ON FINITE GROUPS HAVING A CERTAIN NUMBER OF CYCLIC SUBGROUPS

SAJJAD MAHMOOD ROBATI

Communicated by Mahmut Kuzucuoğlu

Abstract. Let $G$ be a finite group. In this paper, we study the structure of finite groups having $|G| - r$ cyclic subgroups for $3 \leq r \leq 5$.

1. Introduction

Let $G$ be a finite group and $C(G)$ be the poset of cyclic subgroups of $G$. Some results show that the structure of $C(G)$ has an influence on the algebraic structure of $G$. In Main Theorem of [8], Tărnăuceanu proved that the finite group $G$ has $|G| - 1$ cyclic groups if and only if $G$ is isomorphic to $Z_3$, $Z_4$, $S_3$, or $D_8$. In the end of that paper the author states the following problem:

Open Problem. Describe the finite group $G$ satisfying $|C(G)| = |G| - r$ where $2 \leq r \leq |G| - 1$.

In [9], Tărnăuceanu solved this open problem for $|C(G)| = |G| - 2$. In this paper, we describe the structure of finite groups with $|C(G)| = |G| - r$ in which $3 \leq r \leq 5$.

We summarize our notations. $cl(a)$ denotes the conjugacy class of $a$ in $G$, $\pi(G)$ denotes the set of prime numbers dividing the order of $G$, $\phi(n)$ denotes the Euler function that counts the positive integers less than $n$ that are relatively prime to $n$, $F(G)$ denotes the subgroup generated by all normal nilpotent subgroups of $G$, $O_p(G)$ denotes the unique maximal normal $p$-subgroup of $G$, $F_{p,q}$ denotes the Frobenius group of order $pq$ and $o(x)$ denotes the order of $x$.

MSC(2010): Primary: 20D99; Secondary: 20E34.
Keywords: Finite groups, cyclic subgroups, $p$-groups.
Received: 30 November 2017, Accepted: 05 March 2018.

DOI: http://dx.doi.org/10.22108/ijgt.2018.108302.1458
2. Preliminaries

**Lemma 2.1.** Let $G$ be a finite group and $p \in \pi(G)$. Then the number of distinct subgroups of order $p$ in $G$ is $kp + 1$ for some non-negative integer $k$.

We denote with $c_n(G)$ the number of cyclic subgroups of order $2^n$ in a finite 2-group $G$. And, a group $G$ of order $p^m$ is said to be of maximal class if $m > 2$ and $cl(G) = m - 1$.

**Theorem 2.2.** [1, Theorem 1.17] Suppose that a 2-group $G$ is neither cyclic nor of maximal class. Then $c_n(G)$ is even for $n > 1$ and $c_1(G) \equiv 3 \pmod{4}$.

**Remark 2.3** ([1]). For each 2-group $G$, $c_2(G) = 1$ if and only if $G$ is either cyclic or dihedral and $c_2(G) = 3$ if and only if $G$ is either $Q_8$ or $SD_{16}$.

**Theorem 2.4.** [1, Corollary 1.7 and Theorem 1.2] Let $G$ be a 2-group of maximal class. Then it is either $D_{2n}$, $Q_{2n}$, or $SD_{2n+1}$ for $n \geq 3$.

In this paper, $\Omega_n(G) = \langle x \in G | o(x) \leq p^n \rangle$ and $\Omega_n(G) = \langle x^{p^n} | x \in G \rangle$. In the next theorem, 2-groups of order $> 2^3$ with $c_2(G) = 2$ are characterized.

**Theorem 2.5.** [1, Theorem 43.6] and [7, Theorem 5.1 and 5.2 and Proposition 1.4] Suppose that a group $G$ of order $2^m > 2^3$ has exactly two cyclic subgroups $U$ and $V$ of order 4; set $A = \langle U, V \rangle$. Then $A$ is abelian of type $(4, 2)$ and one of the following holds:

(a) $G \cong M_{2^m}$.

(b) $G$ is abelian of type $(2^{m-1}, 2)$.

(c) $G = \langle a, b | a^{2^{m-2}} = b^8 = 1, a^b = a^{-1}, a^{2^{m-3}} = b^4 \rangle$, where $m \geq 5$.

(d) $G = D_{2^{m-1}} \times C_2$.

(e) $G = \langle b, t | b^{2^{m-2}} = t^2 = 1, b^t = b^{-1+2^{m-4}}u, u^2 = [u, t] = 1, b^u = b^{1+2^{m-3}} \rangle$, where $m \geq 5$.

In the next theorems, 2-groups of order $> 2^4$ with $c_2(G) = 4$ are characterized.

**Theorem 2.6.** [6, Theorem 2.1 and Proposition 1.3 and 1.4] and [7, Theorem 2.1] Let $G$ be a 2-group of order $> 2^4$ with $c_2(G) = 4$ and $|\Omega_2(G)| = 2^4$. Then one of the following holds:

(a) $G \cong D_8 \ast C$ (the central product) where $C$ is cyclic of order $\geq 4$.

(b) $G \cong Q_{8S}$ where $S$ is cyclic of order $\geq 16$ and $Q_8$ is normal in $G$.

(c) $G = \langle E, a, b \rangle$, in which $E \cong E_8$, $o(b) = 8$, $o(a) = 2^n$ and $a^{2^{n-2}} = v$ is of order 4. Moreover $A = \Omega_2(G) = \langle E, v \rangle \cong C_4 \times C_2 \times C_2$.

(d) $G = \langle E, a \rangle$, in which $E \cong E_8$, $o(a) = 2^n$ and $a^{2^{n-2}} = v$ is of order 4. Moreover $A = \Omega_2(G) = \langle E, v \rangle \cong C_4 \times C_2 \times C_2$.

In case(d) of Theorem 2.6, if $n = 2$, then $a = v$ and so $G \cong C_4 \times C_2 \times C_2$ contradicting $|G| > 2^4$. Hence, $G$ has some element of order at least 8.

DOI: http://dx.doi.org/10.22108/ijgt.2018.108302.1458
Theorem 2.7. [6, Theorem 2.2] Let $G$ be a 2-group of order $> 2^4$ with $c_2(G) = 4$ and $|\Omega_2(G)| > 2^4$. If $G$ has a quaternion subgroup $Q$, then $Q$ is normal in $G$, $C = C_G(Q)$ is cyclic of order $2^n$, $n \geq 2$, $G = (Q \ast C)(t)$, where $t$ is an involution such that $Q(t) \cong SD_{2^4}$ and $(t)C \cong D_{2^{n+1}}$. We have $|Z(G)| = 2$, $|G| = 2^n$, and $\Omega_2(G) = G$.

In Theorem 2.7, since $Q(t) \cong SD_{2^4}$ is a subgroup of $G$, then $G$ has some element of order 8.

Theorem 2.8. [6, Theorem 2.4, 2.5, and 2.6] Let $G$ be a 2-group of order $> 2^4$ with $c_2(G) = 4$ and $|\Omega_2(G)| > 2^4$. Suppose that $G$ has no subgroup isomorphic to $Q_8$. Then

(a) $G = \Omega_2(G)$ and $\Omega_2(G) = B(t)$ where $B$ is abelian of type $(2^m, 2, 2)$, $m \geq 2$, and $t$ is an involution acting invertingly on $B$.
(b) $|G : \Omega_2(G)| \geq 4$ and $G = \langle a, b, t | a^8 = b^8 = t^2 = 1; a^2 = v; a^4 = z; b^2 = ev; a^b = a^{-1}u; e^2 = a^2 = [e, v] = [u, v] = [e, u] = [a, u] = [t, e] = [t, u] = 1, e^a = ez; e^b = uz; v^t = v^{-1}; a^t = eva^{-1}; b^t = euvb^{-1} \rangle$.
(c) $|G : \Omega_2(G)| = 2$ and $G = \langle b, e, t | b^8 = e^2 = t^2 = 1; (tb)^2 = a; a^{2n} = 1, n \geq 3, a^{2n-2} = v; a^{2n-1} = z, b^2 = uv, u^2 = [b, e] = [a, e] = [a, u] = [u, e] = [t, e] = [t, u] = 1, u^b = uz; a^b = a^{-1}; a^t = a^{-1} \rangle$.

In the previous theorems, we observe that each 2-group of order $> 2^4$ with $c_2(G) = 4$ has some element of order at least 8, except for some cases of Theorem 2.6(a) and Theorem 2.8(a).

Lemma 2.9. Let $G$ be a finite group with the Sylow $p$-subgroup $P$ and $\pi(G) = \{p, q\}$. Assume that $a \in G$ and $p$ divides $o(a)$ if and only if $a$ is of prime power order. Then $G$ is a Frobenius group with kernel $P$.

Proof. Since $P$ is normal in $G$, then $G = PQ$ where $Q$ is a Sylow $q$-subgroup of $G$. On the other hand, if $x \in C_Q(P)$, $p$ and $q$ divide $o(x)$ which is a contradiction. Therefore, By Problem 7.1 of [5], $G$ is a Frobenius group with kernel $P$.

3. Main Theorem

Theorem 3.1. Let $G$ be a finite group. Then

(1) $|C(G)| = |G| - 3$ if and only if $G \cong Z_5$, $Q_8$, or $D_{10}$.
(2) $|C(G)| = |G| - 4$ if and only if $G \cong Z_8$, $Z_3 \times Z_3$, $Z_6 \times Z_2$, $Z_4 \times Z_2 \times Z_2$, $D_{16}$, $(Z_4 \times Z_2) \times Z_2$, $Z_3 \times Z_3$, $D_8 \times Z_2 \times Z_2$, or $D_8 \ast Z_4$ (the central product $D_8$ and $Z_4$).
(3) $|C(G)| = |G| - 5$ if and only if $G \cong Z_7$, $Z_3 \times Z_6$, $Dic_{12}$, $D_{14}$, or $F_{5,4}$ (where $Dic_{12}$ is the dicyclic group of order 12).

Proof. We know that

\begin{equation}
|G| = \sum_{i=1}^{k} n_i \phi(d_i) \quad \text{and} \quad |C(G)| = \sum_{i=1}^{k} n_i \tag{3.1}
\end{equation}

DOI: http://dx.doi.org/10.22108/ijgt.2018.108302.1458
in which $d_i$'s are the positive divisors of $|G|$ and $n_i$'s are the number of cyclic subgroups of order $d_i$ in $G$ for $1 \leq i \leq k$. If $|C(G)| = |G| - r$ then we can obtain that

\[
\sum_{i=1}^{k} n_i(\phi(d_i) - 1) = r.
\]

Suppose that $n = (n_1, \ldots, n_k)$, $d = (d_1, \ldots, d_k)$, $P_q$ is a Sylow $q$-subgroup of $G$, and $A$ is a cyclic subgroup of order 4. Applying equation 3.2 and Lemma 2.1, we distinguish several cases for $r$, $n$, and $d$.

(1) $r = 3$.

Case (1): $n = (1, m)$ and $d = (5, 2)$.

If $m = 0$, then $G \cong Z_5$. Otherwise, by Lemma 2.9, $G$ is a Frobenius group with kernel $P_5$ of order 5. By Theorem 13.3(1),(3) of [3], we deduce that $G \cong D_{10}$.

Case (2): $n = (3, m)$ and $d = (4, 2)$.

$G$ is a 2-group and $c_2(G) = 3$. By Remark 2.3 we obtain that $G \cong Q_8$ or $SD_{16}$ and since $SD_{16}$ has some elements of order 8, then $G \cong Q_8$.

Case (3): $n = (2, 1, m)$ and $d = (4, 3, 2)$.

By Lemma 2.9 and Theorem 13.3(1) of [3] $G \cong S_3$ which contradicts the hypothesis.

Case (4): $n = (1, 1, 1, m)$ and $d = (6, 4, 3, 2)$.

Observe that $A$ and $P_3 \cong Z_3$ are normal in $G$ and so $AP_3 \cong Z_4 \times Z_3 \cong Z_{12}$ is a subgroup of $G$ which is impossible by the hypothesis.

Case (5): $n = (2, 1, m)$ and $d = (6, 3, 2)$.

Assume that $G = P_2P_3$ in which $P_3$ is normal of order 3 and $P_2$ is an elementary abelian 2-group. Since each element of order 6 is a product of an element of order 3 and an element of order 2, then these elements belong to $C_G(P_3)$ and $C_G(P_3) \cong Z_3 \times Z_2 \times \cdots \times Z_2 \subseteq P_2P_3$.

Furthermore, by Theorem 2.2 $c_1(C_G(P_3)) = 1$ or $4k + 3$, therefore the number of cyclic subgroups of order 6 is 1 or $4k + 3$ and so $G$ can not have exactly 2 subgroups of order 6.

(2) $r = 4$.

Case (1): $n = (1, 1, m)$ and $d = (5, 3, 2)$.

Observe that $P_3 \cong Z_3$ and $P_5 \cong Z_5$ are normal in $G$ and so $P_3P_5 \cong Z_{15}$ is a subgroup of $G$, which is impossible.

Case (2): $n = (1, 1, m)$ and $d = (8, 4, 2)$.

DOI: http://dx.doi.org/10.22108/ijgt.2018.108302.1458
Since $c_2(G) = 1$, then by Remark 2.3, $G \cong Z_8$ or $D_{16}$.

**Case (3):** $n = (1,1,m)$ and $d = (5,4,2)$.

Observe that $A$ and $P_5 \cong Z_5$ are normal in $G$ and so $AP_5 \cong Z_{20}$ is a subgroup of $G$, which is impossible.

**Case (4):** $n = (4,m)$ and $d = (3,2)$.

If $P_3 \cong Z_3$, then by Main Theorem of [2] $G$ is a Frobenius group with kernel $P_3$ of order 3 and Theorem 13.3(1) of [3] $G \cong S_3$ which is impossible. Otherwise, $Z_3 \times Z_3 \subseteq P_3$ and $Z_3 \times Z_3$ has 4 subgroups of order 3, then by Main Theorem of [2], either $G \cong Z_3 \times Z_3$ or $G \cong (Z_3 \times Z_3) \rtimes Z_2$.

**Case (5):** $n = (3,1,m)$ and $d = (4,3,2)$.

By Lemma 2.9 $G$ is a Frobenius group with kernel $P_3$ of order 3. On the other hand, by Theorem 13.3(1) of [3] $P \cong Z_2$ which is a contradiction.

**Case (6):** $n = (3,1,m)$ and $d = (6,3,2)$.

Since $P_3 \cong Z_3 \cong \langle x \rangle$ is normal in $G$ and $P_2$ is an elementary abelian 2-group, then $G' \subseteq P_3$ and $|cl(x)| = |G|/|C_G(x)| \leq |G'| \leq 3$, and either $x \in Z(G)$ or $|cl(x)| = 2$. On the other hand, since $G$ has 3 cyclic subgroups of order 6, we can obtain that $C_G(x) \cong Z_2 \times Z_2 \times Z_3$. Thus $G \cong Z_6 \times Z_2$ or $(Z_6 \times Z_2) \rtimes Z_2$, but using GAP [4], we know that $(Z_6 \times Z_2) \rtimes Z_2$ has some element of order 4.

**Case (7):** $n = (1,2,1,m)$ and $d = (6,4,3,2)$.

Since $G$ has 2 cyclic subgroups of order 4, then $c_2(P_2) \leq 2$ and so by Theorem 2.5, $P_2 \cong Z_4 \times Z_2$, $D_8 \times Z_2$, $Z_4$, or $D_8$ and $Q \cong Z_3$. Using GAP [4], we obtain that such group does not exist.

**Case (8):** $n = (2,1,1,m)$ and $d = (6,4,3,2)$.

Observe that $A$ and $P_3 \cong Z_3$ are normal in $G$ and so $AP_3 \cong Z_{12}$ which is impossible by the hypothesis.

**Case (9):** $n = (4,m)$ and $d = (4,2)$.

Observe that $G$ is a 2-group with $c_2(G) = 4$ and since $G$ does not some element of order 8, then by Theorems 2.6-2.8 and using GAP[4], we obtain that $G \cong Z_4 \times Z_2 \times Z_2$, $(Z_4 \times Z_2) \rtimes Z_2$, $D_8 \times Z_2 \times Z_2$, or $D_8 \times Z_4$(Central product).

(3) $r = 5$.

**Case (1):** $n = (1,m)$ and $d = (7,2)$
If \( m = 0 \), then \( G \cong \mathbb{Z}_7 \). Otherwise, by Lemma 2.9, \( G \) is a Frobenius group with kernel \( P_7 \) of order 7. By Theorem 13.3(1) of \([3]\) \( G \) is isomorphic to \( D_{14} \).

**Case (2)**: \( n = (1, 1, 1, m) \) and \( d = (5, 4, 3, 2) \)

We observe that \( P_3 \) and \( P_5 \) are normal in \( G \) and \( P_3P_5 \cong \mathbb{Z}_{15} \) is a subgroup of \( G \) which is impossible.

**Case (3)**: \( n = (1, 2, m) \) and \( d = (5, 4, 2) \)

By Lemma 2.9 \( G \) is a Frobenius group with kernel \( P_5 \) of order 5. By Theorem 13.3(1) of \([3]\), we deduce that \( G \cong F_{5,4} \).

**Case (4)**: \( n = (1, 1, 1, m) \) and \( d = (6, 5, 3, 2) \).

Observe that \( P_3, P_5 \) are normal in \( G \). Thus, \( P_3P_5 \cong \mathbb{Z}_{15} \) which is a contradiction.

**Case (5)**: \( n = (1, 1, 1, m) \) and \( d = (8, 4, 3, 2) \).

Since \( A \) and \( P_3 \) are normal in \( G \), then \( AP_3 \cong \mathbb{Z}_4 \times \mathbb{Z}_3 \cong \mathbb{Z}_{12} \) is a subgroup of \( G \) which is impossible.

**Case (6)**: \( n = (1, 2, m) \) and \( d = (8, 4, 2) \).

Since \( G \) is a 2-group and \( c_3(G) = 1 \), then by Theorem 2.2, \( G \) is of maximal class and so by Theorem 2.4, \( G \) is isomorphic to either \( D_{2n}, Q_{2n}, \) or \( SD_{2n+1} \) for \( n \geq 3 \). However, we have \( c_2(G) = 2 \) and so this case is impossible by Theorem 2.5.

**Case (7)**: \( n = (5, m) \) and \( d = (4, 2) \).

Since \( G \) is a 2-group and \( c_2(G) = 5 \), then by Theorem 2.2, \( G \) is of maximal class and so by Theorem 2.4, \( G \) is isomorphic to either \( D_{2n}, Q_{2n}, \) or \( SD_{2n+1} \) for \( n \geq 3 \). Since \( D_{2n}, Q_{2n}, \) and \( SD_{2n} \) have some element of order 8 for \( n \geq 4 \), then \( G \) is isomorphic to \( D_8 \) or \( Q_8 \) which is impossible by Remark 2.3.

**Case (8)**: \( n = (4, 1, m) \) and \( d = (6, 3, 2) \).

We can see that \( G = P_2P_3 \) in which \( P_3 \) is normal of order 3 and \( P_2 \) is an elementary abelian 2-group. Additionally, \( C_G(P_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \). Since each element of order 6 is a product of an element of order 3 and an element of order 2, then such elements belong to \( C_G(P_2) \). Thus, since by Theorem 2.2 \( c_1(C_Q(P_2)) = 1 \) or \( 4k + 3 \), then \( G \) does not have exactly 4 subgroups of order 6.

**Case (9)**: \( n = (3, 1, 1, m) \) and \( d = (6, 4, 3, 2) \).

DOI: http://dx.doi.org/10.22108/ijgt.2018.108302.1458
Since $A$ and $P_3$ are normal in $G$, then $AP_3 \cong Z_4 \times Z_3 \cong Z_{12}$ is a subgroup of $G$ which is a contradiction.

Case (10): $n = (2, 2, 1, m)$ and $d = (6, 4, 3, 2)$.

We can see that $G = P_2P_3$ in which $P_3$ is normal of order 3 and $P_2$ is a 2-group with $c_2(P) = 2$. If $x$ is of order 4, then $x \not\in C_Q(P_2)$, $C_Q(P_2)$ is an elementary abelian 2-group, and $C_G(P_2) \cong Z_3 \times Z_2 \times \cdots \times Z_2$. Since each element of order 6 belongs to $C_G(P_2)$ and by Theorem 2.2 $c_1(C_Q(P_2)) = 1$ or $4k + 3$, then $G$ does not exactly 2 subgroups of order 6.

Case (11): $n = (1, 4, m)$ and $d = (4, 3, 2)$.

By Lemma 2.9 $AP_3$ is a Frobenius group with kernel $A$ of order 4. By Theorem 13.3(1) of [3] we obtain that $|P_3| = 3$ and so $AP_3$ is a Frobenius group of order 12 contradicting the fact that $A_4$ is the only Frobenius group of order 12 and it does not have any element of order 4.

Case (12): $n = (4, 1, m)$ and $d = (4, 3, 2)$.

By Lemma 2.9 $G$ is a Frobenius group with kernel $P_3$ of order 3 and by Theorem 13.3(1) of [3] $G$ is of order 6 which contradicts the hypothesis.

Case (13): $n = (1, 4, m)$ and $d = (6, 3, 2)$.

Since $G$ has 4 subgroups of order 3, then $P_3 \cong Z_3$ or $Z_3 \times Z_3$. If $P_3 \cong Z_3$, since $G$ has 1 subgroup of order 6, then $C_G(P_3) \cong Z_3$ is a subset of $F(G)$. Therefore $F(G) = O_2(G) \times O_2(G) = O_2(G) \times P_3$ and so $P_3$ is normal in $G$ which is a contradiction.

Now, $P_3 \cong Z_3 \times Z_3$ has 4 subgroups of order 3, then $P_3$ is normal in $G$. Assume that $B$ is the cyclic subgroup of order 6 in $G$. Since $P_2$ is an elementary abelian 2-group, then $G' \subseteq P_3$ and so $|P_2| = |cl(y)| \leq |G'| \leq 9$ for $y \in P_3 \setminus B$. Using GAP [4], we get that $G \cong Z_3 \times Z_6$.

Case (14): $n = (1, 3, 1, m)$ and $d = (6, 4, 3, 2)$.

Observe that $G = P_2P_3$ where $c_2(P_2) \leq 3$ and $P_3 \cong Z_3$ is normal in $G$. By Remark 2.3 and Theorem 2.5 and 2.4, $P_2$ is isomorphic to $Q_8$, $D_8$, $Z_4$, $Z_2 \times D_8$, or $Z_2 \times Z_4$. Using GAP[4], we conclude that $G \cong Dic_{12}$, that is the dicyclic group of order 12.

References


**Sajjad Mahmood Robati**
Department of Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran
Email: sajjad.robati@gmail.com, mahmoodrobati@sci.ikiu.ac.ir