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A CLASSIFICATION OF NILPOTENT 3-BCI GROUPS

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Abstract. Given a finite group G and a subset $S \subseteq G$, the bi-Cayley graph $\mathrm{BCay}(G,S)$ is the graph whose vertex set is $G \times \{0,1\}$ and edge set is $\{\{(x,0),(sx,1)\} : x \in G, s \in S\}$. A bi-Cayley graph $\mathrm{BCay}(G,S)$ is called a BCI-graph if for any bi-Cayley graph $\mathrm{BCay}(G,T)$, $\mathrm{BCay}(G,S) \cong \mathrm{BCay}(G,T)$ implies that $T = gS^\alpha$ for some $g \in G$ and $\alpha \in \mathrm{Aut}(G)$. A group G is called an m-BCI-group if all bi-Cayley graphs of G of valency at most m are BCI-graphs. It was proved by Jin and Liu that, if G is a 3-BCI-group, then its Sylow 2-subgroup is cyclic, or elementary abelian, or $\mathbf{Q}_8$ [European J. Combin. 31 (2010) 1257–1264], and that a Sylow p-subgroup, p is an odd prime, is homocyclic [Util. Math. 86 (2011) 313–320]. In this paper we show that the converse also holds in the case when G is nilpotent, and hence complete the classification of nilpotent 3-BCI-groups.

1. Introduction

In this paper every group and every (di)graph will be finite. Given a group G and a subset $S \subseteq G$, the *bi-Cayley graph* $\mathrm{BCay}(G,S)$ of G with respect to S is the graph whose vertex set is $G \times \{0,1\}$ and edge set is $\{\{(x,0),(sx,1)\} : x \in G, s \in S\}$. We call two bi-Cayley graphs $\mathrm{BCay}(G,S)$ and $\mathrm{BCay}(G,T)$ *bi-Cayley isomorphic* if $T = gS^\alpha$ for some $g \in G$ and $\alpha \in \mathrm{Aut}(\Gamma)$ (here and in what follows for $x \in G$ and $R \subseteq G$, $xR = \{xr : r \in R\}$). It can be easily shown that bi-Cayley isomorphic bi-Cayley graphs are isomorphic as usual graphs. The converse implication is not true in general, and this makes the following definition interesting (see [24]): a bi-Cayley graph $\mathrm{BCay}(G,S)$ is a *BCI-graph* if for any bi-Cayley graph $\mathrm{BCay}(G,T)$, $\mathrm{BCay}(G,S) \cong \mathrm{BCay}(G,T)$ implies that $T = gS^\alpha$ for some $g \in G$ and $\alpha \in \mathrm{Aut}(G)$. A group G is called an *m-BCI-group* if all bi-Cayley graphs of G of valency at most m

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are BCI-graphs, and an $|G|$-BCI-group is simply called a *BCI-group*. It should be remarked that the above concepts are motivated by Cayley digraphs (more details on this will be given in Section 2).

The study of m-BCI-groups was initiated in [24], where it was shown that every group is a 1-BCI-group, and a group is a 2-BCI-group if and only if it has the property that any two elements of the same order are either fused or inverse fused (these groups are described in [18]). The problem of classifying all 3-BCI-groups is still open. Up to our knowledge, it is only known that every cyclic group is a 3-BCI-group (this is a consequence of [23, Theorem 1.1], see also [10]), and that A_5 is the only non-abelian simple 3-BCI-group (see [11]). As for BCI-groups, it was proved by M. Arezoomand and B. Taeri [2] that a finite BCI-group must be solvable. In this paper we make a further step by classifying the nilpotent 3-BCI-groups.

In fact, there is an explicit list of candidates for nilpotent 3-BCI groups, which arises from the earlier works of W. Jin and W. Liu [11, 12] on the Sylow p-subgroups of 3-BCI-groups. In particular, a Sylow 2-subgroup of a 3-BCI-group is $\mathbb{Z}_{2^r}$, $\mathbb{Z}_2^r$ or the quaternion group $\mathbf{Q}_8$ (see [11]), while a Sylow p-subgroup for $p > 2$ is homocyclic (see [12]). A group is said to be *homocyclic* if it is a direct product of cyclic groups of the same order. Consequently, if G is a nilpotent 3-BCI-group, then G decomposes as $G = U \times V$, where U is a homocyclic group of odd order, and V is trivial or one of the groups $\mathbb{Z}_{2^r}$, $\mathbb{Z}_2^r$ and $\mathbf{Q}_8$. In this paper we prove that the converse implication also holds, and hence complete the classification of nilpotent 3-BCI-groups.

Theorem 1.1. *Every finite group $U \times V$ is a 3-BCI-group, where U is a homocyclic group of odd order, and V is trivial or one of the groups $\mathbb{Z}_{2^r}$, $\mathbb{Z}_2^r$ and $\mathbf{Q}_8$.*

In Section 2, following the ideas of [3], we will see that the BCI-property of a given bi-Cayley graph can be read off entirely from its automorphism group (see Lemma 2.2). This was observed for cyclic groups in [13], and this result was later generalized to arbitrary groups in [1]. Theorem 1.1 will be proved in Section 3.

2. A Babai type lemma for bi-Cayley graphs

We start by setting the relevant notations and terminology.

Notations. Let G be a group acting on a finite set V. For $g \in G$ and $v \in V$, the image of v under g will be written as v^g. For a subset $U \subseteq V$, we will denote by G_U the elementwise stabilizer of U in G, while by $G_{\{U\}}$ the setwise stabilizer of U in G. If $U = \{u\}$, then G_u will be written for $G_{\{u\}}$. We say that U is *G-invariant* if G leaves U setwise fixed, or equivalently, when $G_{\{U\}} = G$. If G is transitive on V and $\Delta \subseteq V$ is a *block* for G, then the partition $\delta = \{\Delta^g : g \in G\}$ is called the *system of blocks for G induced by* Δ. The group G acts on δ naturally, the corresponding *kernel* will be denoted by G_δ, i.e., $G_\delta = \{g \in G : \Delta'^g = \Delta' \text{ for all } \Delta' \in \delta\}$. For a graph Γ, we let $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$, and $\mathrm{Aut}(\Gamma)$ denote the vertex set, the edge set, the arc set, and the full group of automorphisms of Γ, respectively. For a subset $U \subseteq V(\Gamma)$, we let $\Gamma[U]$ denote the *subgraph of* Γ *induced by* U. A graph Γ is called *arc-transitive* when $\mathrm{Aut}(\Gamma)$ is transitive on $A(\Gamma)$. By K_n and $K_{n,n}$ we will denote the complete graph on n vertices and the complete bipartite graph on $2n$ vertices respectively. By a *cubic graph* we simply mean a regular graph of valency 3.

Let G be a group and $S \subseteq G$. The *Cayley digraph* $\mathrm{Cay}(G,S)$ is the digraph whose vertex set is G and arc set is $\{(x,sx) : x \in G, s \in S\}$. A Cayley digraph $\mathrm{Cay}(G,S)$ is called a *CI-graph* if for any Cayley digraph $\mathrm{Cay}(G,T)$, $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$ implies that $T = S^\alpha$ for some $\alpha \in \mathrm{Aut}(G)$, G is called an *m-DCI-group* if all Cayley digraphs of G of valency at most m are CI-graphs, and an $(|G|-1)$-CI-group is simply called a *CI-group* (see [19, Definition 1.1]). Finite CI-groups and m-DCI-groups have attracted considerable attention over the last 40 years, for more information on these groups, the reader is referred to the survey [17]. The following result, frequently used in studying CI-graphs, is a special case of a lemma due to Babai [3, Lemma 3.1]:

Lemma 2.1. *The following are equivalent for every Cayley digraph* $\Gamma = \mathrm{Cay}(G,S)$.

1. $\mathrm{Cay}(G,S)$ *is a CI-graph.*
2. *Every two regular subgroups of* $\mathrm{Aut}(\Gamma)$, *isomorphic to* G, *are conjugate in* $\mathrm{Aut}(\Gamma)$.

Given a group G with identity element 1_G, we shall use the symbols $\mathbf{0}$ and $\mathbf{1}$ to denote the elements $(1_G,0)$ and $(1_G,1)$ in $G \times \{0,1\}$ respectively. For a subset $S \subseteq G$, we write $(S,0) = \{(s,0) : s \in S\}$ and $(S,1) = \{(s,1) : s \in S\}$. For $g \in G$, let $\hat{g}$ be the permutation of $G \times \{0,1\}$ defined by

$$(x,i)^{\hat{g}} = (xg,i) \text{ for every } x \in G \text{ and } i \in \{0,1\}.$$

We set $\hat{G} = \{\hat{g} : g \in G\}$. Obviously, $\hat{G} \le \mathrm{Aut}(\mathrm{BCay}(G,S))$, and $\hat{G}$ is semiregular with orbits $(G,0)$ and $(G,1)$. In what follows will we denote by $\mathcal{S}(\mathrm{Aut}(\mathrm{BCay}(G,S))$ the set of all semiregular subgroups of $\mathrm{Aut}(\mathrm{BCay}(G,S))$ whose orbits are $(G,0)$ and $(G,1)$. Finally, we let $G_{\mathrm{right}} \le \mathrm{Sym}((G,1))$ be the permutation group induced by the action of $\hat{G}$ on $(G,1)$.

The next lemma was proved by M. Arezoomand and B. Taeri [1]. For completeness, we give a proof here.

Lemma 2.2. *The following are equivalent for every bi-Cayley graph* $\Gamma = \mathrm{BCay}(G,S)$.

1. $\mathrm{BCay}(G,S)$ *is a BCI-graph.*
2. *The normalizer* $N_{\mathrm{Aut}(\Gamma)}(\hat{G})$ *is transitive on* $V(\Gamma)$, *and every two subgroups in* $\mathcal{S}(\mathrm{Aut}(\Gamma))$, *isomorphic to* G, *are conjugate in* $\mathrm{Aut}(\Gamma)$.

Proof. We start with the part (1) $\Rightarrow$ (2). Let $X \in \mathcal{S}(\mathrm{Aut}(\Gamma))$ such that $X \cong G$. We have to show that X and $\hat{G}$ are conjugate in $\mathrm{Aut}(\Gamma)$. Let $i \in \{0,1\}$, and set $X^{(G,i)}$ and $\hat{G}^{(G,i)}$ for the permutation groups of the set (G,i) induced by X and $\hat{G}$ respectively. The groups $X^{(G,i)}$ and $\hat{G}^{(G,i)}$ are conjugate in $\mathrm{Sym}((G,i))$, because these are isomorphic and regular on (G,i). Thus X and $\hat{G}$ are conjugate by a permutation $\phi \in \mathrm{Sym}(G \times \{0,1\})$ such that $(G,0)$ is ϕ-invariant (here ϕ is viewed as a permutation of $G \times \{0,1\}$). We write $X = \phi\hat{G}\phi^{-1}$. Consider the graph Γ^ϕ, the image of Γ under ϕ. Then $\hat{G} = \phi^{-1}X\phi \le \mathrm{Aut}(\Gamma^\phi)$. Using this and that $(G,0)$ is ϕ-invariant, we obtain that $\Gamma^\phi = \mathrm{BCay}(G,T)$

for some subset $T \subseteq G$. Then $\Gamma \cong \mathrm{BCay}(G,T)$, and by (i), $T = gS^\alpha$ for some $g \in G$ and $\alpha \in \mathrm{Aut}(G)$. Define the permutation σ of $G \times \{0,1\}$ by

$$(x,i)^\sigma = \begin{cases} (x^\alpha, 0) & \text{if } i = 0, \\ (gx^\alpha, 1) & \text{if } i = 1. \end{cases}$$

A direct calculation shows that $\sigma^{-1}\hat{g}\sigma = \hat{g^\sigma}$ if $g \in G$. Thus σ normalizes $\hat{G}$. The vertex $(x,0)$ of $\mathrm{BCay}(G,S)$ has neighborhood $(Sx,1)$. This is mapped by σ to the the set $(gS^\alpha x^\alpha, 1) = (Tx^\alpha, 1)$. This proves that σ is an isomorphism from Γ to Γ^ϕ, and in turn it follows that, $\Gamma^\phi = \Gamma^\sigma$, $\phi\sigma^{-1} \in \mathrm{Aut}(\Gamma)$, and thus $\phi = \rho\sigma$ for some $\rho \in \mathrm{Aut}(\Gamma)$. Finally, $X = \phi\hat{G}\phi^{-1} = \rho\sigma\hat{G}\sigma^{-1}\rho^{-1} = \rho\hat{G}\rho^{-1}$, i.e., X and $\hat{G}$ are conjugate in $\mathrm{Aut}(\Gamma)$.

In order to prove that the normalizer $N_{\mathrm{Aut}(\Gamma)}(\hat{G})$ is transitive on $V(\Gamma)$, it is sufficient to find some automorphism η which switches $(G,0)$ and $(G,1)$ and normalizes $\hat{G}$. Observe that $\mathrm{BCay}(G,S) \cong \mathrm{BCay}(G,S^{-1})$, where $S^{-1} = \{s^{-1} : s \in S\}$. Then by (i), $S^{-1} = gS^\alpha$ for some $g \in G$ and $\alpha \in \mathrm{Aut}(G)$. We leave for the reader to verify that the permutation of $G \times \{0,1\}$ defined below is an appropriate choice for such η:

$$(x,i)^\eta = \begin{cases} (x^\alpha, 1) & \text{if } i = 0, \\ (gx^\alpha, 0) & \text{if } i = 1. \end{cases}$$

We turn to the part (2) $\Rightarrow$ (1). Let $\Gamma' = \mathrm{BCay}(G,T)$ such that $\Gamma' \cong \Gamma$. We have to show that $T = gS^\alpha$ for some $g \in G$ and $\alpha \in \mathrm{Aut}(G)$. We claim the existence of an isomorphism $\phi : \Gamma \to \Gamma'$ for which $\phi : \mathbf{0} \mapsto \mathbf{0}$ and $(G,0)$ is ϕ-invariant (here ϕ is viewed as a permutation of $G \times \{0,1\}$). We construct ϕ in a few steps. To start with, choose an arbitrary isomorphism $\phi_1 : \Gamma \to \Gamma'$. Since the normalizer $N_{\mathrm{Aut}(\Gamma)}(\hat{G})$ is transitive on $V(\Gamma)$, there exists $\rho \in N_{\mathrm{Aut}(\Gamma)}(\hat{G})$ which maps $\mathbf{0}$ to $\mathbf{0}^{\phi_1^{-1}}$. Let $\phi_2 = \rho\phi_1$. Then ϕ_2 is an isomorphism from Γ to Γ', and also $\phi_2 : \mathbf{0} \mapsto \mathbf{0}$. The connected component of Γ containing the vertex $\mathbf{0}$ is equal to the induced subgraph $\Gamma[(H,0) \cup (sH,1)]$, where $s \in S$ and $H \leq G$ is generated by the set $s^{-1}S$. It can be easily checked that

$$\Gamma[(H,0) \cup (sH,1)] \cong \mathrm{BCay}(H, s^{-1}S).$$

Similarly, the connected component of Γ' containing the vertex $\mathbf{0}$ is equal to the induced subgraph $\Gamma'[(K,0) \cup (tK,1)]$, where $t \in T$ and $K \leq G$ is generated by the set $t^{-1}T$, and

$$\Gamma'[(K,0) \cup (tK,1)] \cong \mathrm{BCay}(K, t^{-1}T).$$

Since ϕ_2 fixes $\mathbf{0}$, it induces an isomorphism from $\Gamma[(H,0) \cup (sH,1)]$ to $\Gamma[(K,0) \cup (tK,1)]$; denote this isomorphism by ϕ_3. It follows from the connectedness of these induced subgraphs that ϕ_3 preserves their bipartition classes, moreover, ϕ_3 maps $(H,0)$ to $(K,0)$, since it fixes $\mathbf{0}$. Finally, take $\phi : \Gamma \to \Gamma'$ to be the isomorphism whose restriction to each component of Γ equals ϕ_3. It is clear that $\phi : \mathbf{0} \mapsto \mathbf{0}$ and $(G,0)$ is ϕ-invariant.

Since $\hat{G} \leq \Gamma'$, $\phi\hat{G}\phi^{-1} \leq \mathrm{Aut}(\Gamma)$. The orbit of $\mathbf{0}$ under $\phi\hat{G}\phi^{-1}$ is equal to $(G,0)^{\phi^{-1}} = (G,0)$, and hence $\phi\hat{G}\phi^{-1} \in \mathcal{S}(\mathrm{Aut}(\Gamma))$. By (ii), $\phi\hat{G}\phi^{-1} = \sigma^{-1}\hat{G}\sigma$ for some $\sigma \in \mathrm{Aut}(\Gamma)$. Since $N_{\mathrm{Aut}(\Gamma)}(\hat{G})$ is transitive on $V(\Gamma)$, σ can be chosen so that $\sigma : \mathbf{0} \mapsto \mathbf{0}$. To sum up, we have an isomorphism

$(\sigma\phi) : \Gamma \mapsto \Gamma'$ which fixes $\mathbf{0}$ and also normalizes $\hat{G}$. Thus $(\sigma\phi)$ maps $(G,1)$ to itself. Recall that $G_{\text{right}} \le \text{Sym}((G,1))$ is the permutation group induced by the action of $\hat{G}$ on $(G,1)$. Then, the permutation of $(G,1)$ induced by $(\sigma\phi)$ belongs to the holomorph of G_{right} (cf. [8, Exercise 2.5.6]), and therefore, there exist $g \in G$ and $\alpha \in \text{Aut}(G)$ such that $(\sigma\phi) : (x,1) \mapsto (gx^\alpha, 1)$ for all $x \in G$. On the other hand, being an isomorphism from Γ to Γ', $\sigma\phi$ maps $(S,1)$ to $(T,1)$. These give that $(T,1) = (S,1)^{\sigma\phi} = (gS^\alpha, 1)$, i.e., $T = gS^\alpha$. □

Remark 2.3. *Notice that, we cannot delete the condition on the normalizer $N_{\text{Aut}(\Gamma)}(\hat{G})$ from Lemma 2.2.(ii). To see this, consider the bi-Cayley graph $\Gamma = \text{BCay}(G,S)$, where*

$$G = \langle a, b \mid a^5 = b^4 = 1,\, b^{-1}ab = a^2 \rangle \text{ and } S = \{1, a, b\}.$$

The group G is the unique Frobenius group of order 20, and we find by the help of the computer package Magma [5] *that Γ is arc-transitive. In fact, Γ is the unique arc-transitive cubic graph on 40 points (see [6]). We also compute that any two subgroups in $\mathcal{S}(\text{Aut}(\Gamma))$, isomorphic to G, are conjugate in $\text{Aut}(\Gamma)$. We show below that, for any $g \in G$ and $\alpha \in \text{Aut}(G)$, $S^\alpha \ne gS^{-1}$. Since $\text{BCay}(G,S) \cong \text{BCay}(G,S^{-1})$, this implies that Γ is not a BCI-graph.*

To the contrary assume that $S^\alpha = gS^{-1}$ for some $g \in G$ and $\alpha \in \text{Aut}(G)$. It follows at once that $g \in S$. As no element in $bS^{-1} = \{b, ba^{-1}, 1\}$ is of order 5, $g \ne b$. Since every automorphism of G is inner, α equals the conjugation by some element $c \in G$. Let $g = 1$. Then $S^\alpha = gS^{-1} = S^{-1}$, hence $a^c = a^\alpha = a^{-1}$ and $b^c = b^\alpha = b^{-1}$. From the first equality $c \in C_G(a)b^2 = \langle a \rangle b^2$, where $C_G(a)$ denotes the centraliser of a in G, that is, $C_G(a) = \{x \in G : ax = xa\}$. Thus $c = a^i b^2$ for some $i \in \{0, \ldots, 4\}$. Plugging this in the second equality, we get $b^2 a^{-i} b a^i b^2 = b^{-1}$, hence $a^{3i} b = b^{-1}$, which is impossible. Finally, let $g = a$. Then $S^\alpha = gS^{-1} = aS^{-1}$, hence $a^c = a^\alpha = a$ and $b^c = b^\alpha = ab^{-1}$. The first equality gives that $c = a^i$ for some $i \in \{0, \ldots, 4\}$. Plugging this in the second equality, we get $a^{-i} b a^i = ab^{-1}$, hence $a^{2i} b = ab^{-1}$, which is again impossible. □

As an application of Lemma 2.2, we prove the lemma below in which we connect the BCI-property with the CI-property. This lemma will be used in the proof of Theorem 1.1 in the particular case when the graphs are not arc-transitive.

Lemma 2.4. *Let $\Gamma = \text{BCay}(G,S)$ such that there exists an involution $\tau \in \text{Aut}(\Gamma)$ which normalizes $\hat{G}$ and $\mathbf{0}^\tau = \mathbf{1}$. Suppose, in addition, that $\text{Aut}(\Gamma)_{\mathbf{0}} = \text{Aut}(\Gamma)_{\mathbf{1}}$. Then $\text{BCay}(G,S)$ is a BCI-graph whenever $\text{Cay}(G,S)$ is a CI-graph.*

Proof. Set $A = \text{Aut}(\Gamma)$ and $A^+ = A_{\{(G,0)\}}$, and let us suppose that $\text{Cay}(G,S)$ is a CI-graph. Let $X \in \mathcal{S}(A), X \cong G$. Obviously, $X, \hat{G} \le A^+$. The normalizer $N_A(\hat{G}) \ge \langle \hat{G}, \tau \rangle$, hence it is transitive on $V(\Gamma)$. Thus by Lemma 2.2 we are done if we show that X and $\hat{G}$ are conjugate in A^+. In order to prove this we define a faithful action of A^+ on G as follows. Let $\Delta = \{\mathbf{0}, \mathbf{1}\}$ and consider the setwise stabilizer $A_{\{\Delta\}}$. Since $A_{\mathbf{0}} = A_{\mathbf{1}}$, $A_{\mathbf{0}} \le A_{\{\Delta\}}$. By [8, Theorem 1.5A], the orbit of $\mathbf{0}$ under $A_{\{\Delta\}}$ is a block for A. Since τ switches $\mathbf{0}$ and $\mathbf{1}$, this orbit is equal to Δ, and the system of blocks induced by Δ is

$$\delta = \{\Delta^{\hat{x}} : x \in G\} = \big\{\, \{(x,0),(x,1)\} : x \in G \,\big\}.$$

Now, define the action of A^+ on G by letting $x^\sigma = x'$, where $x \in G$ and $\sigma \in A^+$, if σ maps the block $\{(x,0),(x,1)\}$ to the block $\{(x',0),(x',1)\}$. We will write $\bar{\sigma}$ for the image of σ under the corresponding permutation representation, and let $\bar{B} = \{\bar{\sigma} : \sigma \in B\}$ for a subgroup $B \leq A^+$. It is easily seen that this action is faithful. Therefore, X and $\hat{G}$ are conjugate in A^+ exactly when $\bar{X}$ and $\bar{\hat{G}}$ are conjugate in $\bar{A^+}$. Also, $\bar{\hat{G}} = G_{right}$, and $\bar{X}$ is regular on G. We finish the proof by showing that $\bar{A^+} = \mathrm{Aut}(\mathrm{Cay}(G,S))$. Then the conjugacy of $\bar{X}$ and $\bar{\hat{G}}$ follows by Lemma 2.1 and the assumption that $\mathrm{Cay}(G,S)$ is a CI-graph.

Pick an automorphism $\sigma \in A^+$ and an arc (x, sx) of $\mathrm{Cay}(G,S)$. Then the edge $\{(x,0),(sx,1)\}$ of Γ is mapped by σ to an edge $\{(x',0),(s'x',1)\}$ for some $x' \in G$ and $s' \in S$. Hence $\bar{\sigma} : x \mapsto x'$ and $sx \mapsto s'x'$, i.e., it maps the arc (x, sx) to the arc $(x', s'x')$. We have just proved that $\bar{\sigma} \in \mathrm{Aut}(\mathrm{Cay}(G,S))$, and hence $\bar{A^+} \leq \mathrm{Aut}(\mathrm{Cay}(G,S))$. In order to establish the relation "$\geq$", for an arbitrary automorphism $\rho \in \mathrm{Aut}(\mathrm{Cay}(G,S))$, define the permutation π of $G \times \{0,1\}$ by $(x,i)^\pi = (x^\rho, i)$ for all $x \in G$ and $i \in \{0,1\}$. Repeating the previous argument we obtain that $\pi \in \mathrm{Aut}(\mathrm{Cay}(G,S))$. It is clear that $\pi \in A^+$ and $\bar{\pi} = \rho$. Thus $\bar{A^+} \geq \mathrm{Aut}(\mathrm{Cay}(G,S))$, and so $\bar{A^+} = \mathrm{Aut}(\mathrm{Cay}(G,S))$. The lemma is proved. □

3. Proof of Theorem 1.1

In this section we denote by $\mathcal{C}$ the set of all groups $U \times V$, where U is a homocyclic group of odd order, and V is either trivial or one of $\mathbb{Z}_{2^r}$, $\mathbb{Z}_2^r$ and $\mathbf{Q}_8$; and by $\mathcal{C}_{\mathrm{sub}}$ the set of all groups that have an overgroup in $\mathcal{C}$.

Lemma 3.1. *Let Γ be a cubic bipartite graph with bipartition classes Δ_i, $i = 1,2$, and $X \leq \mathrm{Aut}(\Gamma)$ be a semiregular subgroup whose orbits are Δ_i, $i = 1,2$, and $X \in \mathcal{C}_{\mathrm{sub}}$. Then $\mathrm{Aut}(\Gamma)$ has an element τ_X which satisfies:*

1. *every subgroup of X is normal in $\langle X, \tau_X \rangle$;*
2. *$\langle X, \tau_X \rangle$ is regular on $V(\Gamma)$.*

Proof. It is straightforward to show that $\Gamma \cong \mathrm{BCay}(X,S)$ for some subset $S \subseteq X$ with $1_X \in S$ and $|S| = 3$. Moreover, there is an isomorphism from Γ to $\mathrm{BCay}(X,S)$ which induces a permutation isomorphism from X to $\hat{X}$. Therefore, it is sufficient to find $\tau \in \mathrm{Aut}(\mathrm{BCay}(X,S))$ for which every subgroup of $\hat{X}$ is normal in $\langle \hat{X}, \tau \rangle$; and $\langle \hat{X}, \tau \rangle$ is regular on $V(\mathrm{BCay}(X,S))$.

Since $X \in \mathcal{C}_{\mathrm{sub}}$, $X = U \times V$, where U is an abelian group of odd order, and V is trivial or one of $\mathbb{Z}_{2^r}, \mathbb{Z}_2^r$ and $\mathbf{Q}_8$. We prove below the existence of an automorphism $\iota \in \mathrm{Aut}(X)$, which maps the set S to its inverse S^{-1}. Let π_U and π_V denote the projections $U \times V \to U$ and $U \times V \to V$ respectively. It is sufficient to find an automorphism $\iota_1 \in \mathrm{Aut}(U)$ which maps $\pi_U(S)$ to $\pi_U(S)^{-1}$, and an automorphism $\iota_2 \in \mathrm{Aut}(V)$ which maps $\pi_V(S)$ to $\pi_V(S)^{-1}$. Since U is abelian, we are done by choosing ι_1 to be the automorphism $x \mapsto x^{-1}$. If V is abelian, then let $\iota_2 : x \mapsto x^{-1}$. Otherwise, $V \cong \mathbf{Q}_8$, and since $|\pi_V(S) \setminus \{1_V\}| \leq 2$, it follows that $\pi_V(S)$ is conjugate to $\pi_V(S)^{-1}$ in V. This ensures that ι_2 can be chosen to be some inner automorphism. Now, define ι by setting its restriction $\iota|_U$ to U as $\iota|_U = \iota_1$, and its restriction $\iota|_V$ to V as $\iota|_V = \iota_2$. Define the permutation τ of $X \times \{0,1\}$ by

$$(x,i)^\tau = \begin{cases} (x^\iota, 1) & \text{if } i = 0, \\ (x^\iota, 0) & \text{if } i = 1. \end{cases}$$

The vertex $(x,0)$ of $\mathrm{BCay}(X,S)$ has neighborhood $(Sx,1)$. This is mapped by τ to the set $(S^{-1}x^\iota, 0)$, which is equal to the neighborhood of $(x^\iota, 1)$. We have proved that $\tau \in \mathrm{Aut}(\mathrm{BCay}(X,S))$.

It follows from its construction that τ is an involution. Fix an arbitrary subgroup $Y \leq X$, and pick $y \in Y$. We may write $y = y_U y_V$ for some $y_U \in U$ and $y_V \in V$. Then $\langle y_U, y_V \rangle \leq Y$, since y_U and y_V commute and $\gcd(|U|,|V|) = 1$. Also, $(y_U)^{\iota_1} = y_U^{-1}$ and $(y_V)^{\iota_2} \in \langle y_V \rangle$, implying that $y^\iota = (y_U)^{\iota_1}(y_V)^{\iota_2} \in \langle y_U, u_V \rangle \leq Y$. We conclude that ι maps Y to itself. Thus $\tau^{-1}\hat{y}\tau = \tau\hat{y}\tau = \hat{y}^\iota$ is in $\hat{Y}$, and τ normalizes $\hat{Y}$. Since $X \in \mathcal{C}_{\mathrm{sub}}$, $\hat{Y}$ is also normal in $\hat{X}$, and part (1) follows.

For part (2), observe that $|\langle \hat{X}, \tau \rangle| = 2|X| = |V(\mathrm{BCay}(X,S))|$. Clearly, $\langle \hat{X}, \tau \rangle$ is transitive on $V(\mathrm{BCay}(X,S))$, so it is regular. □

Let Γ be an arbitrary finite graph and $G \leq \mathrm{Aut}(\Gamma)$ which is transitive on $V(\Gamma)$. For a normal subgroup $N \triangleleft G$ which is not transitive on $V(\Gamma)$, the *quotient graph* Γ_N is the graph whose vertices are the N-orbits on $V(\Gamma)$, and two N-orbits $\Delta_i, i = 1,2$, are adjacent if and only if there exist $v_i \in \Delta_i, i = 1,2$, which are adjacent in Γ. For a positive integer s, an *s-arc* of Γ is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of Γ such that, for every $i \in \{1,\ldots,s\}$, v_{i-1} is adjacent to v_i, and for every $i \in \{1,\ldots,s-1\}$, $v_{i-1} \neq v_{i+1}$. The graph Γ is called *(G,s)-arc-transitive* (*(G,s)-arc-regular*) if G is transitive (regular) on the set of s-arcs of Γ. If $G = \mathrm{Aut}(\Gamma)$, then a (G,s)-arc-transitive ((G,s)-arc-regular) graph is simply called *s-transitive* (*s-regular*). The proof of the following lemma is straightforward, hence it is omitted (it can be also deduced from [20, Theorem 9]).

Lemma 3.2. *Let $\Gamma = \mathrm{BCay}(G,S)$ be a connected arc-transitive graph, G be any finite group, $|S| = 3$, and $N < \hat{G}$ be a subgroup which is normal in $\mathrm{Aut}(\Gamma)$. Then the following hold:*

(1) *Γ_N is a cubic connected arc-transitive graph.*
(2) *N is equal to the kernel of $\mathrm{Aut}(\Gamma)$ acting on the set of N-orbits.*
(3) *Γ_N is isomorphic to a bi-Cayley graph of the group $\hat{G}/N$.*

Remark 3.3. *Let Γ and N be as described in Lemma 3.2. The group $\mathrm{Aut}(\Gamma)$ acts on the set of N-orbits, i.e., on the vertex set $V(\Gamma_N)$. Lemma 3.2.(ii) implies that, the induced permutation group on $V(\Gamma_N)$ is isomorphic to $\mathrm{Aut}(\Gamma)/N$, and therefore, by some abuse of notation, this permutation group will also be denoted by $\mathrm{Aut}(\Gamma)/N$. In what follows we shall write $\mathrm{Aut}(\Gamma)/N \leq \mathrm{Aut}(\Gamma_N)$. Also note that, if Γ is s-transitive, then Γ_N is $(\mathrm{Aut}(\Gamma)/N, s)$-arc-transitive.*

The proof of Theorem 1.1 in the case of arc-transitive graphs will be based on three lemmas about cubic connected arc-transitive bi-Cayley graphs to be proved below. In these lemmas we keep the following notation:

($*$) $\Gamma = \mathrm{BCay}(G,S)$ is a connected arc-transitive graph, where $G \in \mathcal{C}_{\mathrm{sub}}$ and $|S| = 3$.

Lemma 3.4. *With notation* $(*)$*, let* δ *be a system of blocks for* $\mathrm{Aut}(\Gamma)$ *induced by a block properly contained in* $(G,0)$*, and* X *be in* $\mathcal{S}(\mathrm{Aut}(\Gamma))$ *such that* $X \in \mathcal{C}_{\mathrm{sub}}$*. Then for the kernel* A_δ *(see Notations),* $A_\delta < X$*. Moreover, if* δ *is non-trivial, then* A_δ *is also non-trivial.*

Proof. Set $A = \mathrm{Aut}(\Gamma)$. Let $Y = X \cap A_{\{\Delta\}}$, where $\Delta \in \delta$ with $\Delta \subset (G,0)$. Then Δ is equal to an orbit of Y, and $|Y| = |\Delta|$ because $\Delta \subset (G,0)$ and X is regular on $(G,0)$. Formally, $\Delta = \mathrm{Orb}_Y(v)$ for some vertex $v \in \Delta$.

Let $\tau_X \in A$ be the automorphism defined in Lemma 3.1, and set $L = \langle X, \tau_X\rangle$. The group L is regular on $V(\Gamma)$, and $Y \trianglelefteq L$. These yield

$$\delta = \{\Delta^l : l \in L\} = \{\mathrm{Orb}_Y(v)^l : l \in L\} = \{\mathrm{Orb}_Y(v^l) : l \in L\}.$$

From this $Y \le A_\delta$. This shows that, if $|Y| = |\Delta| \ne 1$, then A_δ is non-trivial. Since δ has more than 2 blocks, and Γ is a connected and cubic graph, it is known that A_δ is semiregular. These imply that $A_\delta = Y < X$. □

Corollary 3.5. *With notation* $(*)$*, let* $N < \hat{G}$ *be normal in* $\mathrm{Aut}(\Gamma)$*, and* X *be in* $\mathcal{S}(\mathrm{Aut}(\Gamma))$ *such that* $X \in \mathcal{C}_{\mathrm{sub}}$*. Then* $N < X$*.*

Proof. Let δ be the system of blocks for $\mathrm{Aut}(\Gamma)$ consisting of the N-orbits. Then $A_\delta = N$ by Lemma 3.2.(ii), and the corollary follows directly from Lemma 3.4. □

We denote by Q_3 the graph of the cube and by $\mathcal{H}$ the *Heawood graph*, i.e., the unique arc-transitive cubic graph on 14 points (see [6]). Recall that, the *core* of a subgroup $H \le K$ in the group K is the largest normal subgroup of K contained in H.

Lemma 3.6. *With notation* $(*)$*, suppose that* $\hat{G}$ *is not normal in* $\mathrm{Aut}(\Gamma)$*, and let* N *be the core of* $\hat{G}$ *in* $\mathrm{Aut}(\Gamma)$*. Then* $(\hat{G}/N, \Gamma_N)$ *is isomorphic to one of the pairs* $(\mathbb{Z}_3, K_{3,3})$*,* $(\mathbb{Z}_4, Q_3)$*, and* $(\mathbb{Z}_7, \mathcal{H})$*.*

Proof. Set $A = \mathrm{Aut}(\Gamma)$. Consider the quotient graph Γ_N, and suppose that $M \le \hat{G}$ such that $N \le M$ and $M/N \trianglelefteq \mathrm{Aut}(\Gamma_N)$ (here $M/N \le A/N \le \mathrm{Aut}(\Gamma_N)$, see Remark 3.3). This in turn implies that, $M/N \trianglelefteq A/N$, $M \trianglelefteq A$, and $M = N$. We conclude that, Γ_N is a bi-Cayley graph of $\hat{G}/N$, $\hat{G}/N$ is in $\mathcal{C}_{\mathrm{sub}}$, and $\hat{G}/N$ has trivial core in $\mathrm{Aut}(\Gamma_N)$. This shows that it is sufficient to prove Lemma 3.6 in the particular case when N is trivial. For the rest of the proof we assume that the core N is trivial, and we write $N = 1$.

By Tutte Theorem [22], Γ is k-regular for some $k \le 5$. Set $A^+ = \mathrm{Aut}(\Gamma)_{\{(G,0)\}}$. It follows from the connectedness of Γ that $A = \langle A^+, \tau_{\hat{G}}\rangle$, where $\tau_{\hat{G}} \in A$ is the automorphism defined in Lemma 3.1. Let M be the core of $\hat{G}$ in A^+. Then $M \trianglelefteq A$, since M is normalized by $\tau_{\hat{G}}$, see Lemma 3.1.(i), and $A = \langle A^+, \tau_{\hat{G}}\rangle$. Thus $M \le N = 1$, hence M is also trivial.

Let us consider A^+ acting on the set $[A^+ : \hat{G}]$ of right $\hat{G}$-cosets in A^+. This action is faithful because M is trivial. The corresponding degree is equal to $|A^+ : \hat{G}|$. Since $A = \mathrm{Aut}(X)$ is regular on the set of k-arcs of X, thus $|A|$ is equal to the number of k-arcs of X, which is $|V(X)| \cdot 3 \cdot 2^{k-1} = |\hat{G}| \cdot 3 \cdot 2^k$.

Since $|A^+| = |A|/2$, it follows that

$$|A^+ : \hat{G}| = \frac{|\hat{G}| \cdot 3 \cdot 2^k}{2 \cdot |\hat{G}|} = 3 \cdot 2^{k-1}.$$

Since $\hat{G}$ acts as a point stabilizer in this action, we have an embedding of G into $S_{3 \cdot 2^{k-1}-1}$. We will write below that $G \le S_{3 \cdot 2^{k-1}-1}$.

Recall that, $A_{\mathbf{0}}$ is determined uniquely by k, and we have, respectively, $A_{\mathbf{0}} \cong \mathbb{Z}_3$, or S_3, or D_{12}, or S_4, or $S_4 \times \mathbb{Z}_2$. We go through each case.

CASE 1. $k = 1$.

This case can be excluded at once by observing that we have $G \le S_2$ by the above discussion, which contradicts the obvious bound $|G| \ge 3$.

CASE 2. $k = 2$.

In this case $G \le S_5$. Using also that $G \in \mathcal{C}_{\text{sub}}$, we see that G is abelian, hence $|G| \le 6$, $|V(\Gamma)| \le 12$. We obtain by [6, Table] that $\Gamma \cong Q_3$, and $G \cong \mathbb{Z}_4$.

CASE 3. $k = 3$.

Then $A^+ = \hat{G}A_{\mathbf{0}} = \hat{G}D_{12}$, a product of a nilpotent and a dihedral subgroup. Thus A^+ is solvable by Huppert-Itô Theorem (cf. [21, 13.10.1]). Assume for the moment that A^+ is imprimitive on $(G, 0)$. This implies that A is also imprimitive on $V(\Gamma)$ and it has a non-trivial block system δ which has a block properly contained in $(G, 0)$. Lemma 3.4 gives that $A_\delta < \hat{G}$, and A_δ is non-trivial. This, however, contradicts that the core $N = 1$. Thus A^+ is primitive on $(G, 0)$. Using that A^+ is also solvable, we find that G is a p-group. We see that G is either abelian or it is $\mathbf{Q}_8$. In the latter case $|V(\Gamma)| = 16$, and Γ is isomorphic to the Moebius-Kantor graph, which is, however, 2-regular (see [6, Table]). Therefore, G is an abelian p-group. Let $S = \{s_1, s_2, s_3\}$. Since G is abelian, for Γ we have:

$$\mathbf{0} \sim (s_1, 1) \sim (s_2^{-1}s_1, 0) \sim (s_3 s_2^{-1} s_1, 1) = (s_1 s_2^{-1} s_3, 1) \sim (s_2^{-1} s_3, 0) \sim (s_3, 1) \sim \mathbf{0}. \tag{3.1}$$

Thus Γ is of girth at most 6. It was proved in [7, Theorem 2.3] that the Pappus graph on 18 points and the Desargues graph on 20 points are the only 3-regular cubic graphs of girth 6. For the latter graph $|G| = 10$, contradicting that G is a p-group. We exclude the former graph by the help of Magma. We compute that the Pappus graph has no abelian semiregular automorphism group of order 9 which has trivial core in the full automorphism group. Thus Γ is of girth 4 (3 and 5 are impossible as the graph is bipartite). It is well-known that there are only two cubic arc-transitive graphs of girth 4 (see also [14, page 163]): $K_{3,3}$ and Q_3. We get at once that $\Gamma \cong K_{3,3}$ and $G \cong \mathbb{Z}_3$.

CASE 4. $k = 4$.

It is sufficient to show that G is abelian. Then by the above reasoning Γ is of girth 6, and as the Heawood graph is the only cubic 4-regular graph of girth 6 (see [7, Theorem 2.3]), we get at once that $\Gamma \cong \mathcal{H}$ and $G \cong \mathbb{Z}_7$.

Assume, towards a contradiction, that G is non-abelian. Thus $G = U \times V$, where U is an abelian group of odd order, and $V \cong \mathbf{Q}_8$. We have already shown above that A^+ is primitive on $(G, 0)$. In other words, Γ is a 4-transitive bi-primitive cubic graph. Recall that a permutation group on a set Ω is called *bi-primitive* if it is transitive and imprimitive, and Ω has only one nontrivial system of blocks consisting of exactly two blocks.

Two possibilities can be deduced from the list of 4-transitive bi-primitive graphs given in [16, Theorem 1.4]:

- Γ is the standard double cover of a connected vertex-primitive cubic 4-regular graph, in which case $A = A^+ \times \langle \eta \rangle$ for an involution η; or
- Γ isomorphic to the sextet graph $S(p)$ (see [4]), where $p \equiv \pm 7 (\mathrm{mod}\ 16)$, in which case $A \cong PGL(2,p)$, and $A^+ \cong PSL(2,p)$.

The second possibility cannot occur, because then $A^+ \cong PSL(2,p)$, whose Sylow 2-subgroup is a dihedral group (cf. [9, Satz 8.10]), which contradicts that $V \le \hat{G} \le A^+$, and $V \cong \mathbf{Q}_8$. It remains to exclude the first possibility. We may assume, by replacing S with xS for a suitable $x \in G$ if necessary, that η switches $\mathbf{0}$ and $\mathbf{1}$. Since η commutes with $\hat{G}$, we find $(x,1)^\eta = \mathbf{1}^{\hat{x}\eta} = \mathbf{1}^{\eta\hat{x}} = \mathbf{0}^{\hat{x}} = (x,0)$ for every $x \in G$. Let $s \in S$. Then $\mathbf{0} \sim (s,1)$, hence $\mathbf{1} = \mathbf{0}^\eta \sim (s,1)^\eta = (s,0)$, which shows that $s \in S^{-1}$, and thus $S = S^{-1}$. Thus there exists $s \in S$ with $o(s) \le 2$. Put $T = s^{-1}S = sS$. Then $1_G \in T$, and since Γ is connected, $G = \langle T \rangle$. Notice that $s \in Z(G)$. This implies that $T^{-1} = S^{-1}s = sS = T$, and thus $\pi_V(T)$ satisfies $1_V \in \pi_V(T)$ and $\pi_V(T) = \pi_V(T)^{-1}$. Since $V \cong \mathbf{Q}_8$, this implies that $\langle \pi_V(T) \rangle \ne V$, a contradiction to $G = \langle T \rangle$. This completes the proof of this case.

CASE 5. $k = 5$.

In this case Γ is a 5-transitive bi-primitive cubic graph. It was proved in [16, Corollary 1.5] that Γ is isomorphic to either the $P\Gamma L(2,9)$-graph on 30 points (also known as the Tutte's 8-Cage), or the standards double cover of the $PSL(3,3).\mathbb{Z}_2$-graph on 468 points. These graphs are of girth 8 and 12 respectively (see [6, Table]). Also, in both cases $8 \nmid |G|$, hence G is abelian. In this case, however, the graph Γ has a closed walk of length 6, as shown in Eq. (3.1), hence its girth cannot be larger than 6. This proves that this case does not occur. □

For a group A and a prime p dividing $|A|$, we let A_p denote a Sylow p-subgroup of A.

Lemma 3.7. *With notation* $(*)$, *let* $X \in \mathcal{S}(\mathrm{Aut}(\Gamma))$ *such that* $X \in \mathcal{C}_{\mathrm{sub}}$ *and* $X_2 \cong G_2$. *Then* X *and* $\hat{G}$ *are conjugate in* $\mathrm{Aut}(\Gamma)$.

Remark 3.8. *We remark that, the assumption* $X_2 \cong G_2$ *cannot be deleted. The Moebius-Kantor graph is a bi-Cayley graph of the group* $\mathbf{Q}_8$, *which has a semiregular cyclic group of automorphisms of order* 8 *which preserves the bipartition classes.*

Proof. Set $A = \mathrm{Aut}(\Gamma)$. The proof is split into two parts according to whether $\hat{G}$ is normal in A.

CASE 1. $\hat{G}$ is not normal in A.

Let N be the core of $\hat{G}$ in A. By Corollary 3.5, $N < X \cap \hat{G}$. Therefore, it is sufficient to show that

$$X/N \text{ and } \hat{G}/N \text{ are conjugate in } A/N. \tag{3.2}$$

Recall that, the group $A/N \le \mathrm{Aut}(\Gamma_N)$ for the quotient graph Γ_N induced by N (see Remark 3.3 and the preceding paragraph). Both groups X/N and $\hat{G}/N$ are semiregular whose orbits are the bipartition classes of Γ_N. Also notice that, $\hat{G}/N$ cannot be normal in A/N, otherwise $\hat{G}$ were normal in A.

According to Lemma 3.6, $(\hat{G}/N, \Gamma_N) \cong (\mathbb{Z}_3, K_{3,3})$, or $(\mathbb{Z}_4, Q_3)$, or $(\mathbb{Z}_7, \mathcal{H})$. Thus (1) follows immediately from Sylow Theorems when $(\hat{G}/N, \Gamma_N) \cong (\mathbb{Z}_7, \mathcal{H})$.

Let $(\hat{G}/N, \Gamma_N) \cong (\mathbb{Z}_3, K_{3,3})$. Since $\hat{G}/N$ is not normal in A/N, and Γ_N is $(A/N, 1)$-arc-transitive, we compute by Magma that $A/N = \mathrm{Aut}(\Gamma_N)$, or it is a subgroup of $\mathrm{Aut}(\Gamma_N)$ of index 2. In both cases A/N has one conjugacy class of semiregular subgroups whose orbits are the bipartition classes of Γ_N. Thus (1) holds.

Let $(\hat{G}/N, \Gamma_N) \cong (\mathbb{Z}_4, Q_3)$. Since $X_2 \cong G_2$, $X/N \cong \hat{G}/N \cong \mathbb{Z}_4$. Using this and that Γ_N is $(A/N, 1)$-arc-transitive, we compute by Magma that $A/N = \mathrm{Aut}(\Gamma_N)$, and that $\mathrm{Aut}(\Gamma_N)$ has one conjugacy class of semiregular cyclic subgroups whose orbits are the bipartition classes of Γ_N. Thus (1) holds also in this case.

CASE 2. $\hat{G}$ is normal in A.

We have to show that $X = \hat{G}$. Notice that, X contains every proper subgroup $K < \hat{G}$ which is characteristic in $\hat{G}$. Indeed, since $\hat{G} \trianglelefteq A$, we have that $K \trianglelefteq A$, and hence $K < X$ follows from Corollary 3.5. This property will be used often below.

In particular, $\hat{G}_p \le \hat{G}$ is characteristic for every prime p dividing $|\hat{G}|$. If G is not a p-group, then $\hat{G}_p < \hat{G}$, and by the above observation $\hat{G}_p < X$. This gives that $X = \hat{G}$ if G is not a p-group. Let G be a p-group. If $p > 3$, then both $\hat{G}$ and X are Sylow p-subgroups of A, and the statement follows from Sylow Theorems. Notice that, since Γ is connected, G is generated by the set $s^{-1}S$ for some $s \in S$, hence it is generated by two elements.

Let $p = 2$. Assume for the moment that G is cyclic. Then $\hat{G}$ has a characteristic subgroup K such that $\hat{G}/K \cong \mathbb{Z}_4$. Then $K \trianglelefteq A$, $\Gamma_K \cong Q_3$. Moreover, Γ_K is a bi-Cayley graph of $\hat{G}/K$, and $\hat{G}/K$ is normal in $A/K \le \mathrm{Aut}(\Gamma_K)$. A simple computation, using Magma, shows that this situation does not occur. Let G be a non-cyclic 2-group in $\mathcal{C}_{\mathrm{sub}}$. Also using the fact that G is generated by two elements, we conclude that either $G \cong \mathbb{Z}_2^2$ and $\Gamma \cong Q_3$, or $G \cong \mathbf{Q}_8$ and Γ is the Moebius-Kantor graph. Now, $X = X_2 \cong G_2 = G$. Then $X = \hat{G}$ can be verified by the help of Magma in either case.

Let $p = 3$. Observe first that $|G| > 3$. For otherwise, $\Gamma \cong K_{3,3}$, but no semiregular automorphism group of order 3 is normal in $\mathrm{Aut}(K_{3,3})$. Since G is generated by two elements, we may write $G \cong \mathbb{Z}_{3^e} \times \mathbb{Z}_{3^f}$, where $e \ge 1$ and $0 \le f \le e$. If $e = 1$, then $f = 1$, $G \cong \mathbb{Z}_3^2$, and Γ is the Pappus graph. However, this graph has no automorphism group which is isomorphic to $\mathbb{Z}_3^2$ and also normal in the full automorphism group. Therefore, $e \ge 2$. Define $K = \{\hat{x} : x \in G \text{ and } o(x) \le 3^{e-2}\}$. Then K is a characteristic subgroup of $\hat{G}$. Thus $K \triangleleft A$, and Γ_K is a BiCayley graph of $\hat{G}/K$.

Let $f \le e-2$. Then $\hat{G}/K \cong \mathbb{Z}_9$, and Γ_K is the Pappus graph. This graph, however, does not have a cyclic semiregular automorphism group of order 9. We conclude that $f \in \{e-1, e\}$.

Let $f = e-1$. Then $\hat{G}/K \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. It follows that Γ_K is the unique cubic arc-transitive graph on 54 points (see [6, Table]). We have checked by MAGMA that this graph has a unique semiregular abelian automorphism group whose orbits are the bipartition classes. Therefore, $X/K = \hat{G}/K$. This together with $K < X \cap \hat{G}$ yield that $X = \hat{G}$.

Finally, let $f = e$. Then $\hat{G}/K \cong \mathbb{Z}_9 \times \mathbb{Z}_9$. It follows that Γ_K is the unique cubic arc-transitive graph on 162 points (see [6, Table]). A direct computation, using MAGMA, gives that $X/K = \hat{G}/K$, which together with $K < X \cap \hat{G}$ yield that $X = \hat{G}$. □

Recall that, a group H is *homogeneous* if every isomorphism between two subgroups of H can be extended to an automorphism of H. The following result is [15, Proposition 3.2]:

Proposition 3.9. *Every* 2*-DCI-group is homogeneous.*

Since every group in $\mathcal{C}$ is a 2-DCI-group (see [15, Theorem 1.3]), we have the corollary that every group in $\mathcal{C}$ is homogeneous.

Everything is prepared to prove Theorem 1.1.

Proof of Theorem 1.1. Let $G \in \mathcal{C}$ and $\Gamma = \mathrm{BCay}(G, S)$ such that $|S| \le 3$. We have to show that Γ is a BCI-graph. This holds trivially when $|S| = 1$, and follows from the homogeneity of G when $|S| = 2$. Let $|S| = 3$.

CASE 1. Γ is arc-transitive.

Let $\mathrm{BCay}(G, S) \cong \mathrm{BCay}(G, T)$ for some subset $T \subseteq G$. We may assume without loss of generality that $1_G \in S \cap T$. Let $H = \langle S \rangle$ and $K = \langle T \rangle$. Then $H, K \in \mathcal{C}_{\mathrm{sub}}$, both bi-Cayley graphs $\mathrm{BCay}(H, S)$ and $\mathrm{BCay}(K, T)$ are connected, and $\mathrm{BCay}(H, S) \cong \mathrm{BCay}(K, T)$. We claim that $\mathrm{BCay}(H, S)$ is a BCI-graph. In view of Lemma 2.2, this holds if the normalizer of $\hat{H}$ in $\mathrm{Aut}(\mathrm{BCay}(H, S))$ is transitive on the vertex-set $V(\mathrm{BCay}(H, S))$, and for every $X \in \mathcal{S}(\mathrm{Aut}(\mathrm{BCay}(H, S)))$, isomorphic to H, X and $\hat{H}$ are conjugate in $\mathrm{Aut}(\mathrm{BCay}(H, S))$. Now, the first part follows from Lemma 3.1, while the second part follows from Lemma 3.7.

Let ϕ be an isomorphism from $\mathrm{BCay}(K, T)$ to $\mathrm{BCay}(H, S)$, and consider the group $X = \phi^{-1}\hat{K}\phi \le \mathrm{Sym}(H)$. Since ϕ maps the bipartition classes of $\mathrm{BCay}(K, T)$ to the bipartition classes of $\mathrm{BCay}(H, S)$, we have $X \in \mathcal{S}(\mathrm{Aut}(\mathrm{BCay}(H, S)))$. Also, $X_2 \cong \hat{H}_2$, because $X \cong K$, $|H| = |K|$ and H and K are both contained in the group G from $\mathcal{C}$. Thus Lemma 3.7 is applicable, as a result, X and $\hat{H}$ are conjugate in $\mathrm{Aut}(\mathrm{BCay}(H, S))$. In particular, $H \cong K$. Since G is homogeneous, there exists $\alpha_1 \in \mathrm{Aut}(G)$ such that $K^{\alpha_1} = H$. This α_1 induces an isomorphism from $\mathrm{BCay}(K, T)$ to $\mathrm{BCay}(H, T^{\alpha_1})$. Therefore, $\mathrm{BCay}(H, S) \cong \mathrm{BCay}(H, T^{\alpha_1})$, and since $\mathrm{BCay}(H, S)$ is a BCI-graph, $T^{\alpha_1} = gS^{\alpha_2}$ for some $g \in H$ and $\alpha_2 \in \mathrm{Aut}(H)$. By the homogeneity of G, α_2 extends to an automorphism of G, implying that $\mathrm{BCay}(G, S)$ is a BCI-graph.

CASE 2. Γ is not arc-transitive.

Since Γ is vertex-transitive (see Lemma 3.1), but not arc-transitive, we have $A_{\mathbf{0}} = A_{(s,1)}$ for some $s \in S$. We show below that $\mathrm{BCay}(G, s^{-1}S)$ is a BCI-graph, this obviously yields that the same holds for $\mathrm{BCay}(G, S)$. Define the permutation ϕ of $G \times \{0,1\}$ by

$$(x,i)^{\phi} = \begin{cases} (x,0) & \text{if } i = 0, \\ (s^{-1}x, 1) & \text{if } i = 1. \end{cases}$$

The vertex $(x,0)$ of $\mathrm{BCay}(G,S)$ has neighborhood $(Sx,1)$. This is mapped by ϕ to the the set $(s^{-1}Sx, 1)$. This shows that ϕ is an isomorphism from Γ to $\Gamma' = \mathrm{BCay}(G, s^{-1}S)$. Then we have $\mathrm{Aut}(\Gamma')_{\mathbf{0}} = \phi^{-1}A_{\mathbf{0}}\phi = \phi^{-1}A_{(s,1)}\phi = \mathrm{Aut}(\Gamma')_{\mathbf{1}}$. Let $\tau_{\hat{G}}$ be the automorphism of Γ' defined in Lemma 3.1. It follows that $\tau_{\hat{G}}$ is an involution (see the proof of Lemma 3.1), which normalizes $\hat{G}$ and maps $\mathbf{0}$ to $\mathbf{1}$. Now, Lemma 2.4 is applicable to Γ', as a result, it is sufficient to show that $\mathrm{Cay}(G, s^{-1}S)$ is a CI-graph. This follows because $|s^{-1}S \setminus \{1_G\}| = 2$ and that G is a 2-DCI-group (see [15, Theorem 1.3]). This completes the proof of the theorem. □

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