A CLASSIFICATION OF NILPOTENT 3-BCI GROUPS

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Abstract. Given a finite group $G$ and a subset $S \subseteq G$, the bi-Cayley graph $BCay(G, S)$ is the graph whose vertex set is $G \times \{0, 1\}$ and edge set is $\{(x, 0), (sx, 1) : x \in G, s \in S\}$. A bi-Cayley graph $BCay(G, S)$ is called a BCI-graph if for any bi-Cayley graph $BCay(G, T)$, $BCay(G, S) \cong BCay(G, T)$ implies that $T = gS^\alpha$ for some $g \in G$ and $\alpha \in Aut(G)$. A group $G$ is called an $m$-BCI-group if all bi-Cayley graphs of $G$ of valency at most $m$ are BCI-graphs. It was proved by Jin and Liu that, if $G$ is a 3-BCI-group, then its Sylow 2-subgroup is cyclic, or elementary abelian, or $Q_8$ [European J. Combin. 31 (2010) 1257–1264], and that a Sylow $p$-subgroup, $p$ is an odd prime, is homocyclic [Util. Math. 86 (2011) 313–320]. In this paper we show that the converse also holds in the case when $G$ is nilpotent, and hence complete the classification of nilpotent 3-BCI-groups.

1. Introduction

In this paper every group and every (di)graph will be finite. Given a group $G$ and a subset $S \subseteq G$, the bi-Cayley graph $BCay(G, S)$ of $G$ with respect to $S$ is the graph whose vertex set is $G \times \{0, 1\}$ and edge set is $\{(x, 0), (sx, 1) : x \in G, s \in S\}$. We call two bi-Cayley graphs $BCay(G, S)$ and $BCay(G, T)$ bi-Cayley isomorphic if $T = gS^\alpha$ for some $g \in G$ and $\alpha \in Aut(G)$ (here and in what follows for $x \in G$ and $R \subseteq G$, $xR = \{xr : r \in R\}$). It can be easily shown that bi-Cayley isomorphic bi-Cayley graphs are isomorphic as usual graphs. The converse implication is not true in general, and this makes the following definition interesting (see [24]): a bi-Cayley graph $BCay(G, S)$ is a $BCI$-graph if for any bi-Cayley graph $BCay(G, T)$, $BCay(G, S) \cong BCay(G, T)$ implies that $T = gS^\alpha$ for some $g \in G$ and $\alpha \in Aut(G)$. A group $G$ is called an $m$-$BCI$-group if all bi-Cayley graphs of $G$ of valency at most $m$
are BCI-graphs, and an $|G|$-BCI-group is simply called a $BCI$-group. It should be remarked that the above concepts are motivated by Cayley digraphs (more details on this will be given in Section 2).

The study of $m$-BCI-groups was initiated in [24], where it was shown that every group is a 1-BCI-group, and a group is a 2-BCI-group if and only if it has the property that any two elements of the same order are either fused or inverse fused (these groups are described in [18]). The problem of classifying all 3-BCI-groups is still open. Up to our knowledge, it is only known that every cyclic group is a 3-BCI-group (this is a consequence of [23, Theorem 1.1], see also [10]), and that $A_5$ is the only non-abelian simple 3-BCI-group (see [11]). As for BCI-groups, it was proved by M. Arezoomand and B. Taeri [2] that a finite BCI-group must be solvable. In this paper we make a further step by classifying the nilpotent 3-BCI-groups.

In fact, there is an explicit list of candidates for nilpotent 3-BCI groups, which arises from the earlier works of W. Jin and W. Liu [11, 12] on the Sylow $p$-subgroups of 3-BCI-groups. In particular, a Sylow 2-subgroup of a 3-BCI-group is $\mathbb{Z}_{2^r}$, $\mathbb{Z}_{2^r}^+$ or the quaternion group $\mathbb{Q}_8$ (see [11]), while a Sylow $p$-subgroup for $p > 2$ is homocyclic (see [12]). A group is said to be homocyclic if it is a direct product of cyclic groups of the same order. Consequently, if $G$ is a nilpotent 3-BCI-group, then $G$ decomposes as $G = U \times V$, where $U$ is a homocyclic group of odd order, and $V$ is trivial or one of the groups $\mathbb{Z}_{2^r}$, $\mathbb{Z}_{2^r}^+$ and $\mathbb{Q}_8$. In this paper we prove that the converse implication also holds, and hence complete the classification of nilpotent 3-BCI-groups.

**Theorem 1.1.** Every finite group $U \times V$ is a 3-BCI-group, where $U$ is a homocyclic group of odd order, and $V$ is trivial or one of the groups $\mathbb{Z}_{2^r}$, $\mathbb{Z}_{2^r}^+$ and $\mathbb{Q}_8$.

In Section 2, following the ideas of [3], we will see that the BCI-property of a given bi-Cayley graph can be read off entirely from its automorphism group (see Lemma 2.2). This was observed for cyclic groups in [13], and this result was later generalized to arbitrary groups in [1]. Theorem 1.1 will be proved in Section 3.

2. A Babai type lemma for bi-Cayley graphs

We start by setting the relevant notations and terminology.

**Notations.** Let $G$ be a group acting on a finite set $V$. For $g \in G$ and $v \in V$, the image of $v$ under $g$ will be written as $v^g$. For a subset $U \subseteq V$, we will denote by $G_U$ the elementwise stabilizer of $U$ in $G$, while by $G_{\{U\}}$ the setwise stabilizer of $U$ in $G$. If $U = \{u\}$, then $G_u$ will be written for $G_{\{u\}}$. We say that $U$ is $G$-invariant if $G$ leaves $U$ setwise fixed, or equivalently, when $G_{\{U\}} = G$. If $G$ is transitive on $V$ and $\Delta \subseteq V$ is a block for $G$, then the partition $\delta = \{\Delta^g : g \in G\}$ is called the system of blocks for $G$ induced by $\Delta$. The group $G$ acts on $\delta$ naturally, the corresponding kernel will be denoted by $G_\delta$, i.e., $G_\delta = \{g \in G : \Delta^g = \Delta' \text{ for all } \Delta' \in \delta\}$. For a graph $\Gamma$, we let $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$, and $\text{Aut}(\Gamma)$ denote the vertex set, the edge set, the arc set, and the full group of automorphisms of $\Gamma$, respectively. For a subset $U \subseteq V(\Gamma)$, we let $\Gamma[U]$ denote the subgraph of $\Gamma$ induced by $U$. A graph $\Gamma$ is called arc-transitive when $\text{Aut}(\Gamma)$ is transitive on $A(\Gamma)$. By $K_n$ and $K_{n,n}$ we will denote the complete
graph on \( n \) vertices and the complete bipartite graph on \( 2n \) vertices respectively. By a cubic graph we simply mean a regular graph of valency 3.

Let \( G \) be a group and \( S \subseteq G \). The Cayley digraph \( \text{Cay}(G, S) \) is the digraph whose vertex set is \( G \) and arc set is \( \{(x, sx) : x \in G, s \in S \} \). A Cayley digraph \( \text{Cay}(G, S) \) is called a CI-graph if for any Cayley digraph \( \text{Cay}(G, T) \), \( \text{Cay}(G, S) \cong \text{Cay}(G, T) \) implies that \( T = S^\alpha \) for some \( \alpha \in \text{Aut}(G) \), \( G \) is called an \( m \)-DCI-group if all Cayley digraphs of \( G \) of valency at most \( m \) are CI-graphs, and an \(|G| \)-CI-group is simply called a CI-graph (see [19, Definition 1.1]). Finite CI-groups and \( m \)-DCI-groups have attracted considerable attention over the last 40 years, for more information on these groups, the reader is referred to the survey [17]. The following result, frequently used in studying CI-graphs, is a special case of a lemma due to Babai [3, Lemma 3.1]:

**Lemma 2.1.** The following are equivalent for every Cayley digraph \( \Gamma = \text{Cay}(G, S) \).

1. \( \text{Cay}(G, S) \) is a CI-graph.
2. Every two regular subgroups of \( \text{Aut}(\Gamma) \), isomorphic to \( G \), are conjugate in \( \text{Aut}(\Gamma) \).

Given a group \( G \) with identity element \( 1_G \), we shall use the symbols \( 0 \) and \( 1 \) to denote the elements \((1_G, 0)\) and \((1_G, 1)\) in \( G \times \{0, 1\} \) respectively. For a subset \( S \subseteq G \), we write \((S, 0) = \{(s, 0) : s \in S\}\) and \((S, 1) = \{(s, 1) : s \in S\}\). For \( g \in G \), let \( \hat{g} \) be the permutation of \( G \times \{0, 1\} \) defined by

\[
(x, i)^{\hat{g}} = (xg, i) \quad \text{for every } x \in G \text{ and } i \in \{0, 1\}.
\]

We set \( \hat{G} = \{\hat{g} : g \in G\} \). Obviously, \( \hat{G} \leq \text{Aut}(\text{BCay}(G, S)) \), and \( \hat{G} \) is semiregular with orbits \( (G, 0) \) and \( (G, 1) \). In what follows will we denote by \( \mathcal{S}(\text{Aut}(\text{BCay}(G, S))) \) the set of all semiregular subgroups of \( \text{Aut}(\text{BCay}(G, S)) \) whose orbits are \( (G, 0) \) and \( (G, 1) \). Finally, we let \( G_{\text{right}} \leq \text{Sym}((G, 1)) \) be the permutation group induced by the action of \( \hat{G} \) on \( (G, 1) \).

The next lemma was proved by M. Arezoomand and B. Taeri [1]. For completeness, we give a proof here.

**Lemma 2.2.** The following are equivalent for every bi-Cayley graph \( \Gamma = \text{BCay}(G, S) \).

1. \( \text{BCay}(G, S) \) is a BCI-graph.
2. The normalizer \( N_{\text{Aut}(\Gamma)}(\hat{G}) \) is transitive on \( V(\Gamma) \), and every two subgroups in \( \mathcal{S}(\text{Aut}(\Gamma)) \), isomorphic to \( G \), are conjugate in \( \text{Aut}(\Gamma) \).

**Proof.** We start with the part \((1) \Rightarrow (2)\). Let \( X \in \mathcal{S}(\text{Aut}(\Gamma)) \) such that \( X \cong G \). We have to show that \( X \) and \( \hat{G} \) are conjugate in \( \text{Aut}(\Gamma) \). Let \( i \in \{0, 1\} \), and set \( X^{(G, i)} \) and \( \hat{G}^{(G, i)} \) for the permutation groups of the set \((G, i)\) induced by \( X \) and \( \hat{G} \) respectively. The groups \( X^{(G, i)} \) and \( \hat{G}^{(G, i)} \) are conjugate in \( \text{Sym}((G, i)) \), because these are isomorphic and regular on \((G, i)\). Thus \( X \) and \( \hat{G} \) are conjugate by a permutation \( \phi \in \text{Sym}(G \times \{0, 1\}) \) such that \((G, 0)\) is \( \phi \)-invariant (here \( \phi \) is viewed as a permutation of \( G \times \{0, 1\} \)). We write \( X = \phi \hat{G} \phi^{-1} \). Consider the graph \( \Gamma^\phi \), the image of \( \Gamma \) under \( \phi \). Then \( \hat{G} = \phi^{-1} X \phi \leq \text{Aut}((\Gamma)^\phi) \). Using this and that \((G, 0)\) is \( \phi \)-invariant, we obtain that \( \Gamma^\phi = \text{BCay}(G, T) \)
for some subset $T \subseteq G$. Then $\Gamma \cong \text{BCay}(G,T)$, and by (i), $T = gS^\alpha$ for some $g \in G$ and $\alpha \in \text{Aut}(G)$.

Define the permutation $\sigma$ of $G \times \{0,1\}$ by

$$(x,i)^\sigma = \begin{cases} (x^\alpha,0) & \text{if } i = 0, \\ (gx^\alpha,1) & \text{if } i = 1. \end{cases}$$

A direct calculation shows that $\sigma^{-1} \hat{\gamma} \sigma = \hat{\phi}$ if $g \in G$. Thus $\sigma$ normalizes $\hat{G}$. The vertex $(x,0)$ of $\text{BCay}(G,S)$ has neighborhood $(Sx,1)$. This is mapped by $\sigma$ to the the set $(gS^\alpha x^\alpha,1) = (Tx^\alpha,1)$. This proves that $\sigma$ is an isomorphism from $\Gamma$ to $\Gamma^\phi$, and in turn it follows that, $\Gamma^\phi = \Gamma^\sigma$, $\phi \sigma^{-1} \in \text{Aut}(\Gamma)$, and thus $\phi = \rho \sigma$ for some $\rho \in \text{Aut}(\Gamma)$. Finally, $X = \phi \hat{G} \phi^{-1} = \rho \sigma \hat{G} \sigma^{-1} \rho^{-1} = \rho \hat{G} \rho^{-1}$, i.e., $X$ and $\hat{G}$ are conjugate in $\text{Aut}(\Gamma)$.

In order to prove that the normalizer $N_{\text{Aut}(\Gamma)}(\hat{G})$ is transitive on $V(\Gamma)$, it is sufficient to find some automorphism $\eta$ which switches $(G,0)$ and $(G,1)$ and normalizes $\hat{G}$. Observe that $\text{BCay}(G,S) \cong \text{BCay}(G,S^{-1})$, where $S^{-1} = \{s^{-1} : s \in S\}$. Then by (i), $S^{-1} = gS^\alpha$ for some $g \in G$ and $\alpha \in \text{Aut}(G)$. We leave for the reader to verify that the permutation of $G \times \{0,1\}$ defined below is an appropriate choice for such $\eta$:

$$(x,i)^\eta = \begin{cases} (x^\alpha,1) & \text{if } i = 0, \\ (gx^\alpha,0) & \text{if } i = 1. \end{cases}$$

We turn to the part (2) $\Rightarrow$ (1). Let $\Gamma' = \text{BCay}(G,T)$ such that $\Gamma' \cong \Gamma$. We have to show that $T = gS^\alpha$ for some $g \in G$ and $\alpha \in \text{Aut}(G)$. We claim the existence of an isomorphism $\phi : \Gamma \to \Gamma'$ for which $\phi : (0) \to (0)$ and $(G,0)$ is $\phi$-invariant (here $\phi$ is viewed as a permutation of $G \times \{0,1\}$). We construct $\phi$ in a few steps. To start with, choose an arbitrary isomorphism $\phi_1 : \Gamma \to \Gamma'$. Since the normalizer $N_{\text{Aut}(\Gamma)}(\hat{G})$ is transitive on $V(\Gamma)$, there exists $\rho \in N_{\text{Aut}(\Gamma)}(\hat{G})$ which maps $0$ to $0^{\phi_1^{-1}}$. Let $\phi_2 = \rho \phi_1$. Then $\phi_2$ is an isomorphism from $\Gamma$ to $\Gamma'$, and also $\phi_2 : 0 \to 0$. The connected component of $\Gamma$ containing the vertex $0$ is equal to the induced subgraph $\Gamma[(H,0) \cup (sH,1)]$, where $s \in S$ and $H \leq G$ is generated by the set $s^{-1}S$. It can be easily checked that

$$\Gamma[(H,0) \cup (sH,1)] \cong \text{BCay}(H,s^{-1}S).$$

Similarly, the connected component of $\Gamma'$ containing the vertex $0$ is equal to the induced subgraph $\Gamma'[(K,0) \cup (tK,1)]$, where $t \in T$ and $K \leq G$ is generated by the set $t^{-1}T$, and

$$\Gamma'[(K,0) \cup (tK,1)] \cong \text{BCay}(K,t^{1-T}).$$

Since $\phi_2$ fixes $0$, it induces an isomorphism from $\Gamma[(H,0) \cup (sH,1)]$ to $\Gamma'[(K,0) \cup (tK,1)]$; denote this isomorphism by $\phi_3$. It follows from the connectedness of these induced subgraphs that $\phi_3$ preserves their bipartition classes, moreover, $\phi_3$ maps $(H,0)$ to $(K,0)$, since it fixes $0$. Finally, take $\phi : \Gamma \to \Gamma'$ to be the isomorphism whose restriction to each component of $\Gamma$ equals $\phi_3$. It is clear that $\phi : 0 \to 0$ and $(G,0)$ is $\phi$-invariant.

Since $\hat{G} \leq \Gamma'$, $\phi \hat{G} \phi^{-1} \leq \text{Aut}(\Gamma)$. The orbit of $0$ under $\phi \hat{G} \phi^{-1}$ is equal to $(G,0)^{\phi^{-1}} = (G,0)$, and hence $\phi \hat{G} \phi^{-1} \in S(\text{Aut}(\Gamma))$. By (ii), $\phi \hat{G} \phi^{-1} = \sigma^{-1} \hat{G} \sigma$ for some $\sigma \in \text{Aut}(\Gamma)$. Since $N_{\text{Aut}(\Gamma)}(\hat{G})$ is transitive on $V(\Gamma)$, $\sigma$ can be chosen so that $\sigma : 0 \to 0$. To sum up, we have an isomorphism
$(\sigma \phi) : \Gamma \mapsto \Gamma'$ which fixes 0 and also normalizes $\hat{G}$. Thus $(\sigma \phi)$ maps $(G, 1)$ to itself. Recall that $G_{\text{right}} \leq \text{Sym}((G, 1))$ is the permutation group induced by the action of $\hat{G}$ on $(G, 1)$. Then, the permutation of $(G, 1)$ induced by $(\sigma \phi)$ belongs to the holomorph of $G_{\text{right}}$ (cf. [8, Exercise 2.5.6]), and therefore, there exist $g \in G$ and $\alpha \in \text{Aut}(G)$ such that $(\sigma \phi) : (x, 1) \mapsto (gx^\alpha, 1)$ for all $x \in G$. On the other hand, being an isomorphism from $\Gamma$ to $\Gamma'$, $(\sigma \phi)$ maps $(S, 1)$ to $(T, 1)$. These give that $(T, 1) = (S, 1)^{\sigma \phi} = (gS^\alpha, 1)$, i.e., $T = gS^\alpha$.

\begin{proof}

Lemma 2.4. Whenever $\phi$ is an isomorphism from $\hat{G}$ to itself. Recall that $G_{\text{right}} \leq \text{Sym}((G, 1))$ is the permutation group induced by the action of $\hat{G}$ on $(G, 1)$. Then, the permutation of $(G, 1)$ induced by $(\sigma \phi)$ belongs to the holomorph of $G_{\text{right}}$ (cf. [8, Exercise 2.5.6]), and therefore, there exist $g \in G$ and $\alpha \in \text{Aut}(G)$ such that $(\sigma \phi) : (x, 1) \mapsto (gx^\alpha, 1)$ for all $x \in G$. On the other hand, being an isomorphism from $\Gamma$ to $\Gamma'$, $(\sigma \phi)$ maps $(S, 1)$ to $(T, 1)$. These give that $(T, 1) = (S, 1)^{\sigma \phi} = (gS^\alpha, 1)$, i.e., $T = gS^\alpha$.

\begin{remark}

Notice that, we cannot delete the condition on the normalizer $N_{\text{Aut}(\Gamma)}(\hat{G})$ from Lemma 2.2.(ii). To see this, consider the bi-Cayley graph $\Gamma = \text{BCay}(G, S)$, where

$$G = \langle a, b \mid a^5 = b^4 = 1, b^{-1}ab = a^2 \rangle \text{ and } S = \{1, a, b\}.$$ 

The group $G$ is the unique Frobenius group of order 20, and we find by the help of the computer package MAGMA [5] that $\Gamma$ is arc-transitive. In fact, $\Gamma$ is the unique arc-transitive cubic graph on 40 points (see [6]). We also compute that any two subgroups in $S(\text{Aut}(\Gamma))$, isomorphic to $G$, are conjugate in $\text{Aut}(\Gamma)$. We show below that, for any $g \in G$ and $\alpha \in \text{Aut}(G)$, $S^\alpha \neq gS^{-1}$. Since $\text{BCay}(G, S) \cong \text{BCay}(G, S^{-1})$, this implies that $\Gamma$ is not a BCI-graph.

To the contrary assume that $S^\alpha = gS^{-1}$ for some $g \in G$ and $\alpha \in \text{Aut}(G)$. It follows at once that $g \in S$. As no element in $bS^{-1} = \{b, ba^{-1}, 1\}$ is of order 5, $g \neq b$. Since every automorphism of $G$ is inner, $\alpha$ equals the conjugation by some element $c \in G$. Let $g = 1$. Then $S^\alpha = gS^{-1} = S^{-1}$, hence $a^c = a^\alpha = a^{-1}$ and $b^c = b^\alpha = b^{-1}$. From the first equality $c \in C_G(a)b^2 = \langle a \rangle b^2$, where $C_G(a)$ denotes the centraliser of $a$ in $G$, that is, $C_G(a) = \{x \in G : ax = xa\}$. Thus $c = a^ib^2$ for some $i \in \{0, \ldots, 4\}$. Plugging this in the second equality, we get $b^2a^{-i}ba^{-1}b^{-1}$, hence $a^{3i}b = b^{-1}$, which is impossible. Finally, let $g = a$. Then $S^\alpha = gS^{-1} = aS^{-1}$, hence $a^c = a^\alpha = a$ and $b^c = b^\alpha = ab^{-1}$. The first equality gives that $c = a^i$ for some $i \in \{0, \ldots, 4\}$. Plugging this in the second equality, we get $a^{-i}ba^i = ab^{-1}$, hence $a^{2i}b = ab^{-1}$, which is again impossible.

\end{proof}

As an application of Lemma 2.2, we prove the lemma below in which we connect the BCI-property with the CI-property. This lemma will be used in the proof of Theorem 1.1 in the particular case when the graphs are not arc-transitive.

\begin{lemma}

Let $\Gamma = \text{BCay}(G, S)$ such that there exists an involution $\tau \in \text{Aut}(\Gamma)$ which normalizes $\hat{G}$ and $0^\tau = 1$. Suppose, in addition, that $\text{Aut}(\Gamma)_0 = \text{Aut}(\Gamma)_1$. Then $\text{BCay}(G, S)$ is a BCI-graph whenever $\text{Cay}(G, S)$ is a CI-graph.

\begin{proof}

Set $A = \text{Aut}(\Gamma)$ and $A^+ = A_\{(G, 0)\}$, and let us suppose that $\text{Cay}(G, S)$ is a CI-graph. Let $X \in S(A), X \cong G$. Obviously, $X, \hat{G} \leq A^+$. The normalizer $N_A(\hat{G}) \geq \langle \hat{G}, \tau \rangle$, hence it is transitive on $V(\Gamma)$. Thus by Lemma 2.2 we are done if we show that $X$ and $\hat{G}$ are conjugate in $A^+$. In order to prove this we define a faithful action of $A^+$ on $G$ as follows. Let $\Delta = \{0, 1\}$ and consider the setwise stabilizer $A_{\{\Delta\}}$. Since $A_0 = A_1$, $A_0 \leq A_{\{\Delta\}}$. By [8, Theorem 1.5A], the orbit of 0 under $A_{\{\Delta\}}$ is a block for $A$. Since $\tau$ switches 0 and 1, this orbit is equal to $\Delta$, and the system of blocks induced by $\Delta$ is $\delta = \{\Delta^x : x \in G\} = \{(x, 0), (x, 1) : x \in G\}$.

\end{proof}

\end{lemma}
Now, define the action of \( A^+ \) on \( G \) by letting \( x^\sigma = x' \), where \( x \in G \) and \( \sigma \in A^+ \), if \( \sigma \) maps the block \( \{(x,0),(x,1)\} \) to the block \( \{(x',0),(x',1)\} \). We will write \( \tilde{\sigma} \) for the image of \( \sigma \) under the corresponding permutation representation, and let \( \tilde{B} = \{\tilde{\sigma} : \sigma \in B\} \) for a subgroup \( B \leq A^+ \). It is easily seen that this action is faithful. Therefore, \( X \) and \( \tilde{G} \) are conjugate in \( A^+ \) exactly when \( \tilde{X} \) and \( \tilde{G} \) are conjugate in \( A^+ \). Also, \( \tilde{G} = G_{\text{right}} \), and \( \tilde{X} \) is regular on \( G \). We finish the proof by showing that \( A^+ = \text{Aut}(\text{Cay}(G,S)) \). Then the conjugacy of \( \tilde{X} \) and \( \tilde{G} \) follows by Lemma 2.1 and the assumption that \( \text{Cay}(G,S) \) is a CI-graph.

Pick an automorphism \( \sigma \in A^+ \) and an arc \( (x,sx) \) of \( \text{Cay}(G,S) \). Then the edge \( \{(x,0),(sx,1)\} \) of \( \Gamma \) is mapped by \( \sigma \) to an edge \( \{(x',0),(s'x',1)\} \) for some \( x' \in G \) and \( s' \in S \). Hence \( \tilde{\sigma} : x \mapsto x' \) and \( sx \mapsto s'x' \), i.e., it maps the arc \( (x,sx) \) to the arc \( (x',s'x') \). We have just proved that \( \tilde{\sigma} \in \text{Aut}(\text{Cay}(G,S)) \), and hence \( A^+ \leq \text{Aut}(\text{Cay}(G,S)) \). In order to establish the relation \( \geq \), for an arbitrary automorphism \( \rho \in \text{Aut}(\text{Cay}(G,S)) \), define the permutation \( \pi \) of \( G \times \{0,1\} \) by \( (x,i)\pi = (x^\rho,i) \) for all \( x \in G \) and \( i \in \{0,1\} \). Repeating the previous argument we obtain that \( \pi \in \text{Aut}(\text{Cay}(G,S)) \). It is clear that \( \pi \in A^+ \) and \( \tilde{\pi} = \rho \). Thus \( A^+ \geq \text{Aut}(\text{Cay}(G,S)) \), and so \( A^+ = \text{Aut}(\text{Cay}(G,S)) \). The lemma is proved.

3. Proof of Theorem 1.1

In this section we denote by \( C \) the set of all groups \( U \times V \), where \( U \) is a homocyclic group of odd order, and \( V \) is either trivial or one of \( \mathbb{Z}_{2^r} \), \( \mathbb{Z}_2^\infty \) and \( \mathbb{Q}_8 \); and by \( C_{\text{sub}} \) the set of all groups that have an overgroup in \( C \).

**Lemma 3.1.** Let \( \Gamma \) be a cubic bipartite graph with bipartition classes \( \Delta_i, i = 1,2 \), and \( X \leq \text{Aut}(\Gamma) \) be a semiregular subgroup whose orbits are \( \Delta_i, i = 1,2 \), and \( X \in C_{\text{sub}} \). Then \( \text{Aut}(\Gamma) \) has an element \( \tau_X \) which satisfies:

1. every subgroup of \( X \) is normal in \( \langle X, \tau_X \rangle \);
2. \( \langle X, \tau_X \rangle \) is regular on \( V(\Gamma) \).

**Proof.** It is straightforward to show that \( \Gamma \cong \text{BCay}(X,S) \) for some subset \( S \subseteq X \) with \( 1_X \in S \) and \( |S| = 3 \). Moreover, there is an isomorphism from \( \Gamma \) to \( \text{BCay}(X,S) \) which induces a permutation isomorphism from \( X \) to \( \tilde{X} \). Therefore, it is sufficient to find \( \tau \in \text{Aut}(\text{BCay}(X,S)) \) for which every subgroup of \( \tilde{X} \) is normal in \( \langle \tilde{X}, \tau \rangle \); and \( \langle \tilde{X}, \tau \rangle \) is regular on \( V(\text{BCay}(X,S)) \).

Since \( X \in C_{\text{sub}}, X = U \times V \), where \( U \) is an abelian group of odd order, and \( V \) is trivial or one of \( \mathbb{Z}_{2^r}, \mathbb{Z}_2^\infty \) and \( \mathbb{Q}_8 \). We prove below the existence of an automorphism \( \iota \in \text{Aut}(X) \), which maps the set \( S \) to its inverse \( S^{-1} \). Let \( \pi_U \) and \( \pi_V \) denote the projections \( U \times V \to U \) and \( U \times V \to V \) respectively. It is sufficient to find an automorphism \( \iota_1 \in \text{Aut}(U) \) which maps \( \pi_U(S) \) to \( \pi_U(S)^{-1} \), and an automorphism \( \iota_2 \in \text{Aut}(V) \) which maps \( \pi_V(S) \) to \( \pi_V(S)^{-1} \). Since \( U \) is abelian, we are done by choosing \( \iota_1 \) to be the automorphism \( x \mapsto x^{-1} \). If \( V \) is abelian, then let \( \iota_2 : x \mapsto x^{-1} \). Otherwise, \( V \cong \mathbb{Q}_8 \), and since \( |\pi_{V}(S) \setminus \{1_{V}\}| \leq 2 \), it follows that \( \pi_{V}(S) \) is conjugate to \( \pi_{V}(S)^{-1} \) in \( V \). This ensures that \( \iota_2 \) can be chosen to be some inner automorphism. Now, define \( \iota \) by setting its restriction \( \iota|_U \) to \( U \) as \( \iota|_U = \iota_1 \),
and its restriction \( \iota|_V \) to \( V \) as \( \iota|_V = \iota_2 \). Define the permutation \( \tau \) of \( X \times \{0, 1\} \) by

\[
(x, i)^\tau = \begin{cases} (x^i, 1) & \text{if } i = 0, \\
(x^i, 0) & \text{if } i = 1.
\end{cases}
\]

The vertex \( (x, 0) \) of \( BCay(X, S) \) has neighborhood \((Sx, 1)\). This is mapped by \( \tau \) to the set \((S^{-1}x^i, 0)\), which is equal to the neighborhood of \((x^i, 1)\). We have proved that \( \tau \in Aut(BCay(X, S)) \).

It follows from its construction that \( \tau \) is an involution. Fix an arbitrary subgroup \( Y \leq X \), and pick \( y \in Y \). We may write \( y = y_Uy_V \) for some \( y_U \in U \) and \( y_V \in V \). Then \( \langle y_U, y_V \rangle \leq Y \), since \( y_U \) and \( y_V \) commute and \( \gcd(|U|, |V|) = 1 \). Also, \( (y_U)^{t_1} = y_U^{-1} \) and \( (y_V)^{t_2} \in \langle y_V \rangle \), implying that \( y' = (y_U)^{t_1}(y_V)^{t_2} \in \langle y_U, y_V \rangle \leq Y \). We conclude that \( \iota \) maps \( Y \) to itself. Thus \( \tau^{-1}y\tau = \tau y \tau = y' \) is in \( \hat{Y} \), and \( \tau \) normalizes \( \hat{Y} \). Since \( X \in C_{\text{sub}} \), \( \hat{Y} \) is also normal in \( \hat{X} \), and part (1) follows.

For part (2), observe that \( |\langle \hat{X}, \tau \rangle| = 2|X| = |V(BCay(X, S))| \). Clearly, \( \langle \hat{X}, \tau \rangle \) is transitive on \( V(BCay(X, S)) \), so it is regular.  

\[ \square \]

Let \( \Gamma \) be an arbitrary finite graph and \( G \leq Aut(\Gamma) \) which is transitive on \( V(\Gamma) \). For a normal subgroup \( N \triangleleft G \) which is not transitive on \( V(\Gamma) \), the quotient graph \( \Gamma_N \) is the graph whose vertices are the \( N \)-orbits on \( V(\Gamma) \), and two \( N \)-orbits \( \Delta_i, i = 1, 2 \), are adjacent if and only if there exist \( v_i \in \Delta_i, i = 1, 2 \), which are adjacent in \( \Gamma \). For a positive integer \( s \), an \( s \)-arc of \( \Gamma \) is an ordered \((s+1)\)-tuple \( (v_0, v_1, \ldots, v_s) \) of vertices of \( \Gamma \) such that, for every \( i \in \{1, \ldots, s\} \), \( v_{i-1} \) is adjacent to \( v_i \), and for every \( i \in \{1, \ldots, s-1\} \), \( v_{i-1} \neq v_{i+1} \). The graph \( \Gamma \) is called \((G, s)\)-arc-transitive \(((G, s)\)-arc-regular\) if \( G \) is transitive (regular) on the set of \( s \)-arcs of \( \Gamma \). If \( G = Aut(\Gamma) \), then a \((G, s)\)-arc-transitive \(((G, s)\)-arc-regular\) graph is simply called \( s \)-transitive \((s\)-regular\). The proof of the following lemma is straightforward, hence it is omitted (it can be also deduced from \cite[Theorem 9]{20}).

**Lemma 3.2.** Let \( \Gamma = BCay(G, S) \) be a connected arc-transitive graph, \( G \) be any finite group, \( |S| = 3 \), and \( N < \hat{G} \) be a subgroup which is normal in \( Aut(\Gamma) \). Then the following hold:

1. \( \Gamma_N \) is a cubic connected arc-transitive graph.
2. \( N \) is equal to the kernel of \( Aut(\Gamma) \) acting on the set of \( N \)-orbits.
3. \( \Gamma_N \) is isomorphic to a bi-Cayley graph of the group \( \hat{G}/N \).

**Remark 3.3.** Let \( \Gamma \) and \( N \) be as described in Lemma 3.2. The group \( Aut(\Gamma) \) acts on the set of \( N \)-orbits, i.e., on the vertex set \( V(\Gamma_N) \). Lemma 3.2.(ii) implies that, the induced permutation group on \( V(\Gamma_N) \) is isomorphic to \( Aut(\Gamma)/N \), and therefore, by some abuse of notation, this permutation group will also be denoted by \( Aut(\Gamma)/N \). In what follows we shall write \( Aut(\Gamma)/N < Aut(\Gamma_N) \). Also note that, if \( \Gamma \) is \( s \)-transitive, then \( \Gamma_N \) is \((Aut(\Gamma)/N, s)\)-arc-transitive.

The proof of Theorem 1.1 in the case of arc-transitive graphs will be based on three lemmas about cubic connected arc-transitive bi-Cayley graphs to be proved below. In these lemmas we keep the following notation:

\[(*) \quad \Gamma = BCay(G, S) \text{ is a connected arc-transitive graph, where } G \in C_{\text{sub}} \text{ and } |S| = 3.\]
Lemma 3.4. With notation (⋆), let $\delta$ be a system of blocks for $\text{Aut}(\Gamma)$ induced by a block properly contained in $(G,0)$, and $X$ be in $\mathcal{S}(\text{Aut}(\Gamma))$ such that $X \in \mathcal{C}_{\text{sub}}$. Then for the kernel $A_\delta$ (see Notations), $A_\delta < X$. Moreover, if $\delta$ is non-trivial, then $A_\delta$ is also non-trivial.

Proof. Set $A = \text{Aut}(\Gamma)$. Let $Y = X \cap A_\delta(\Delta)$, where $\Delta \in \delta$ with $\Delta \subset (G,0)$. Then $\Delta$ is equal to an orbit of $Y$, and $|Y| = |\Delta|$ because $\Delta \subset (G,0)$ and $X$ is regular on $(G,0)$. Formally, $\Delta = \text{Orb}_Y(v)$ for some vertex $v \in \Delta$.

Let $\tau_X \in A$ be the automorphism defined in Lemma 3.1, and set $L = \langle X, \tau_X \rangle$. The group $L$ is regular on $V(\Gamma)$, and $Y \not\subseteq L$. These yield

$$\delta = \{\Delta^l : l \in L\} = \{\text{Orb}_Y(v)^l : l \in L\} = \{\text{Orb}_Y(v^l) : l \in L\}.$$ 

From this $Y \leq A_\delta$. This shows that, if $|Y| = |\Delta| \neq 1$, then $A_\delta$ is non-trivial. Since $\delta$ has more than 2 blocks, and $\Gamma$ is a connected and cubic graph, it is known that $A_\delta$ is semiregular. These imply that $A_\delta = Y \leq X$. □

Corollary 3.5. With notation (⋆), let $N < \hat{G}$ be normal in $\text{Aut}(\Gamma)$, and $X$ be in $\mathcal{S}(\text{Aut}(\Gamma))$ such that $X \in \mathcal{C}_{\text{sub}}$. Then $N < X$.

Proof. Let $\delta$ be the system of blocks for $\text{Aut}(\Gamma)$ consisting of the $N$-orbits. Then $A_\delta = N$ by Lemma 3.2.(ii), and the corollary follows directly from Lemma 3.4. □

We denote by $Q_3$ the graph of the cube and by $H$ the Heawood graph, i.e., the unique arc-transitive cubic graph on 14 points (see [6]). Recall that, the core of a subgroup $H \leq K$ in the group $K$ is the largest normal subgroup of $K$ contained in $H$.

Lemma 3.6. With notation (⋆), suppose that $\hat{G}$ is not normal in $\text{Aut}(\Gamma)$, and let $N$ be the core of $\hat{G}$ in $\text{Aut}(\Gamma)$. Then $(\hat{G}/N, \Gamma_N)$ is isomorphic to one of the pairs $(Z_3, K_{3,3})$, $(Z_4, Q_3)$, and $(Z_7, H)$.

Proof. Set $A = \text{Aut}(\Gamma)$. Consider the quotient graph $\Gamma_N$, and suppose that $M \leq \hat{G}$ such that $N \leq M$ and $M/N \leq \text{Aut}(\Gamma_N)$ (here $M/N \leq A/N \leq \text{Aut}(\Gamma_N)$), see Remark 3.3). This in turn implies that, $M/N \leq A/N$, $M \leq A$, and $M = N$. We conclude that, $\Gamma_N$ is a bi-Cayley graph of $\hat{G}/N$, $\hat{G}/N$ is in $\mathcal{C}_{\text{sub}}$, and $\hat{G}/N$ has trivial core in $\text{Aut}(\Gamma_N)$. This shows that it is sufficient to prove Lemma 3.6 in the particular case when $N$ is trivial. For the rest of the proof we assume that the core $N$ is trivial, and we write $N = 1$.

By Tutte Theorem [22], $\Gamma$ is $k$-regular for some $k \leq 5$. Set $A^+ = \text{Aut}(\Gamma)_{\{(G,0)\}}$. It follows from the connectedness of $\Gamma$ that $A = \langle A^+, \tau_{\hat{G}} \rangle$, where $\tau_{\hat{G}} \in A$ is the automorphism defined in Lemma 3.1. Let $M$ be the core of $\hat{G}$ in $A^+$. Then $M \leq A$, since $M$ is normalized by $\tau_{\hat{G}}$, see Lemma 3.1.(i), and $A = \langle A^+, \tau_{\hat{G}} \rangle$. Thus $M \leq N = 1$, hence $M$ is also trivial.

Let us consider $A^+$ acting on the set $[A^+ : \hat{G}]$ of right $\hat{G}$-cosets in $A^+$. This action is faithful because $M$ is trivial. The corresponding degree is equal to $|A^+ : \hat{G}|$. Since $A = \text{Aut}(X)$ is regular on the set of $k$-arcs of $X$, thus $|A|$ is equal to the number of $k$-arcs of $X$, which is $|V(X)| \cdot 3 \cdot 2^{k-1} = |\hat{G}| \cdot 3 \cdot 2^k$. 

Since $|A^+| = |A|/2$, it follows that

$$|A^+ : \hat{G}| = \frac{|\hat{G}| \cdot 3 \cdot 2^k}{2 \cdot |\hat{G}|} = 3 \cdot 2^{k-1}.$$ 

Since $\hat{G}$ acts as a point stabilizer in this action, we have an embedding of $G$ into $S_{3 \cdot 2^k-1-1}$. We will write below that $G \leq S_{3 \cdot 2^k-1-1}$.

Recall that, $A_0$ is determined uniquely by $k$, and we have, respectively, $A_0 \cong \mathbb{Z}_4$, or $S_3$, or $D_{12}$, or $S_4$, or $S_4 \times \mathbb{Z}_2$. We go through each case.

CASE 1. $k = 1$.

This case can be excluded at once by observing that we have $G \leq S_2$ by the above discussion, which contradicts the obvious bound $|G| \geq 3$.

CASE 2. $k = 2$.

In this case $G \leq S_5$. Using also that $G \in C_{\text{sub}}$, we see that $G$ is abelian, hence $|G| \leq 6$, $|V(G)| \leq 12$. We obtain by [6, Table] that $\Gamma \cong Q_3$, and $G \cong \mathbb{Z}_4$.

CASE 3. $k = 3$.

Then $A^+ = \hat{G}A_0 = \hat{G}D_{12}$, a product of a nilpotent and a dihedral subgroup. Thus $A^+$ is solvable by Huppert-Itô Theorem (cf. [21, 13.10.1]). Assume for the moment that $A^+$ is imprimitive on $(G, 0)$. This implies that $A$ is also imprimitive on $V(G)$ and it has a non-trivial block system $\delta$ which has a block properly contained in $(G, 0)$. Lemma 3.4 gives that $A_\delta < \hat{G}$, and $A_\delta$ is non-trivial. This, however, contradicts that the core $N = 1$. Thus $A^+$ is primitive on $(G, 0)$. Using that $A^+$ is also solvable, we find that $G$ is a $p$-group. We see that $G$ is either abelian or it is $Q_8$. In the latter case $|V(G)| = 16$, and $\Gamma$ is isomorphic to the Moebius-Kantor graph, which is, however, 2-regular (see [6, Table]). Therefore, $G$ is an abelian $p$-group. Let $S = \{s_1, s_2, s_3\}$. Since $G$ is abelian, for $\Gamma$ we have:

$$0 \sim (s_1, 1) \sim (s_2^{-1}s_1, 0) \sim (s_3s_2^{-1}s_1, 1) = (s_1s_2^{-1}s_3, 1) \sim (s_2^{-1}s_3, 0) \sim (s_3, 1) \sim 0.$$ 

Thus $\Gamma$ is of girth at most 6. It was proved in [7, Theorem 2.3] that the Pappus graph graph on 18 points and the Desargues graph graph on 20 points are the only 3-regular cubic graphs of girth 6. For the latter graph $|G| = 10$, contradicting that $G$ is a $p$-group. We exclude the former graph by the help of MAGMA. We compute that the Pappus graph has no abelian semiregular automorphism group of order 9 which has trivial core in the full automorphism group. Thus $\Gamma$ is of girth 4 (3 and 5 are impossible as the graph is bipartite). It is well-known that there are only two cubic arc-transitive graphs of girth 4 (see also [14, page 163]): $K_{3,3}$ and $Q_3$. We get at once that $\Gamma \cong K_{3,3}$ and $G \cong \mathbb{Z}_3$.

CASE 4. $k = 4$.

It is sufficient to show that $G$ is abelian. Then by the above reasoning $\Gamma$ is of girth 6, and as the Heawood graph is the only cubic 4-regular graph of girth 6 (see [7, Theorem 2.3]), we get at once that $\Gamma \cong H$ and $G \cong \mathbb{Z}_7$. 


Assume, towards a contradiction, that $G$ is non-abelian. Thus $G = U \times V$, where $U$ is an abelian group of odd order, and $V \cong Q_8$. We have already shown above that $A^+$ is primitive on $(G, 0)$. In other words, $\Gamma$ is a 4-transitive bi-primitive cubic graph. Recall that a permutation group on a set $\Omega$ is called bi-primitive if it is transitive and imprimitive, and $\Omega$ has only one nontrivial system of blocks consisting of exactly two blocks.

Two possibilities can be deduced from the list of 4-transitive bi-primitive graphs given in [16, Theorem 1.4]:

- $\Gamma$ is the standard double cover of a connected vertex-primitive cubic 4-regular graph, in which case $A = A^+ \times \langle \eta \rangle$ for an involution $\eta$; or
- $\Gamma$ isomorphic to the sextet graph $S(p)$ (see [4]), where $p \equiv \pm 7 \pmod{16}$, in which case $A \cong PGL(2, p)$, and $A^+ \cong PSL(2, p)$.

The second possibility cannot occur, because then $A^+ \cong PSL(2, p)$, whose Sylow 2-subgroup is a dihedral group (cf. [9, Satz 8.10]), which contradicts that $V \leq \hat{G} \leq A^+$, and $V \cong Q_8$. It remains to exclude the first possibility. We may assume, by replacing $S$ with $xS$ for a suitable $x \in G$ if necessary, that $\eta$ switches 0 and 1. Since $\eta$ commutes with $\hat{G}$, we find $(x, 1)^\eta = 1^\eta x = (x, 0)$ for every $x \in G$. Let $s \in S$. Then $0 \sim (s, 1)$, hence $1 = 0^\eta \sim (s, 1)^\eta = (s, 0)$, which shows that $s \in S^{-1}$, and thus $S = S^{-1}$. Thus there exists $s \in S$ with $o(s) \leq 2$. Put $T = s^{-1}S = sS$. Then $1_G \in T$, and since $\Gamma$ is connected, $G = \langle T \rangle$. Notice that $s \in Z(G)$. This implies that $T^{-1} = S^{-1}s = sS = T$, and thus $\pi_V(T)$ satisfies $1_V \in \pi_V(T)$ and $\pi_V(T) = \pi_V(T)^{-1}$. Since $V \cong Q_8$, this implies that $\langle \pi_V(T) \rangle \neq V$, a contradiction to $G = \langle T \rangle$. This completes the proof of this case.

CASE 5. $k = 5$.

In this case $\Gamma$ is a 5-transitive bi-primitive cubic graph. It was proved in [16, Corollary 1.5] that $\Gamma$ is isomorphic to either the $PTL(2, 9)$-graph on 30 points (also known as the Tutte’s 8-Cage), or the standards double cover of the $PSL(3, 3)_2$-graph on 468 points. These graphs are of girth 8 and 12 respectively (see [6, Table]). Also, in both cases $8 \nmid |G|$, hence $G$ is abelian. In this case, however, the graph $\Gamma$ has a closed walk of length 6, as shown in Eq. (3.1), hence its girth cannot be larger than 6. This proves that this case does not occur. \hfill \Box

For a group $A$ and a prime $p$ dividing $|A|$, we let $A_p$ denote a Sylow $p$-subgroup of $A$.

Lemma 3.7. With notation (*), let $X \in S(Aut(\Gamma))$ such that $X \in C_{sub}$ and $X_2 \cong G_2$. Then $X$ and $\hat{G}$ are conjugate in $Aut(\Gamma)$.

Remark 3.8. We remark that, the assumption $X_2 \cong G_2$ cannot be deleted. The M"obius-Kantor graph is a bi-Cayley graph of the group $Q_8$, which has a semiregular cyclic group of automorphisms of order 8 which preserves the bipartition classes.

Proof. Set $A = Aut(\Gamma)$. The proof is split into two parts according to whether $\hat{G}$ is normal in $A$.

CASE 1. $\hat{G}$ is not normal in $A$. 

Let $N$ be the core of $\hat{G}$ in $A$. By Corollary 3.5, $N < X \cap \hat{G}$. Therefore, it is sufficient to show that
\begin{equation}
X/N \text{ and } \hat{G}/N \text{ are conjugate in } A/N.
\end{equation}
Recall that, the group $A/N \leq \text{Aut}(\Gamma_N)$ for the quotient graph $\Gamma_N$ induced by $N$ (see Remark 3.3 and the preceding paragraph). Both groups $X/N$ and $\hat{G}/N$ are semiregular whose orbits are the bipartition classes of $\Gamma_N$. Also notice that, $\hat{G}/N$ cannot be normal in $A/N$, otherwise $\hat{G}$ were normal in $A$.

According to Lemma 3.6, $(\hat{G}/N, \Gamma_N) \cong (\mathbb{Z}_3, K_{3,3})$, or $(\mathbb{Z}_4, Q_3)$, or $(\mathbb{Z}_7, \mathcal{H})$. Thus (1) follows immediately from Sylow Theorems when $(\hat{G}/N, \Gamma_N) \cong (\mathbb{Z}_7, \mathcal{H})$.

Let $(\hat{G}/N, \Gamma_N) \cong (\mathbb{Z}_3, K_{3,3})$. Since $\hat{G}/N$ is not normal in $A/N$, and $\Gamma_N$ is $(A/N,1)$-arc-transitive, we compute by Magma that $A/N = \text{Aut}(\Gamma_N)$, or it is a subgroup of $\text{Aut}(\Gamma_N)$ of index 2. In both cases $A/N$ has one conjugacy class of semiregular subgroups whose orbits are the bipartition classes of $\Gamma_N$. Thus (1) holds.

Let $(\hat{G}/N, \Gamma_N) \cong (\mathbb{Z}_4, Q_3)$. Since $X_2 \cong G_2$, $X/N \cong \hat{G}/N \cong \mathbb{Z}_4$. Using this and that $\Gamma_N$ is $(A/N,1)$-arc-transitive, we compute by Magma that $A/N = \text{Aut}(\Gamma_N)$, and that $\text{Aut}(\Gamma_N)$ has one conjugacy class of semiregular cyclic subgroups whose orbits are the bipartition classes of $\Gamma_N$. Thus (1) holds also in this case.

**CASE 2.** $\hat{G}$ is normal in $A$.

We have to show that $X = \hat{G}$. Notice that, $X$ contains every proper subgroup $K < \hat{G}$ which is characteristic in $\hat{G}$. Indeed, since $\hat{G} \trianglelefteq A$, we have that $K \trianglelefteq A$, and hence $K < X$ follows from Corollary 3.5. This property will be used often below.

In particular, $\hat{G}_p < \hat{G}$ is characteristic for every prime $p$ dividing $|\hat{G}|$. If $G$ is not a $p$-group, then $\hat{G}_p < \hat{G}$, and by the above observation $\hat{G}_p < X$. This gives that $X = \hat{G}$ if $G$ is not a $p$-group. Let $G$ be a $p$-group. If $p > 3$, then both $\hat{G}$ and $X$ are Sylow $p$-subgroups of $A$, and the statement follows from Sylow Theorems. Notice that, since $\Gamma$ is connected, $G$ is generated by the set $s^{-1}S$ for some $s \in S$, hence it is generated by two elements.

Let $p = 2$. Assume for the moment that $G$ is cyclic. Then $\hat{G}$ has a characteristic subgroup $K$ such that $\hat{G}/K \cong \mathbb{Z}_4$. Then $K \trianglelefteq A$, $\Gamma_K \cong Q_3$. Moreover, $\Gamma_K$ is a bi-Cayley graph of $\hat{G}/K$, and $\hat{G}/K$ is normal in $A/K \trianglelefteq \text{Aut}(\Gamma_K)$. A simple computation, using Magma, shows that this situation does not occur. Let $G$ be a non-cyclic 2-group in $\mathcal{C}_{\text{sub}}$. Also using the fact that $G$ is generated by two elements, we conclude that either $G \cong \mathbb{Z}_2^2$ and $\Gamma \cong Q_3$, or $G \cong Q_8$ and $\Gamma$ is the Moebsius-Kantor graph. Now, $X = X_2 \cong G_2 = G$. Then $X = \hat{G}$ can be verified by the help of Magma in either case.

Let $p = 3$. Observe first that $|G| > 3$. For otherwise, $\Gamma \cong K_{3,3}$, but no semiregular automorphism group of order 3 is normal in $\text{Aut}(K_{3,3})$. Since $G$ is generated by two elements, we may write $G \cong \mathbb{Z}_e \times \mathbb{Z}_3^f$, where $e \geq 1$ and $0 \leq f \leq e$. If $e = 1$, then $f = 1$, $G \cong \mathbb{Z}_3^2$, and $\Gamma$ is the Pappus graph. However, this graph has no automorphism group which is isomorphic to $\mathbb{Z}_3^2$ and also normal in the full automorphism group. Therefore, $e \geq 2$. Define $K = \{ \hat{x} : x \in G \text{ and } o(x) \leq 3^{e-2} \}$. Then $K$ is a characteristic subgroup of $\hat{G}$. Thus $K \trianglelefteq A$, and $\Gamma_K$ is a BiCayley graph of $\hat{G}/K$. 


Let $f \leq e - 2$. Then $\hat{G}/K \cong \mathbb{Z}_9$, and $\Gamma_K$ is the Pappus graph. This graph, however, does not have a cyclic semiregular automorphism group of order 9. We conclude that $f \in \{e - 1, e\}$.

Let $f = e - 1$. Then $\hat{G}/K \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. It follows that $\Gamma_K$ is the unique cubic arc-transitive graph on 54 points (see [6, Table]). We have checked by Magma that this graph has a unique semiregular abelian automorphism group whose orbits are the bipartition classes. Therefore, $X/K = \hat{G}/K$. This together with $K < X \cap \hat{G}$ yield that $X = \hat{G}$.

Finally, let $f = e$. Then $\hat{G}/K \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. It follows that $\Gamma_K$ is the unique cubic arc-transitive graph on 162 points (see [6, Table]). A direct computation, using Magma, gives that $X/K = \hat{G}/K$, which together with $K < X \cap \hat{G}$ yield that $X = \hat{G}$. \hfill \Box

Recall that, a group $H$ is homogeneous if every isomorphism between two subgroups of $H$ can be extended to an automorphism of $H$. The following result is [15, Proposition 3.2]:

**Proposition 3.9.** Every 2-DCI-group is homogeneous.

Since every group in $\mathcal{C}$ is a 2-DCI-group (see [15, Theorem 1.3]), we have the corollary that every group in $\mathcal{C}$ is homogeneous.

Everything is prepared to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $G \in \mathcal{C}$ and $\Gamma = \text{BCay}(G, S)$ such that $|S| \leq 3$. We have to show that $\Gamma$ is a BCI-graph. This holds trivially when $|S| = 1$, and follows from the homogeneity of $G$ when $|S| = 2$. Let $|S| = 3$.

CASE 1. $\Gamma$ is arc-transitive.

Let $\text{BCay}(G, S) \cong \text{BCay}(G, T)$ for some subset $T \subseteq G$. We may assume without loss of generality that $1_G \in S \cap T$. Let $H = \langle S \rangle$ and $K = \langle T \rangle$. Then $H, K \in \mathcal{C}_{\text{sub}}$, both bi-Cayley graphs $\text{BCay}(H, S)$ and $\text{BCay}(K, T)$ are connected, and $\text{BCay}(H, S) \cong \text{BCay}(K, T)$. We claim that $\text{BCay}(H, S)$ is a BCI-graph. In view of Lemma 2.2, this holds if the normalizer of $\hat{H}$ in $\text{Aut}(\text{BCay}(H, S))$ is transitive on the vertex-set $V(\text{BCay}(H, S))$, and for every $X \in \text{S}(\text{Aut}(\text{BCay}(H, S)))$, isomorphic to $H$, $X$ and $\hat{H}$ are conjugate in $\text{Aut}(\text{BCay}(H, S))$. Now, the first part follows from Lemma 3.1, while the second part follows from Lemma 3.7.

Let $\phi$ be an isomorphism from $\text{BCay}(K, T)$ to $\text{BCay}(H, S)$, and consider the group $X = \phi^{-1}\hat{K}\phi \leq \text{Sym}(H)$. Since $\phi$ maps the bipartition classes of $\text{BCay}(K, T)$ to the bipartition classes of $\text{BCay}(H, S)$, we have $X \in \text{S}(\text{Aut}(\text{BCay}(H, S)))$. Also, $X_2 \cong \hat{H}_2$, because $X \cong K, |H| = |K|$ and $H$ and $K$ are both contained in the group $G$ from $\mathcal{C}$. Thus Lemma 3.7 is applicable, as a result, $X$ and $\hat{H}$ are conjugate in $\text{Aut}(\text{BCay}(H, S))$. In particular, $H \cong K$. Since $G$ is homogeneous, there exists $\alpha_1 \in \text{Aut}(G)$ such that $K^{\alpha_1} = H$. This $\alpha_1$ induces an isomorphism from $\text{BCay}(K, T)$ to $\text{BCay}(H, T^{\alpha_1})$. Therefore, $\text{BCay}(H, S) \cong \text{BCay}(H, T^{\alpha_1})$, and since $\text{BCay}(H, S)$ is a BCI-graph, $T^{\alpha_1} = gS^{\alpha_2}$ for some $g \in H$ and $\alpha_2 \in \text{Aut}(H)$. By the homogeneity of $G$, $\alpha_2$ extends to an automorphism of $G$, implying that $\text{BCay}(G, S)$ is a BCI-graph.

CASE 2. $\Gamma$ is not arc-transitive.
Since $\Gamma$ is vertex-transitive (see Lemma 3.1), but not arc-transitive, we have $A_0 = A_{(s,1)}$ for some $s \in S$. We show below that $\text{BCay}(G, s^{-1}S)$ is a BCI-graph, this obviously yields that the same holds for $\text{BCay}(G, S)$. Define the permutation $\phi$ of $G \times \{0, 1\}$ by

$$\phi(x, i) = \begin{cases} (x, 0) & \text{if } i = 0, \\ (s^{-1}x, 1) & \text{if } i = 1. \end{cases}$$

The vertex $(x, 0)$ of $\text{BCay}(G, S)$ has neighborhood $(Sx, 1)$. This is mapped by $\phi$ to the the set $(s^{-1}Sx, 1)$. This shows that $\phi$ is an isomorphism from $\Gamma$ to $\Gamma' = \text{BCay}(G, s^{-1}S)$. Then we have $\text{Aut}(\Gamma_0) = \phi^{-1}A_0\phi = \phi^{-1}A_{(s,1)}\phi = \text{Aut}(\Gamma')_1$. Let $\tau_G^*$ be the automorphism of $\Gamma'$ defined in Lemma 3.1. It follows that $\tau_G^*$ is an involution (see the proof of Lemma 3.1), which normalizes $G$ and maps 0 to 1. Now, Lemma 2.4 is applicable to $\Gamma'$, as a result, it is sufficient to show that $\text{Cay}(G, s^{-1}S)$ is a CI-graph. This follows because $|s^{-1}S \setminus \{1_G\}| = 2$ and that $G$ is a 2-DCI-group (see [15, Theorem 1.3]). This completes the proof of the theorem. $\square$

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References


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