(2r + 2k - 1)/2$*.*

Proof. Clearly B is nilpotent and we let **Tor** (B) denote the torsion subgroup of B. We first suppose that B is torsion-free and let

$$1 = Z_0 \leq Z_1 \leq \cdots \leq Z_m = B$$

be the upper central series of B. Since $A \leq B \cap \zeta(G) \leq Z_1$ it follows that B/Z_1 has finite 0-rank at most r. The factors Z_j/Z_{j-1} are torsion-free, for $1 \leq j \leq m$, and hence $m - 1 \leq r$.

Let $\mathbf{r}_0(Z_{j+1}/Z_j) = r_{j+1}$, for $1 \leq j \leq m-1$. Clearly we have $\mathbf{r}_0(B/Z_2) = r - r_2$. A result of V. M. Glushkov [9] shows that the special rank of B/Z_2 is also $r - r_2$ and hence G/Z_2 has special rank at most $k + r - r_2$. Then Corollary 2.5 implies that $C_2 = [Z_2, G]$ has finite 0-rank at most $r_2(r_2 + 2k + 2r - 2r_2 - 1)/2 = r_2(2k + 2r - r_2 - 1)/2$.

The centre of G/C_2 contains Z_2/C_2 and we now apply the above argument to G/C_2. We have $\mathbf{r}_0((Z_3/C_2)/(Z_2/C_2)) = \mathbf{r}_0(Z_3/Z_2) = r_3$. Also G/Z_3 has special rank at most $k+r-r_2-r_3$, so at most $k + r - r_3$. Then, as above, $C_3/C_2 = [Z_3/C_2, G/C_2]$ has finite 0-rank at most $r_3(2k + 2r - r_3 - 1)/2$. Since $C_2 = [Z_2, G] \leq [Z_3, G]$ we have $[Z_3/C_2, G/C_2] = [Z_3, G]/C_2$. and we deduce that

$$\begin{aligned}\mathbf{r}_0([Z_3, G]) &\leq r_2(2k + 2r - r_2 - 1)/2 + r_3(2k + 2r - r_3 - 1)/2 \\ &= (r_2 + r_3)(2k + 2r - 1)/2 - (r_2^2 + r_3^2)/2.\end{aligned}$$

We now repeat this argument and after finitely many steps we deduce that

$$\begin{aligned}\mathbf{r}_0([B,G]) &\leq r(2k+2r-1)/2-(r_2^2+r_3^2+\cdots r_m^2)/2\\ &\leq r(2k+2r-1)/2.\end{aligned}$$

Finally we note that if B is not torsion-free then we can apply the argument above to the group $G/\mathbf{Tor}\,(B)$ to deduce that

$$\mathbf{r}_0([B/\mathbf{Tor}\,(B), G/\mathbf{Tor}\,(B)] \leq r(2k+2r-1)/2.$$

Since $\mathbf{r}_0([B,G]) = \mathbf{r}_0([B/\mathbf{Tor}\,(B), G/\mathbf{Tor}\,(B)])$, the result follows. □

A variation of the next result appears in [19, Corollary 5] and follows from results of Kargapolov [11] and Baer and Heineken [3]. We omit the proof since is similar to that given in [19]. In any case the next few results are probably well-known.

Lemma 2.7. *Let p be a prime and let G be a finite p-group. If every abelian subgroup of G has finite special rank at most r, then G has special rank at most $r(5r+1)/2$.*

The following result was observed in [5, Lemma 1.2] and we omit its proof.

Proposition 2.8. *Let G be a finite group. If every abelian subgroup of G has special rank at most r, then G has special rank at most $r(5r+3)/2$.*

Corollary 2.9. *Let F be a field and let G be a finite subgroup of $GL_n(F)$. If F has characteristic 0 then G has special rank at most $n(5n+3)/2$.*

Proof. By Lemma 2.7 every abelian subgroup of G has special rank at most n, so we can apply Proposition 2.8 to G to obtain the result. □

Corollary 2.10. *Let A be a torsion-free abelian group with finite 0-rank r and let G be a finite subgroup of $Aut(A)$. Then G has special rank at most $r(5r+3)/2$.*

Proof. Let D be the divisible envelope of A. Every automorphism of A extends to an automorphism of D. Also $D=\oplus_{j=1}^{r} D_j$ where $D_j \cong \mathbb{Q}$, for $1 \leq j \leq r$ and $\mathrm{Aut}\,(D) \cong GL_r(\mathbb{Q})$. The result now follows by Corollary 2.9. □

Corollary 2.11. *Let G be a group and let A be a torsion-free abelian normal subgroup of G. Let C be a subgroup of $A\cap\zeta(G)$ such that A/C is torsion-free of finite 0-rank r. Then every finite subgroup of $G/C_G(A)$ has special rank at most $r(5r+3)/2$.*

Proof. Let $H/C_G(A)$ be a finite subgroup of $G/C_G(A)$ and let $Z = C_H(A/C)$. If $z \in Z, a \in A$, then $a^z = ac$, for some element $c \in C$ and for every natural number m we have $z^{-m}az^m = ac^m$. Since $H/C_G(A)$ is finite there is a natural number t such that $z^t \in C_G(A)$, so we have $a = z^{-t}az^t = ac^t$

and hence $c^t = 1$. However C is torsion-free, so $c = 1$ and hence $z \in C_H(A)$. Consequently, $C_H(A/C) = C_H(A)$. Corollary 2.10 implies that $H/C_H(A/C)$ has special rank at most $r(5r+3)/2$, as required. □

We may now prove the main result of this section.

Proof of Theorem A. Let $R = \textbf{Tor}\,(G)$ and note that $\mathbf{r}_0(G) = \mathbf{r}_0(G/R)$. Hence it is sufficient to prove the theorem for G/R and in so doing we may suppose that $R = 1$.

By Theorem [6, Theorem A], G has a series of normal subgroups $C \le T \le L \le K \le G$ such that T/C is locally finite, L/T is a torsion-free nilpotent group, K/L is a finitely generated torsion-free abelian group and G/K is finite. By [4, Proposition 3.2], $[T,G]$ is locally finite. By our assumption we have $[T,G] = 1$ and hence we may assume that $T = C$.

Let $\mathbf{r}_0(L/C) = r_1$ and $\mathbf{r}_0(K/L) = r_2$. Corollary 2.6 applied to L implies that L' has 0-rank at most $r_1(2r_1-1)/2$. Applying Corollary 2.3 to K/L' shows that $[L/L', K/L'] = [L,K]/L'$ has finite 0-rank at most r_1r_2 and we deduce that

$$\mathbf{r}_0([L,K]) \le r_1(2r_1-1)/2 + r_1r_2.$$

Next we note that $L/[L,K]$ lies in the centre of $K/[L,K]$ so we may apply Lemma 2.4 to $K/[L,K]$ to deduce that $(K/[L,K])' = K'/[L,K]$ has finite 0-rank at most $r_2(r_2-1)/2$. Hence we have, observing that $r = r_1 + r_2$,

$$\begin{aligned}\mathbf{r}_0(K') &\le r_1(2r_1-1)/2 + r_1r_2 + r_2(r_2-1)/2\\ &= r_1^2 + r_1r_2 + r_2^2/2 - r_1/2 - r_2/2 = r^2 - r/2 - r_2(r - r_2/2)\\ &\le r(2r-1)/2.\end{aligned}$$

Let $S/K' = \textbf{Tor}\,(K/K')$ and note that $\mathbf{r}_0(S) = \mathbf{r}_0(K')$. Furthermore, K/S is a torsion-free abelian group. Let $P/CS = \textbf{Tor}\,(K/CS)$, a locally finite group. Proposition 3.2 of [4] implies that $[P/S, G/S] = [P,G]S/S$ is locally finite, so we deduce that $[P,G] \le S$. Consequently, $P/S \le \zeta(G/S)$. The factor group K/P is torsion-free and has 0-rank at most r. Applying Corollary 2.11 to G/S, we see that the finite group $(G/S)/C_{G/S}(K/S)$ has special rank at most $r(5r+3)/2$. Corollary 2.3 can be applied again and this now implies that $[K/S, G/S] = [K,G]S/S$ has 0-rank at most $r^2(5r+3)/2$. We now have

$$\mathbf{r}_0([K,G]) \le r^2(5r+3)/2 + r(2r-1)/2 = r(5r^2+5r-1)/2.$$

The centre of $G/[K,G]$ contains $K/[K,G]$ which is of finite index in $G/[K,G]$. Consequently we may apply Schur's Theorem to deduce that $G'/[K,G] = (G/[K,G])'$ is finite and hence $\mathbf{r}_0(G') = \mathbf{r}_0([K,G])$. Finally we have $\mathbf{r}_0(G') = r(5r^2+5r-1)/2$, as required. □

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3. An extension of Baer's Theorem

Our next goal is to establish an analogue of Baer's Theorem for groups of finite 0-rank. Thus we shall be considering groups in which $G/\zeta_k(G)$ has finite 0-rank for some natural number k and we shall be interested in obtaining information concerning the 0-rank of $\gamma_{k+1}(G)$. There are two stages required to establish this goal, the first being the important special case when $G/\zeta_k(G)$ is locally finite.

We consider a slightly more general situation. We let A be a normal subgroup of the group G. Write $A = \gamma_1(A,G), \gamma_2(A,G) = [A,G]$ and recursively $\gamma_{\alpha+1}(A,G) = [\gamma_\alpha(A,G),G]$ for all ordinals α and $\gamma_\lambda(A,G) = \cap_{\mu<\lambda}\gamma_\mu(A,G)$ for all limit ordinals λ. Our next result is really a generalization of [4, Proposition 3.2].

Theorem 3.1. *Let G be a group and let A be a normal subgroup of G. If $A/(A\cap\zeta_k(G))$ is locally finite, then $\gamma_{k+1}(A,G)$ is also locally finite and $\Pi(\gamma_{k+1}(A,G)) \subseteq \Pi(A/A\cap\zeta_k(G))$.*

Proof. Let $Z_j = \zeta_j(G)$ and let

$$1 = Z_0 \leq Z_1 \leq \cdots \leq Z_{k-1} \leq Z_k = Z$$

be a part of the upper central series of G. We use induction on k. If $k = 1$ then $A/(A\cap Z_1)$ is locally finite and an application of [4, Proposition 3.2] shows that $\gamma_2(A,G) = [A,G]$ is locally finite and that $\Pi(\gamma_2(A,G)) \subseteq \Pi(A/A\cap Z_1)$.

Assume now that $k > 1$ and consider G/Z_1. Suppose we have already proved that $\gamma_k(AZ_1/Z_1, G/Z_1)$ is locally finite and that

$$\Pi(\gamma_k(AZ_1/Z_1, G/Z_1)) \subseteq \Pi(A/A\cap\zeta_k(G)).$$

Let $K/Z_1 = \gamma_k(AZ_1/Z_1, G/Z_1)$ and apply [4, Proposition 3.2] to deduce that $[K,G]$ is locally finite and $\Pi([K,G]) \subseteq \Pi(K/Z_1)$. We note that $\gamma_k(AZ_1/Z_1, G/Z_1) = \gamma_k(A,G)Z_1/Z_1$ and hence $\gamma_k(A,G) \leq K$. Then $\gamma_{k+1}(A,G) = [\gamma_k(A,G),G] \leq [K,G]$. It follows that $\gamma_{k+1}(A,G)$ is locally finite and $\Pi(\gamma_{k+1}(A,G)) \subseteq \Pi(K/Z_1) \subseteq \Pi(A/(A\cap\zeta_k(G)))$. □

Corollary 3.2. *Let G be a group and suppose that $G/\zeta_k(G)$ is locally finite for some natural number k. Then $\gamma_{k+1}(G)$ is also locally finite and $\Pi(\gamma_{k+1}(G)) \subseteq \Pi(G/\zeta_k(G))$.*

The following result is a version of [8, Lemma 2] and a special case of it occurs in [5, Theorem B], so we omit the proof which follows verbatim from that result.

Theorem 3.3. *Let G be group and suppose that $G/\zeta_\infty(G)$ is locally finite. Then G contains a normal locally finite subgroup R such that G/R is locally nilpotent. Furthermore $\Pi(R) \subseteq \Pi(G/\zeta_\infty(G))$. In particular the locally nilpotent residual $G^{\mathbf{L}\mathfrak{N}}$ of G is locally finite and $G/G^{\mathbf{L}\mathfrak{N}}$ is locally nilpotent.*

We obtain the following corollary which occurs as [8, Lemma 1].

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Corollary 3.4. *Let G be a group and let C be a G-invariant subgroup of $\zeta_\infty(G)$ such that G/C is locally finite. If $\textbf{Tor}\,(G) = 1$, then G is a torsion-free hypercentral group.*

Proof. Since $\mathbf{Tor}\,(G) = 1$, $Z = \zeta_\infty(G)$ is torsion-free. Using Theorem 3.3 we deduce that G is locally nilpotent and hence is torsion-free. Now a theorem of Maltsev [17] implies that G is hypercentral, as required. □

To continue further we need to determine the connection between the 0-rank and the special rank.

Lemma 3.5. *Let G be a nilpotent-by-finite group with finite 0-rank r. If $\textbf{Tor}\,(G) = 1$ then G has special rank at most $5r^2(r+1)/2$.*

Proof. Let L be the maximal normal nilpotent subgroup of G and let

$$1 = Z_0 \le Z_1 \le \cdots \le Z_k = L$$

be the upper central series of L. The hypotheses on G imply that $\mathbf{Tor}\,(L) = 1$ and it is well-known that each factor Z_j/Z_{j-1} is torsion-free for $1 \le j \le k$ and hence $k \le \mathbf{r}_0(L)$. Furthermore, G/L is finite so $\mathbf{r}_0(G) = \mathbf{r}_0(L)$.

Let $C = Z_{k-1}$ and note that G/C is centre-by-finite. Let $V_k/C = C_{G/C}(Z_k/C)$. Since Z_k/C is torsion-free it follows that $T_{k-1}/C = \mathbf{Tor}\,(V_k/C)$ is finite. We may apply Schur's Theorem to G/C and deduce that $(V_k/C)'$ is finite. Corollary 2.10 shows that G/V_k has finite special rank at most $r(5r+3)/2$ so

$$\begin{aligned}\mathbf{r}(G/T_{k-1}) &\le \mathbf{r}(V_k/T_{k-1}) + \mathbf{r}(G/V_k)\\ &\le r + r(5r+3)/2 = r(5r+5)/2.\end{aligned}$$

Next we consider the factor group T_{k-1}/Z_{k-2}. By the same arguments as those used above we deduce that there is a G-invariant subgroup T_{k-2} containing Z_{k-2} such that T_{k-2}/Z_{k-2} is finite and T_{k-1}/T_{k-2} has special rank at most $r(5r+5)/2$. Then

$$\mathbf{r}(G/T_{k-2}) \le \mathbf{r}(T_{k-1}/T_{k-2}) + \mathbf{r}(G/T_{k-1}) \le r(5r+5)/2 + r(5r+5)/2.$$

We may repeat these arguments to construct a sequence of G-invariant subgroups, T_j, such that $T_j \le T_{j+1}$, T_j/Z_j is finite and $\mathbf{r}(T_{j+1}/T_j) \le r(5r+5)/2$, for $2 \le j \le k-2$. This means that G/T_2 has special rank at most $(k-1)r(5r+5)/2$. Likewise $T_2/\mathbf{Tor}\,(T_2)$ has special rank at most $r(5r+5)/2$. However $\mathbf{Tor}\,(T_2) = 1$ so G has special rank at most

$$(k-1)r(5r+5)/2 + r(5r+5)/2 = kr(5r+5)/2 \le 5r^2(r+1)/2,$$

as required. □

Theorem 3.6. *Let G be a locally generalized radical group. If G has finite 0-rank r, then $G/\textbf{Tor}\,(G)$ has special rank at most $5r^2(r+1)/2$.*

Proof. Since $\mathbf{Tor}\,(G/\mathbf{Tor}\,(G)) = 1$ and we are obtaining the result for $G/\mathbf{Tor}\,(G)$, we shall assume that $\mathbf{Tor}\,(G) = 1$. Then, by [6, Theorem A], G has normal subgroups $L \leq K$ such that L is torsion-free nilpotent, K/L is a finitely generated torsion-free abelian group and G/K is finite.

If $L = K$, then Lemma 3.5 shows that G has special rank at most $5r^2(r+1)/2$, so suppose that $K \neq L$. Let $\mathbf{r}_0(L) = r_1$ and $\mathbf{r}_0(K/L) = r_2$, so $r_1 + r_2 = r$. Let $T_1/L = \mathbf{Tor}\,(G/L)$. Applying Lemma 3.5 we deduce that G/T_1 has special rank at most $5r_2^2(r_2+1)/2$. We note that T_1/L is finite so we may apply Lemma 3.5 to T_1 to deduce that T_1 has special rank at most $5r_1^2(r_1+1)/2$. Hence G has special rank at most

$$\begin{aligned}
5r_1^2(r_1+1)/2 + 5r_2^2(r_2+1)/2 &= 5(r_1^3 + r_2^3 + r_1^2 + r_2^2)/2 \\
&= 5((r-r_2)^3 + r_2^3 + (r-r_2)^2 + r_2^2)/2 \\
&= 5(r^3 + r^2 + 3rr_2(r_2 - r) + 2r_2(r_2 - r))/2 \\
&\leq 5r^2(r+1)2,
\end{aligned}$$

since $r \geq r_2$. In either case $\mathbf{r}(G) \leq 5r^2(r+1)/2$, as required. □

Proof of Theorem B. Let $T = \mathbf{Tor}\,(G)$, let $Z_i/T = \zeta_i(G/T)$ and let

$$1 = Z_0/T \leq Z_1/T \leq \cdots \leq Z_{k-1}/T \leq Z_k/T = Z/T$$

be a part of the upper central series of G/T. By [6, Theorem A] G has a series $Z \leq P \leq Q \leq K \leq G$ of normal subgroups such that P/Z is locally finite, Q/P is torsion-free nilpotent, K/Q is finitely generated torsion-free abelian and G/K is finite. By Theorem 3.1 $S/T = \gamma_{k+1}(P/T, G/T)$ is locally finite and, since T is locally finite, it is clear that S is locally finite. Hence $S = T$ and so $P/T \leq \zeta_k(G/T) = Z_k/T = Z/T$. Hence $P = Z$ and Q/Z is a torsion-free nilpotent group of finite 0-rank.

The remainder of the argument is to be applied to the group G/T. We shall show that $\gamma_{k+1}(G/T)$ has finite zero rank. We note that $\gamma_{k+1}(G/T) = \gamma_{k+1}(G)T/T$. Since T is periodic it follows that

$$\mathbf{r}_0(\gamma_{k+1}(G/T)) = \mathbf{r}_0(\gamma_{k+1}(G)T) = \mathbf{r}_0(\gamma_{k+1}(G)).$$

Thus in order to simplify our notation we may assume that $T = 1$ in the remainder of the argument and hence assume that Z is torsion-free. As we saw above, $\mathbf{Tor}\,(G/Z) = 1$ and Theorem 3.6 implies that $\mathbf{r}(G/Z) \leq 5r^2(r+1)/2$.

We use induction on k. If $k = 1$ then the central factor group G/Z_1 has finite 0-rank and then Theorem A implies that G' has finite 0-rank at most $r(5r^2 + 5r - 1)/2 = \nu(r, 1)$.

Assume now that $k > 1$ and that there is a function ν, depending on r and k such that $\gamma_k(G/Z_1)$ has finite 0-rank at most $\nu(r, k-1)$. Set $V/Z_1 = \gamma_k(G/Z_1)$, let $L = \gamma_k(G)$ and note that $V = LZ_1$. Then $V' = [LZ_1, LZ_1] = L'$, since $Z_1 = \zeta_1(G)$. Also $[V, G] = [L, G] = \gamma_{k+1}(G)$. An application of Theorem A shows that L' has finite 0-rank at most $\nu(r, k-1)(5\nu(r, k-1)^2 + 5\nu(r, k-1) - 1)$

The factor group L/L' is abelian and we have

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$$\begin{aligned}(L/L')/(L/L' \cap Z_1L'/L') &\cong (L/L')(Z_1L'/L')/(Z_1L'/L') \\ &\cong LZ_1/L'Z_1 \cong L/(L \cap Z_1L'),\end{aligned}$$

an epimorphic image of $L/L \cap Z_1$. Certainly,

$$\mathbf{r}_0(L/L \cap Z_1) = \mathbf{r}_0(LZ_1/Z_1) = \mathbf{r}_0(V/Z_1) \leq \nu(r, k-1)$$

and hence

$$\mathbf{r}_0((L/L')/(L/L' \cap Z_1L'/L')) \leq \nu(r, k-1).$$

It is well-known that $[L, Z] = 1$ and hence $G/C_G(L)$ is an image of G/Z. It follows that $G/C_G(H)$ has special rank at most $5r^2(r+1)/2$. We apply Corollary 2.3 to G/L' and the work above shows that $[L/L', G/L'] = [L, G]/L' = \gamma_{k+1}(G)/L'$ has finite 0-rank at most $5\nu(r, k-1)r^2(r+1)/2$.

It follows that $\gamma_{k+1}(G)$ has finite 0-rank and that

$$\begin{aligned}\mathbf{r}_0(\gamma_{k+1}(G)) &\leq \mathbf{r}_0(\gamma_{k+1}(G)/L') + \mathbf{r}_0(L') \\ &\leq 5\nu(r, k-1)r^2(r+1)/2 \\ &+ \nu(r, k-1)(5\nu(r, k-1)^2 + 5\nu(r, k-1) - 1).\end{aligned}$$

Consequently we let

$$\nu(r, k) = 5\nu(r, k-1)r^2(r+1)/2 + \nu(r, k-1)(5\nu(r, k-1)^2 + 5\nu(r, k-1) - 1).$$

The result follows. □

The previous result shows that $\gamma_{k+1}(G)$ has finite 0-rank and this implies that the nilpotent residual of G also has finite 0-rank. However the bound for the 0-rank of the nilpotent residual turns out to be simpler than that for $\gamma_{k+1}(G)$ as we shall now see, although it requires some further ideas. A key role here will be played by the Z-decomposition of certain normal abelian subgroups. However, for torsion-free subgroups these Z-decompositions can only exist in certain extensions. It is convenient to use the language of modules over group rings.

Proposition 3.7. *Let G be a group and let A be a torsion-free normal abelian subgroup of G. Suppose that ,for some natural number k, $A/(A \cap \zeta_k(G))$ has finite 0-rank r. If $G/C_G(A)$ is hypercentral, then A contains a pure G-invariant subgroup D satisfying the following conditions:*

(i) *D has a finite series of G -invariant pure subgroups*

$$1 = D_0 \leq D_1 \leq \cdots \leq D_n = D$$

such that D_j/D_{j-1} is rationally irreducible, for $1 \leq j \leq n$;

(ii) *A/D is G-nilpotent and $\mathbf{r}_0(D) \leq r$.*

Proof. The group G acts on A by conjugation and this action can be extended naturally to the action of the group ring $\mathbb{Z}G$ on A, making A into a $\mathbb{Z}G$-module. Let B denote the divisible envelope of A. The action of G on A can be naturally extended to an action of G on B and B can then be made into a $\mathbb{Q}G$-module. Let C be the divisible envelope of $A \cap \zeta_k(G)$ in B, so that C is a $\mathbb{Q}G$-submodule of B. Furthermore, $\dim_{\mathbb{Q}}(B/C) = \mathbf{r}_0(A/(A \cap \zeta_k(G) = r$ is finite so that B/C has a finite $\mathbb{Q}G$-composition series. It follows from Corollary [12, Corollary 2.6] that B has a Z-decomposition, $B = U \oplus E$, where U is the upper $\mathbb{Q}G$-hypercentre of B and E is a hypereccentric $\mathbb{Q}G$-submodule of B. Furthermore, $\mathbf{cl}_{\mathbb{Q}G}(E) \leq \mathbf{cl}_{\mathbb{Q}G}(B/C) \leq r$. Thus E has a finite series

$$1 = E_0 \leq E_1 \leq \cdots \leq E_n = E$$

of $\mathbb{Q}G$-submodules whose factors are simple $\mathbb{Q}G$-modules. Let $D = A \cap E$, $D_j = A \cap E_j$, for $1 \leq j \leq n$. Then every subgroup D_j is pure and G-invariant. Every factor D_j/D_{j-1} is G-rationally irreducible for $1 \leq j \leq n$. Finally, $\mathbf{r}_0(D) = \dim_{\mathbb{Q}}(E) \leq \mathbf{cl}_{\mathbb{Q}G}(E) \leq \mathbf{cl}_{\mathbb{Q}G}(B/C) \leq r$. Since B/E is G-nilpotent, A/D is also G-nilpotent. □

Proof of Theorem C. As in the proof of Theorem B we may assume that $\mathbf{Tor}\,(G) = 1$. Then the group G has a series $\zeta_k(G) = Z \leq V \leq K \leq G$ of normal subgroups such that V/Z is a torsion-free nilpotent group, K/V is finitely generated torsion-free abelian and G/K is finite. Let

$$1 = Z_0 \leq Z_1 \leq \cdots \leq Z_{k-1} \leq Z_k = Z$$

be a part of the upper central series of G. Now $G/C_G(Z)$ is nilpotent of class at most $k-1$, by [10]. Let $C = C_G(Z)$. Since $Z \leq C_G(C)$ it follows that $G/C_G(C)$ has finite 0-rank at most r. Clearly $C \cap Z \leq \zeta(C)$ and $C/(Z \cap C) \cong CZ/Z$ has finite 0-rank at most r. Using Theorem A we deduce that C' has finite 0-rank at most $r(5r^2 + 5r - 1)/2$. Let $D/C' = \mathbf{Tor}\,(C/C')$ and note that $\mathbf{r}_0(D) = \mathbf{r}_0(C')$. The subgroup D is G-invariant and C/D is a torsion-free abelian group. Hence $C/D \leq C_{G/D}(C/D)$ so that $(G/D)/C_{G/D}(C/D)$ is nilpotent and has finite 0-rank. We have $(C \cap Z)D/D \leq \zeta^{\infty}_{G/D}(C/D)$ and $(C/D)/((C \cap Z)D/D) \cong C/(C \cap Z)D$ has finite 0-rank at most r. By Corollary 3.7, C/D contains a pure G-invariant subgroup V/D of 0-rank at most r such that $(C/D)/(V/D)$ is G-nilpotent. Since G/C is nilpotent, G/V is also nilpotent, so $G^{\mathfrak{N}} \leq V$. Finally,

$$\mathbf{r}_0(V) = \mathbf{r}_0(D) + \mathbf{r}_0(V/D) \leq r(5r^2 + 5r - 1)/2 + r = r(5r^2 + 5r + 1)/2,$$

as required. □

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References

[1] R. Baer, Representations of groups as quotient groups. II. minimal central chains of a group, *Trans. Amer. Math. Soc.*, **58** (1945) 348–389.

[2] ———, Endlichkeitskriterien für Kommutatorgruppen, *Math. Ann.*, **124** (1952) 161–177.

[3] R. Baer and H. Heineken, Radical groups of finite abelian subgroup rank, *Illinois J. Math.*, **16** (1972) 533–580.

[4] A. Ballester-Bolinches, S. Camp-Mora, L. A. Kurdachenko and J. Otal, Extension of a Schur theorem to groups with a central factor with a bounded section rank, *J. Algebra*, **393** (2013) 1–15.

[5] M. R. Dixon, L. A. Kurdachenko and J. Otal, On groups whose factor-group modulo the hypercentre has finite section p-rank, *J. Algebra*, **440** (2015) 489–503.

[6] M. R. Dixon, L. A. Kurdachenko and N. V. Polyakov, Locally generalized radical groups satisfying certain rank conditions, *Ricerche di Matematica*, **56** (2007) 43–59.

[7] M. R. Dixon, L. A. Kurdachenko and A. A. Pypka, The theorems of Schur and Baer: a survey, *Int. J. Group Theory*, **4** (2015) 21–32.

[8] M. De Falco, F. de Giovanni, C. Musella and Y. P. Sysak, On the upper central series of infinite groups, *Proc. Amer. Math. Soc.*, **139** (2011) 385–389.

[9] V. M. Glushkov, On some questions of the theory of nilpotent and locally nilpotent groups without torsion, *Mat. Sbornik N. S.*, **30** (1952) 79–104.

[10] L. Kaloujnine, *Über gewisse Beziehungen zwischen einer Gruppe und ihren Automorphismen*, Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, Deutscher Verlag der Wissenschaften, Berlin, 1953 164–172.

[11] M. I. Kargapolov, On solvable groups of finite rank, *Algebra i Logika* Sem., **1** (1962) 37–44.

[12] L. A. Kurdachenko and J. Otal, The rank of the factor-group modulo the hypercenter and the rank of the some hypocenter of a group, *Cent. Eur. J. Math.*, **11** (2013) 1732–1741.

[13] ———, Groups with Chernikov factor-group by hypercentral, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, **109** (2015) 569–579.

[14] L. A. Kurdachenko, J. Otal and A. A. Pypka, On some properties of central and generalized series of groups, *Reports of the National Academy of Sciences of Ukraine*, **1** (2015) 20–24.

[15] L. A. Kurdachenko, J. Otal and I. Ya. Subbotin, *Artinian Modules over Group Rings*, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2007.

[16] L. A. Kurdachenko and P. Shumyatsky, The ranks of central factor and commutator groups, *Math. Proc. Cambridge Philos. Soc.*, **154** (2013) 63–69.

[17] A. I. Maltsev, Nilpotent torsion-free groups, *Izvestiya Akad. Nauk. SSSR. Ser. Mat.*, **13** (1949) 201–212.

[18] B. H. Neumann, Groups with finite classes of conjugate elements, *Proc. London Math. Soc. (3)*, **1** (1951) 178–187.

[19] D. J. S. Robinson, A new treatment of soluble groups with a finiteness condition on their abelian subgroups, *Bull. London Math. Soc.*, **8** (1976) 113–129.

[20] A. Schlette, Artinian, almost abelian groups and their groups of automorphisms, *Pacific J. Math.*, **29** (1969) 403–425.

[21] I. Schur, Über die Darstellungen der endlichen Gruppen durch gebrochene lineare Substitutionen, *J. reine angew. Math.*, **127** (1904) 20–50.

[22] D. I. Zaitsev, Hypercyclic extensions of abelian groups, Groups defined by properties of a system of subgroups (Russian), *Akad. Nauk Ukrain. SSR, Inst. Mat. Kiev*, **152** (1979) 16–37.

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