A NOTE ON ENGEL ELEMENTS IN THE FIRST GRIGORCHUK GROUP

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Abstract. Let $\Gamma$ be the first Grigorchuk group. According to a result of Bartholdi, the only left Engel elements of $\Gamma$ are the involutions. This implies that the set of left Engel elements of $\Gamma$ is not a subgroup. The natural question arises whether this is also the case for the sets of bounded left Engel elements, right Engel elements and bounded right Engel elements of $\Gamma$. Motivated by this, we prove that these three subsets of $\Gamma$ coincide with the identity subgroup.

1. Introduction

Let $G$ be a group. An element $g \in G$ is called a left Engel element if for any $x \in G$ there exists a positive integer $n = n(g, x)$ such that $[x, n, g] = 1$. As usual, the commutator $[x, n, g]$ is defined inductively by the rules

$$[x, 1, g] = [x, g] = x^{-1}x^g$$

and, for $n \geq 2$, $[x, n, g] = [[[x, n-1, g], g]$.

If $n$ can be chosen independently of $x$, then $g$ is called a left $n$-Engel element, or less precisely a bounded left Engel element. Similarly, $g$ is a right Engel element or a bounded right Engel element if the variable $x$ appears on the right. The group $G$ is then called Engel (or bounded Engel, resp.) if all its elements are both left and right Engel (or bounded Engel, resp.). We denote by $L(G), L(G), R(G)$

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and $\overline{R}(G)$ respectively the sets of left Engel elements, bounded left Engel elements, right Engel elements, and bounded right Engel elements of $G$. It is clear that these four subsets are invariant under automorphisms of $G$. Furthermore, by a well-known result of Heineken (see [10, 12.3.1]), we have

\[(*) \quad R(G)^{-1} \subseteq L(G) \quad \text{and} \quad \overline{R}(G)^{-1} \subseteq \overline{L}(G).\]

It is a long-standing question whether the sets $L(G), \overline{L}(G), R(G)$ and $\overline{R}(G)$ are subgroups of $G$ (see Problems 16.15 and 16.16 in [8]). There are several classes of groups for which this is true (see [1] and also [2, 11]). The question is however still open in general, except for $L(G)$ when $G$ is a 2-group. For 2-groups it is in fact easy to see that the involutions are left Engel elements [1, Proposition 3.3]. However, according to an example of Bludov, there exists a 2-group generated by involutions with an element of order four which is not left Engel ([5], see [9] for a proof). This suggests the following question.

**Question** (Bludov). *Assuming that $G$ is not a 2-group, is $L(G)$ a subgroup of $G$?*

We point out that the group $G$ considered by Bludov is based on the (first) Grigorchuk group [7], that we denote throughout by $\Gamma$. More precisely, $G$ is the wreath product $D_8 \rtimes \Gamma^4$ where $D_8$ is the dihedral group of order 8. Since $\Gamma$ is a 2-group generated by involutions, one might wonder whether $\Gamma$ is an Engel group but the answer is negative, as shown by Bartholdi:

**Theorem 1.1** ([3], see also [4]). *Let $\Gamma$ be the first Grigorchuk group. Then

\[L(\Gamma) = \{g \in \Gamma \mid g^2 = 1\}.\]

In particular, $\Gamma$ is not an Engel group.*

The following natural question now arises: are $\overline{L}(\Gamma), R(\Gamma)$ and $\overline{R}(\Gamma)$ subgroups of $\Gamma$? Recall that $\Gamma$ is just-infinite, that is, $\Gamma$ is an infinite group all of whose proper quotients are finite. As a consequence, if $\overline{L}(\Gamma)$ were a (proper) subgroup of $\Gamma$, then $\overline{L}(\Gamma)$ would be finitely generated and, by Theorem 1.1, also abelian. Hence $\overline{L}(\Gamma)$ would be finite and then trivial as otherwise $\Gamma$ would be an extension of a finite group by a finite group giving the contradiction that $\Gamma$ is finite. Notice also that, by $(*)$, the same holds for $R(\Gamma)$ and $\overline{R}(\Gamma)$.

Motivated by this, in the present note we prove the following theorem.

**Theorem 1.2.** *Let $\Gamma$ be the first Grigorchuk group. Then

\[\overline{L}(\Gamma) = R(\Gamma) = \overline{R}(\Gamma) = \{1\}.\]

The proof of Theorem 1.2 will be given in the next section.*
2. The proof

Before proving Theorem 1.2, we recall how the Grigorchuk group $\Gamma$ is defined. We also collect some properties of $\Gamma$ on which our proof depends. For a more detailed account on $\Gamma$, we refer to [6, Chapter 8].

Let $T$ be the regular binary rooted tree with vertices indexed by $X^*$, the free monoid on the alphabet $X = \{0, 1\}$. An automorphism of $T$ is a bijection of the vertices that preserves incidence. The set $\text{Aut} T$ of all automorphisms of $T$ is a group with respect to composition. The stabilizer $\text{st}(n)$ of the $n$th level of $T$ is the normal subgroup of $\text{Aut} T$ consisting of the automorphisms leaving fixed all words of length $n$.

If an automorphism $g$ fixes a vertex, then the restriction $g_i$ of $g$ to the subtree hanging from this vertex induces an automorphism of $T$. In particular, if $g \in \text{st}(n)$ then $g_i$ is defined for $i = 1, \ldots, 2^n$, and one can consider the injective homomorphism

$$\psi_n : g \in \text{st}(n) \mapsto (g_1, \ldots, g_{2^n}) \in \text{Aut} T \times \cdots \times \text{Aut} T.$$ 

We write $\psi$ instead of $\psi_1$. If $\psi(g) = (g_1, g_2)$, it is easy to see that

$$\psi(g^a) = (g_2, g_1),$$

where $a$ is the rooted automorphism of $T$ corresponding to the permutation $(01)$; this will be used frequently in the sequel.

The Grigorchuk group $\Gamma$ is the subgroup of $\text{Aut} T$ generated by the rooted automorphism $a$, and the automorphisms $b, c, d \in \text{st}(1)$ which are defined recursively as follows:

$$\psi(b) = (a, c), \: \psi(c) = (a, d), \: \psi(d) = (1, b).$$

Moreover,

$$\Gamma = \langle a \rangle \rtimes \text{st}_T(1)$$

where $\text{st}_T(1) = \Gamma \cap \text{st}(1)$. Recall also that $\Gamma$ is spherically transitive (i.e., it acts transitively on each level of $T$) and it has a subgroup $K$ of finite index such that $\psi(K) \cong K \times K$. In other words, $\Gamma$ is regular branch over $K$.

For the proof of Theorem 1.2 we require two lemmas concerning commutators between specific elements of $\Gamma$.

**Lemma 2.1.** Let $x = ag$ be an involution in $\Gamma$ where $g \in \text{st}_T(1)$ and $\psi(g) = (g_1, g_2)$. Let $y \in \text{st}_T(1)$ where $\psi(y) = (k, 1)$. Then for every $m \geq 1$ we have

$$\psi([y, m x]) = (k^{1-m} 2^{m-1}, (k^{g_2})^{2^{m-1}}).$$

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Proof. Since $x$ is an involution we have $[y, m x] = [y, x]^{-(2)^{m-1}}$ for every $m \geq 1$ (see [1, Proposition 3.3]). Thus

$$
\psi([y, m x]) = \psi([y, x])^{-(2)^{m-1}} = \psi(y^{-1} y^{ag})^{-(2)^{m-1}} = \left(\psi(y^{-1}) \psi(y^{ag})\right)^{-(2)^{m-1}}
$$

$$
= (k^{-1}, k^2)^{-(2)^{m-1}} = (k^{-1})^{m 2^{m-1}}, (k^2)^{-(2)^{m-1}},
$$
as desired. \hfill \square

Lemma 2.2. Let $x = ag$ where $g \in \text{st}_1(1)$ and $\psi(g) = (g_1, g_2)$. Let $y \in \text{st}_1(1)$ with $\psi(y) = (y_1, y_2)$. Then for every $m \geq 1$ we have

$$
\psi([x, m+1 y]) = ((y_2^{-1})^{g_1, m + 1} y_1, [(y_2^{-1})^{g_2, m} y_2]^{y_2}).
$$

Proof. Of course, $[x, m y] \in \text{st}_1(1)$ for every $n \geq 1$. Thus

$$
\psi([x, y]) = \psi((y^{-1})^{x} y) = \psi((y^{-1})^{a} \psi(y)) = ((y_2^{-1})^{g_1}, (y_1^{-1})^{g_2})(y_1, y_2)
$$

$$
= ((y_2^{-1})^{g_1} y_1, (y_1^{-1})^{g_2} y_2).
$$

It follows that

$$
\psi([x, y, y]) = [\psi([x, y]), \psi(y)]
$$

$$
= [((y_2^{-1})^{g_1} y_1, (y_1^{-1})^{g_2} y_2), (y_1, y_2)]
$$

$$
= [((y_2^{-1})^{g_1} y_1, y_1], [(y_1^{-1})^{g_2} y_2, y_2])
$$

$$
= [((y_2^{-1})^{g_1} y_1 y_1], [(y_1^{-1})^{g_2} y_2 y_2]].
$$

This proves the result when $m = 1$. Let $m > 1$. Then, by induction, we conclude that

$$
\psi([x, m+1 y]) = [\psi([x, m y]), \psi(y)]
$$

$$
= [((y_2^{-1})^{g_1, m-1} y_1, [(y_1^{-1})^{g_2, m-1} y_2]^{y_2}), (y_1, y_2)]
$$

$$
= [((y_2^{-1})^{g_1, m} y_1 y_1], [(y_1^{-1})^{g_2, m} y_2 y_2]].
$$

\hfill \square

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $x$ be a nontrivial element of $\Gamma$ where $x$ is either in $\overline{L}(\Gamma)$ or $R(\Gamma)$. First, notice that we may assume $x \not\in \text{st}_1(1)$. In fact, if $x \in \text{st}_1(n) \setminus \text{st}_1(n + 1)$ then

$$
\psi_n(x) = (x_1, \ldots, x_2^n)
$$

where all the $x_i$’s are Engel elements of the same kind of $x$ and one of $x_i$’s does not belong to $\text{st}_1(1)$. Hence $x = ag$, for some $g \in \text{st}_1(1)$ with $\psi(g) = (g_1, g_2)$. We distinguish two cases: $x \in \overline{L}(\Gamma)$ and $x \in R(\Gamma)$.
Assume $x \in \mathcal{L}(\Gamma)$. Then $[y, m x] = 1$ for every $y \in \Gamma$. Also $x^2 = 1$, by Theorem 1.1. Since $K$ is not of finite exponent, we can take $k \in K$ of order $2^{m-1}$. On the other hand $\psi(K) \supseteq K \times K$, so there exists $y \in K \subseteq \operatorname{st}_\Gamma(1)$ such that $(k, 1) = \psi(y)$. Thus, by Lemma 2.1, we have

$$(1, 1) = \psi(1) = \psi([y, m x]) = \left(k^{-1} m 2^{m-1}, (k^{g_2})(-2)^{m-1}\right).$$

It follows that $k^{2^{m-1}} = 1$, a contradiction. This proves that $\mathcal{L}(\Gamma) = \{1\}$.

Assume $x \in R(\Gamma)$. Since $K$ is not abelian, it cannot be an Engel group by Theorem 1.1. Thus $[h, y_1] \neq 1$ for some $h, y_1 \in K$ and for every $m \geq 1$. Put $y_2 = [y_1, h]^{g_1}$. Obviously, $y_2 \in K$ and $(y_2^{-1})^{g_1} = [h, y_1]$. Now $\Gamma$ is regular branch over $K$, so there exists $y \in K \subseteq \operatorname{st}_\Gamma(1)$ such that $\psi(y) = (y_1, y_2)$. Furthermore, there is $m = m(x, y) \geq 1$ such that $[x, m y] = 1$. Applying Lemma 2.2, we get

$$(1, 1) = \psi(1) = \psi([x, m + 1 y])$$

$$= \left([y_2^{-1}]^{g_1} m y_1]^{g_1}, [(y_1^{-1})^{g_2} m y_2]^{g_2}\right)$$

$$= \left([h, m + 1 y_1]^{g_1}, [(y_1^{-1})^{g_2} m y_2]^{g_2}\right).$$

This implies that $[h, m + 1 y_1] = 1$, which is a contradiction. Therefore $R(\Gamma) = \overline{R}(\Gamma) = \{1\}$, and the proof of Theorem 1.2 is complete. □

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