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A NOTE ON ENGEL ELEMENTS IN THE FIRST GRIGORCHUK GROUP

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Abstract. Let Γ be the first Grigorchuk group. According to a result of Bartholdi, the only left Engel elements of Γ are the involutions. This implies that the set of left Engel elements of Γ is not a subgroup. The natural question arises whether this is also the case for the sets of bounded left Engel elements, right Engel elements and bounded right Engel elements of Γ. Motivated by this, we prove that these three subsets of Γ coincide with the identity subgroup.

1. **Introduction**

Let G be a group. An element $g \in G$ is called a left Engel element if for any $x \in G$ there exists a positive integer $n = n(g, x)$ such that $[x,_n g] = 1$. As usual, the commutator $[x,_n g]$ is defined inductively by the rules

$$[x,_1 g] = [x, g] = x^{-1}x^g \quad \text{and, for } n \geq 2, \quad [x,_n g] = [[x,_{n-1} g], g].$$

If n can be chosen independently of x, then g is called a left n-Engel element, or less precisely a bounded left Engel element. Similarly, g is a right Engel element or a bounded right Engel element if the variable x appears on the right. The group G is then called Engel (or bounded Engel, resp.) if all its elements are both left and right Engel (or bounded Engel, resp.). We denote by $L(G), \overline{L}(G), R(G)$

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and $\overline{R}(G)$ respectively the sets of left Engel elements, bounded left Engel elements, right Engel elements, and bounded right Engel elements of G. It is clear that these four subsets are invariant under automorphisms of G. Furthermore, by a well-known result of Heineken (see [10, 12.3.1]), we have

$$(*)\qquad R(G)^{-1} \subseteq L(G) \quad \text{and} \quad \overline{R}(G)^{-1} \subseteq \overline{L}(G).$$

It is a long-standing question whether the sets $L(G), \overline{L}(G), R(G)$ and $\overline{R}(G)$ are subgroups of G (see Problems 16.15 and 16.16 in [8]). There are several classes of groups for which this is true (see [1] and also [2, 11]). The question is however still open in general, except for $L(G)$ when G is a 2-group. For 2-groups it is in fact easy to see that the involutions are left Engel elements [1, Proposition 3.3]. However, according to an example of Bludov, there exists a 2-group generated by involutions with an element of order four which is not left Engel ([5], see [9] for a proof). This suggests the following question.

Question (Bludov)**.** *Assuming that G is not a 2-group, is $L(G)$ a subgroup of G?*

We point out that the group G considered by Bludov is based on the (first) Grigorchuk group [7], that we denote throughout by Γ. More precisely, G is the wreath product $D_8 \ltimes \Gamma^4$ where D_8 is the dihedral group of order 8. Since Γ is a 2-group generated by involutions, one might wonder whether Γ is an Engel group but the answer is negative, as shown by Bartholdi:

Theorem 1.1 ([3], see also [4])**.** *Let Γ be the first Grigorchuk group. Then*

$$L(\Gamma) = \{g \in \Gamma \mid g^2 = 1\}.$$

In particular, Γ is not an Engel group.

The following natural question now arises: *are $\overline{L}(\Gamma), R(\Gamma)$ and $\overline{R}(\Gamma)$ subgroups of Γ?*
Recall that Γ is just-infinite, that is, Γ is an infinite group all of whose proper quotients are finite. As a consequence, if $\overline{L}(\Gamma)$ were a (proper) subgroup of Γ, then $\overline{L}(\Gamma)$ would be finitely generated and, by Theorem 1.1, also abelian. Hence $\overline{L}(\Gamma)$ would be finite and then trivial as otherwise Γ would be an extension of a finite group by a finite group giving the contradiction that Γ is finite. Notice also that, by (*), the same holds for $R(\Gamma)$ and $\overline{R}(\Gamma)$.

Motivated by this, in the present note we prove the following theorem.

Theorem 1.2. *Let Γ be the first Grigorchuk group. Then*

$$\overline{L}(\Gamma) = R(\Gamma) = \overline{R}(\Gamma) = \{1\}.$$

The proof of Theorem 1.2 will be given in the next section.

2. The proof

Before proving Theorem 1.2, we recall how the Grigorchuk group Γ is defined. We also collect some properties of Γ on which our proof depends. For a more detailed account on Γ, we refer to [6, Chapter 8].

Let $\mathcal{T}$ be the regular binary rooted tree with vertices indexed by X^*, the free monoid on the alphabet $X = \{0,1\}$. An automorphism of $\mathcal{T}$ is a bijection of the vertices that preserves incidence. The set $\operatorname{Aut}\mathcal{T}$ of all automorphisms of $\mathcal{T}$ is a group with respect to composition. The stabilizer $\operatorname{st}(n)$ of the nth level of $\mathcal{T}$ is the normal subgroup of $\operatorname{Aut}\mathcal{T}$ consisting of the automorphisms leaving fixed all words of length n.

If an automorphism g fixes a vertex, then the restriction g_i of g to the subtree hanging from this vertex induces an automorphism of $\mathcal{T}$. In particular, if $g \in \operatorname{st}(n)$ then g_i is defined for $i = 1, \dots, 2^n$, and one can consider the injective homomorphism

$$\psi_n : g \in \operatorname{st}(n) \longmapsto (g_1, \dots, g_{2^n}) \in \operatorname{Aut}\mathcal{T} \times \overbrace{\cdots}^{2^n} \times \operatorname{Aut}\mathcal{T}.$$

We write ψ instead of ψ_1. If $\psi(g) = (g_1, g_2)$, it is easy to see that

$$\psi(g^a) = (g_2, g_1),$$

where a is the rooted automorphism of $\mathcal{T}$ corresponding to the permutation $(0\,1)$; this will be used frequently in the sequel.

The Grigorchuk group Γ is the subgroup of $\operatorname{Aut}\mathcal{T}$ generated by the rooted automorphism a, and the automorphisms $b, c, d \in \operatorname{st}(1)$ which are defined recursively as follows:

$$\psi(b) = (a, c),\ \psi(c) = (a, d),\ \psi(d) = (1, b).$$

Moreover,

$$\Gamma = \langle a \rangle \ltimes \operatorname{st}_\Gamma(1)$$

where $\operatorname{st}_\Gamma(1) = \Gamma \cap \operatorname{st}(1)$. Recall also that Γ is spherically transitive (i.e., it acts transitively on each level of $\mathcal{T}$) and it has a subgroup K of finite index such that $\psi(K) \supseteq K \times K$. In other words, Γ is regular branch over K.

For the proof of Theorem 1.2 we require two lemmas concerning commutators between specific elements of Γ.

Lemma 2.1. *Let $x = ag$ be an involution in Γ where $g \in \operatorname{st}_\Gamma(1)$ and $\psi(g) = (g_1, g_2)$. Let $y \in \operatorname{st}_\Gamma(1)$ where $\psi(y) = (k, 1)$. Then for every $m \geq 1$ we have*

$$\psi([y,_m x]) = (k^{(-1)^m 2^{m-1}}, (k^{g_2})^{(-2)^{m-1}}).$$

Proof. Since x is an involution we have $[y,_m x] = [y,x]^{(-2)^{m-1}}$ for every $m \geq 1$ (see [1, Proposition 3.3]). Thus

$$\begin{aligned}\psi([y,_m x]) &= \psi([y,x])^{(-2)^{m-1}} = \psi(y^{-1}y^{ag})^{(-2)^{m-1}} = \left(\psi(y^{-1})\psi(y^a)^{\psi(g)}\right)^{(-2)^{m-1}} \\ &= (k^{-1}, k^{g_2})^{(-2)^{m-1}} = (k^{(-1)^m 2^{m-1}}, (k^{g_2})^{(-2)^{m-1}}),\end{aligned}$$

as desired. □

Lemma 2.2. *Let $x = ag$ where $g \in \mathrm{st}_\Gamma(1)$ and $\psi(g) = (g_1, g_2)$. Let $y \in \mathrm{st}_\Gamma(1)$ with $\psi(y) = (y_1, y_2)$. Then for every $m \geq 1$ we have*

$$\psi([x,_{m+1} y]) = ([(y_2^{-1})^{g_1},_m y_1]^{y_1}, [(y_1^{-1})^{g_2},_m y_2]^{y_2}).$$

Proof. Of course, $[x,_n y] \in \mathrm{st}_\Gamma(1)$ for every $n \geq 1$. Thus

$$\begin{aligned}\psi([x,y]) &= \psi((y^{-1})^x y) = \psi((y^{-1})^a)^{\psi(g)}\psi(y) = ((y_2^{-1})^{g_1}, (y_1^{-1})^{g_2})(y_1, y_2) \\ &= ((y_2^{-1})^{g_1} y_1, (y_1^{-1})^{g_2} y_2).\end{aligned}$$

It follows that

$$\begin{aligned}\psi([x,y,y]) &= [\psi([x,y]), \psi(y)] \\ &= [((y_2^{-1})^{g_1} y_1, (y_1^{-1})^{g_2} y_2), (y_1, y_2)] \\ &= ([(y_2^{-1})^{g_1} y_1, y_1], [(y_1^{-1})^{g_2} y_2, y_2]) \\ &= ([(y_2^{-1})^{g_1}, y_1]^{y_1}, [(y_1^{-1})^{g_2}, y_2]^{y_2}).\end{aligned}$$

This proves the result when $m = 1$. Let $m > 1$. Then, by induction, we conclude that

$$\begin{aligned}\psi([x,_{m+1} y]) &= [\psi([x,_m y]), \psi(y)] \\ &= [([(y_2^{-1})^{g_1},_{m-1} y_1]^{y_1}, [(y_1^{-1})^{g_2},_{m-1} y_2]^{y_2}), (y_1, y_2)] \\ &= ([(y_2^{-1})^{g_1},_m y_1]^{y_1}, [(y_1^{-1})^{g_2},_m y_2]^{y_2}).\end{aligned}$$

□

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let x be a nontrivial element of Γ where x is either in $\overline{L}(\Gamma)$ or $R(\Gamma)$. First, notice that we may assume $x \notin \mathrm{st}_\Gamma(1)$. In fact, if $x \in \mathrm{st}_\Gamma(n) \setminus \mathrm{st}_\Gamma(n+1)$ then

$$\psi_n(x) = (x_1, \dots, x_{2^n})$$

where all the x_i's are Engel elements of the same kind of x and one of x_i's does not belong to $\mathrm{st}_\Gamma(1)$. Hence $x = ag$, for some $g \in \mathrm{st}_\Gamma(1)$ with $\psi(g) = (g_1, g_2)$. We distinguish two cases: $x \in \overline{L}(\Gamma)$ and $x \in R(\Gamma)$.

Assume $x \in \overline{L}(\Gamma)$. Then $[y,_m x] = 1$ for every $y \in \Gamma$. Also $x^2 = 1$, by Theorem 1.1. Since K is not of finite exponent, we can take $k \in K$ of order $> 2^{m-1}$. On the other hand $\psi(K) \supseteq K \times K$, so there exists $y \in K \subseteq \mathrm{st}_\Gamma(1)$ such that $(k, 1) = \psi(y)$. Thus, by Lemma 2.1, we have

$$(1,1) = \psi(1) = \psi([y,_m x]) = \left(k^{(-1)^m \, 2^{m-1}}, (k^{g_2})^{(-2)^{m-1}}\right).$$

It follows that $k^{2^{m-1}} = 1$, a contradiction. This proves that $\overline{L}(\Gamma) = \{1\}$.

Assume $x \in R(\Gamma)$. Since K is not abelian, it cannot be an Engel group by Theorem 1.1. Thus $[h,_m y_1] \neq 1$ for some $h, y_1 \in K$ and for every $m \geq 1$. Put $y_2 = [y_1, h]^{g_1^{-1}}$. Obviously, $y_2 \in K$ and $(y_2^{-1})^{g_1} = [h, y_1]$. Now Γ is regular branch over K, so there exists $y \in K \subseteq \mathrm{st}_\Gamma(1)$ such that $\psi(y) = (y_1, y_2)$. Furthermore, there is $m = m(x, y) \geq 1$ such that $[x,_m y] = 1$. Applying Lemma 2.2, we get

$$\begin{aligned}(1,1) = \psi(1) &= \psi([x,_{m+1} y]) \\ &= ([(y_2^{-1})^{g_1},_m y_1]^{y_1}, [(y_1^{-1})^{g_2},_m y_2]^{y_2}) \\ &= ([h,_{m+1} y_1]^{y_1}, [(y_1^{-1})^{g_2},_m y_2]^{y_2}).\end{aligned}$$

This implies that $[h,_{m+1} y_1] = 1$, which is a contradiction. Therefore $R(\Gamma) = \overline{R}(\Gamma) = \{1\}$, and the proof of Theorem 1.2 is complete. □

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