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A NOTE ON ENGEL ELEMENTS IN THE FIRST GRIGORCHUK GROUP

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ABSTRACT. Let Γ be the first Grigorchuk group. According to a result of Bartholdi, the only left Engel elements of Γ are the involutions. This implies that the set of left Engel elements of Γ is not a subgroup. The natural question arises whether this is also the case for the sets of bounded left Engel elements, right Engel elements and bounded right Engel elements of Γ . Motivated by this, we prove that these three subsets of Γ coincide with the identity subgroup.

1. Introduction

Let G be a group. An element $g \in G$ is called a left Engel element if for any $x \in G$ there exists a positive integer $n = n(g, x)$ such that $[x, n g] = 1$. As usual, the commutator $[x, n g]$ is defined inductively by the rules

$$[x, 1 g] = [x, g] = x^{-1}x^g \quad \text{and, for } n \geq 2, \quad [x, n g] = [[x, n-1 g], g].$$

If n can be chosen independently of x , then g is called a left n -Engel element, or less precisely a bounded left Engel element. Similarly, g is a right Engel element or a bounded right Engel element if the variable x appears on the right. The group G is then called Engel (or bounded Engel, resp.) if all its elements are both left and right Engel (or bounded Engel, resp.). We denote by $L(G), \bar{L}(G), R(G)$

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and $\overline{R}(G)$ respectively the sets of left Engel elements, bounded left Engel elements, right Engel elements, and bounded right Engel elements of G . It is clear that these four subsets are invariant under automorphisms of G . Furthermore, by a well-known result of Heineken (see [10, 12.3.1]), we have

$$(*) \quad R(G)^{-1} \subseteq L(G) \quad \text{and} \quad \overline{R}(G)^{-1} \subseteq \overline{L}(G).$$

It is a long-standing question whether the sets $L(G)$, $\overline{L}(G)$, $R(G)$ and $\overline{R}(G)$ are subgroups of G (see Problems 16.15 and 16.16 in [8]). There are several classes of groups for which this is true (see [1] and also [2, 11]). The question is however still open in general, except for $L(G)$ when G is a 2-group. For 2-groups it is in fact easy to see that the involutions are left Engel elements [1, Proposition 3.3]. However, according to an example of Bludov, there exists a 2-group generated by involutions with an element of order four which is not left Engel ([5], see [9] for a proof). This suggests the following question.

Question (Bludov). *Assuming that G is not a 2-group, is $L(G)$ a subgroup of G ?*

We point out that the group G considered by Bludov is based on the (first) Grigorchuk group [7], that we denote throughout by Γ . More precisely, G is the wreath product $D_8 \times \Gamma^4$ where D_8 is the dihedral group of order 8. Since Γ is a 2-group generated by involutions, one might wonder whether Γ is an Engel group but the answer is negative, as shown by Bartholdi:

Theorem 1.1 ([3], see also [4]). *Let Γ be the first Grigorchuk group. Then*

$$L(\Gamma) = \{g \in \Gamma \mid g^2 = 1\}.$$

In particular, Γ is not an Engel group.

The following natural question now arises: *are $\overline{L}(\Gamma)$, $R(\Gamma)$ and $\overline{R}(\Gamma)$ subgroups of Γ ?* Recall that Γ is just-infinite, that is, Γ is an infinite group all of whose proper quotients are finite. As a consequence, if $\overline{L}(\Gamma)$ were a (proper) subgroup of Γ , then $\overline{L}(\Gamma)$ would be finitely generated and, by Theorem 1.1, also abelian. Hence $\overline{L}(\Gamma)$ would be finite and then trivial as otherwise Γ would be an extension of a finite group by a finite group giving the contradiction that Γ is finite. Notice also that, by (*), the same holds for $R(\Gamma)$ and $\overline{R}(\Gamma)$.

Motivated by this, in the present note we prove the following theorem.

Theorem 1.2. *Let Γ be the first Grigorchuk group. Then*

$$\overline{L}(\Gamma) = R(\Gamma) = \overline{R}(\Gamma) = \{1\}.$$

The proof of Theorem 1.2 will be given in the next section.

2. The proof

Before proving Theorem 1.2, we recall how the Grigorchuk group Γ is defined. We also collect some properties of Γ on which our proof depends. For a more detailed account on Γ , we refer to [6, Chapter 8].

Let \mathcal{T} be the regular binary rooted tree with vertices indexed by X^* , the free monoid on the alphabet $X = \{0, 1\}$. An automorphism of \mathcal{T} is a bijection of the vertices that preserves incidence. The set $\text{Aut } \mathcal{T}$ of all automorphisms of \mathcal{T} is a group with respect to composition. The stabilizer $\text{st}(n)$ of the n th level of \mathcal{T} is the normal subgroup of $\text{Aut } \mathcal{T}$ consisting of the automorphisms leaving fixed all words of length n .

If an automorphism g fixes a vertex, then the restriction g_i of g to the subtree hanging from this vertex induces an automorphism of \mathcal{T} . In particular, if $g \in \text{st}(n)$ then g_i is defined for $i = 1, \dots, 2^n$, and one can consider the injective homomorphism

$$\psi_n : g \in \text{st}(n) \mapsto (g_1, \dots, g_{2^n}) \in \text{Aut } \mathcal{T} \times \overbrace{\dots}^{2^n} \times \text{Aut } \mathcal{T}.$$

We write ψ instead of ψ_1 . If $\psi(g) = (g_1, g_2)$, it is easy to see that

$$\psi(g^a) = (g_2, g_1),$$

where a is the rooted automorphism of \mathcal{T} corresponding to the permutation $(0\ 1)$; this will be used frequently in the sequel.

The Grigorchuk group Γ is the subgroup of $\text{Aut } \mathcal{T}$ generated by the rooted automorphism a , and the automorphisms $b, c, d \in \text{st}(1)$ which are defined recursively as follows:

$$\psi(b) = (a, c), \quad \psi(c) = (a, d), \quad \psi(d) = (1, b).$$

Moreover,

$$\Gamma = \langle a \rangle \rtimes \text{st}_\Gamma(1)$$

where $\text{st}_\Gamma(1) = \Gamma \cap \text{st}(1)$. Recall also that Γ is spherically transitive (i.e., it acts transitively on each level of \mathcal{T}) and it has a subgroup K of finite index such that $\psi(K) \supseteq K \times K$. In other words, Γ is regular branch over K .

For the proof of Theorem 1.2 we require two lemmas concerning commutators between specific elements of Γ .

Lemma 2.1. *Let $x = ag$ be an involution in Γ where $g \in \text{st}_\Gamma(1)$ and $\psi(g) = (g_1, g_2)$. Let $y \in \text{st}_\Gamma(1)$ where $\psi(y) = (k, 1)$. Then for every $m \geq 1$ we have*

$$\psi([y, {}_m x]) = (k^{(-1)^m 2^{m-1}}, (k^{g_2})^{(-2)^{m-1}}).$$

Proof. Since x is an involution we have $[y, {}_m x] = [y, x]^{(-2)^{m-1}}$ for every $m \geq 1$ (see [1, Proposition 3.3]). Thus

$$\begin{aligned}\psi([y, {}_m x]) &= \psi([y, x])^{(-2)^{m-1}} = \psi(y^{-1} y^{ag})^{(-2)^{m-1}} = \left(\psi(y^{-1}) \psi(y^a) \psi(g) \right)^{(-2)^{m-1}} \\ &= (k^{-1}, k^{g_2})^{(-2)^{m-1}} = (k^{(-1)^m 2^{m-1}}, (k^{g_2})^{(-2)^{m-1}}),\end{aligned}$$

as desired. \square

Lemma 2.2. *Let $x = ag$ where $g \in \text{st}_\Gamma(1)$ and $\psi(g) = (g_1, g_2)$. Let $y \in \text{st}_\Gamma(1)$ with $\psi(y) = (y_1, y_2)$. Then for every $m \geq 1$ we have*

$$\psi([x, {}_{m+1} y]) = ([y_2^{-1}]^{g_1}, {}_m y_1]^{y_1}, [y_1^{-1}]^{g_2}, {}_m y_2]^{y_2}).$$

Proof. Of course, $[x, {}_n y] \in \text{st}_\Gamma(1)$ for every $n \geq 1$. Thus

$$\begin{aligned}\psi([x, y]) &= \psi((y^{-1})^x y) = \psi((y^{-1})^a) \psi(g) \psi(y) = ((y_2^{-1})^{g_1}, (y_1^{-1})^{g_2})(y_1, y_2) \\ &= ((y_2^{-1})^{g_1} y_1, (y_1^{-1})^{g_2} y_2).\end{aligned}$$

It follows that

$$\begin{aligned}\psi([x, y, y]) &= [\psi([x, y]), \psi(y)] \\ &= [((y_2^{-1})^{g_1} y_1, (y_1^{-1})^{g_2} y_2), (y_1, y_2)] \\ &= ([y_2^{-1}]^{g_1} y_1, y_1], [(y_1^{-1})^{g_2} y_2, y_2]) \\ &= ([y_2^{-1}]^{g_1}, y_1]^{y_1}, [(y_1^{-1})^{g_2}, y_2]^{y_2}).\end{aligned}$$

This proves the result when $m = 1$. Let $m > 1$. Then, by induction, we conclude that

$$\begin{aligned}\psi([x, {}_{m+1} y]) &= [\psi([x, {}_m y]), \psi(y)] \\ &= [([y_2^{-1}]^{g_1}, {}_{m-1} y_1]^{y_1}, [(y_1^{-1})^{g_2}, {}_{m-1} y_2]^{y_2}), (y_1, y_2)] \\ &= ([y_2^{-1}]^{g_1}, {}_m y_1]^{y_1}, [(y_1^{-1})^{g_2}, {}_m y_2]^{y_2}).\end{aligned}$$

\square

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let x be a nontrivial element of Γ where x is either in $\bar{L}(\Gamma)$ or $R(\Gamma)$. First, notice that we may assume $x \notin \text{st}_\Gamma(1)$. In fact, if $x \in \text{st}_\Gamma(n) \setminus \text{st}_\Gamma(n+1)$ then

$$\psi_n(x) = (x_1, \dots, x_{2^n})$$

where all the x_i 's are Engel elements of the same kind of x and one of x_i 's does not belong to $\text{st}_\Gamma(1)$. Hence $x = ag$, for some $g \in \text{st}_\Gamma(1)$ with $\psi(g) = (g_1, g_2)$. We distinguish two cases: $x \in \bar{L}(\Gamma)$ and $x \in R(\Gamma)$.

Assume $x \in \overline{L}(\Gamma)$. Then $[y, {}_m x] = 1$ for every $y \in \Gamma$. Also $x^2 = 1$, by Theorem 1.1. Since K is not of finite exponent, we can take $k \in K$ of order $> 2^{m-1}$. On the other hand $\psi(K) \supseteq K \times K$, so there exists $y \in K \subseteq \text{st}_\Gamma(1)$ such that $(k, 1) = \psi(y)$. Thus, by Lemma 2.1, we have

$$(1, 1) = \psi(1) = \psi([y, {}_m x]) = \left(k^{(-1)^m 2^{m-1}}, (k^{g_2})^{(-2)^{m-1}} \right).$$

It follows that $k^{2^{m-1}} = 1$, a contradiction. This proves that $\overline{L}(\Gamma) = \{1\}$.

Assume $x \in R(\Gamma)$. Since K is not abelian, it cannot be an Engel group by Theorem 1.1. Thus $[h, {}_m y_1] \neq 1$ for some $h, y_1 \in K$ and for every $m \geq 1$. Put $y_2 = [y_1, h]^{g_1^{-1}}$. Obviously, $y_2 \in K$ and $(y_2^{-1})^{g_1} = [h, y_1]$. Now Γ is regular branch over K , so there exists $y \in K \subseteq \text{st}_\Gamma(1)$ such that $\psi(y) = (y_1, y_2)$. Furthermore, there is $m = m(x, y) \geq 1$ such that $[x, {}_m y] = 1$. Applying Lemma 2.2, we get

$$\begin{aligned} (1, 1) &= \psi(1) = \psi([x, {}_{m+1} y]) \\ &= ([(y_2^{-1})^{g_1}, {}_m y_1]^{y_1}, [(y_1^{-1})^{g_2}, {}_m y_2]^{y_2}) \\ &= ([h, {}_{m+1} y_1]^{y_1}, [(y_1^{-1})^{g_2}, {}_m y_2]^{y_2}). \end{aligned}$$

This implies that $[h, {}_{m+1} y_1] = 1$, which is a contradiction. Therefore $R(\Gamma) = \overline{R}(\Gamma) = \{1\}$, and the proof of Theorem 1.2 is complete. □

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