UPPER BOUNDS ON THE UNIFORM SPREADS OF THE SPORADIC SIMPLE GROUPS

ALI REZA RAHIMIPOUR* AND YOUSOF FARZANEH

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Abstract. A finite group $G$ has uniform spread $k$ if there exists a fixed conjugacy class $C$ of elements in $G$ with the property that for any $k$ nontrivial elements $s_1, s_2, \ldots, s_k$ in $G$ there exists $y \in C$ such that $G = \langle s_i, y \rangle$ for $i = 1, 2, \ldots, k$. Further, the exact uniform spread of $G$ is the largest $k$ such that $G$ has the uniform spread $k$. In this paper we give upper bounds on the exact uniform spreads of thirteen sporadic simple groups.

1. Introduction

It is well-known that every finite simple group can be generated by two suitable elements [2, 17, 18]. In this case the group is called 2-generated. Binder showed that for any two non-trivial elements $x_1$ and $x_2$ of the symmetric group $S_n$ there exists an element $y$ such that $S_n = \langle x_1, y \rangle = \langle x_2, y \rangle$ [3]. From this Brenner and Wiegold made the following definition in [6].

Definition 1.1. Let $r$ be any positive integer. A finite non-abelian group $G$ is said to have spread $r$, if for every set $S = \{s_1, s_2, \ldots, s_r\}$ of distinct non-trivial elements of $G$, there exists an element $y \in G$ such that $G = \langle s_i, y \rangle$ for $i = 1, 2, \ldots, r$. In this case $y$ is called the mate of $S$. $G$ has exact spread $r$ if it has spread $r$ but not $r + 1$ and this is denoted by $s(G) = r$.

The stronger notion of uniform spread was introduced in [14].

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Definition 1.2. A finite group $G$ has uniform spread $k$ if there exists a fixed conjugacy class $C$ of elements in $G$ with the property that for every set $S = \{s_1, s_2, \ldots, s_k\}$ of distinct non-trivial elements of $G$ there exists $y \in C$ such that $G = \langle s_i, y \rangle$ for $i = 1, 2, \ldots, k$. Further, the exact uniform spread of $G$, denoted by $u(G)$, is the largest $k$ such that $G$ has the uniform spread $k$.

The following lemma gives an equivalent definition of spread, was presented by Bradley and Moori in [5], such that is useful for computational purposes.

Lemma 1.3. [5, Lemma 1.1] A finite non-abelian group $G$ has spread $r$, if for every set $S = \{s_1, s_2, \ldots, s_r\}$ of distinct elements of prime order in $G$, there exists an element $y \in G$ such that $G = \langle s_i, y \rangle$ for $i = 1, 2, \ldots, r$.

The following is a version of Lemma 1.3 adapted to uniform spreads.

Lemma 1.4. A finite group $G$ has uniform spread $k$ if there exists a fixed conjugacy class $C$ in $G$ such that for every set $S = \{s_1, s_2, \ldots, s_k\}$ of distinct elements of prime order in $G$, there exists an element $y \in C$ such that $G = \langle s_i, y \rangle$ for $i = 1, 2, \ldots, k$.

Proof. Let $S = \{s_1, s_2, \ldots, s_k\}$ be any set of distinct non-trivial elements of $G$. Then there exist positive integers $m_i$ for $i = 1, 2, \ldots, k$ such that $s_i^{m_i} = x_i$ and order of $x_i$ for $i = 1, 2, \ldots, k$ is prime. Thus by assumption there exists $y \in C$ such that $G = \langle x_i, y \rangle = \langle s_i^{m_i}, y \rangle \subseteq \langle s_i, y \rangle \subseteq G$. Therefore $G = \langle s_i, y \rangle$ for all $i = 1, 2, \ldots, k$.  

Clearly $u(G) \leq s(G)$, and in general these numbers are distinct. The exact spread and uniform spread of finite simple groups has been studied in [3, 6, 7, 9, 15, 16]. The exact spread of only two sporadic simple groups have been determined. For the remaining twenty four sporadic simple groups, bounds of the exact spread are known [4, 5, 7, 11, 13, 21]. In [4, 21] it was proved that $s(M_{11}) = 3$ and Fairbairn has shown that $s(M_{23}) = 8064$ [12].

In this paper we give upper bounds on the exact uniform spreads of thirteen sporadic simple groups. First we give some lemmas that present upper bounds and then offer three algorithms to calculate these upper bounds of the exact uniform spread.

For information on the sporadic simple groups and their maximal subgroups we use Atlas [10]. All calculations were done with the aid of GAP [19] and Magma [8].

The rest of the paper is organized as follows: in Section 2, we give some preliminary results. Three algorithms to calculate upper bounds of the exact uniform spread are offered in Section 3, and in Section 4 we prove our Main Theorem.

2. Preliminaries

Let $G$ be a finite group and let $M$ be a maximal subgroup of $G$. As is standard we write $M^G$ for the maximal subgroups of $G$ that are conjugate to $M$ and $\mathcal{M}$ for the set of all maximal subgroups.
of $G$. We write $cl(G)$ for the collection of all conjugacy classes of elements of $G$ and $cl_p(G)$ for the collection of all conjugacy classes of elements of $G$ of prime order. We write $nX$ for a conjugacy class of elements in $G$ of order $n$ which $n > 1$. For a conjugacy class $nX$ we define $\text{supp}(nX) = \{ M \mid M \in \mathcal{M} \text{ and } M \cap nX \neq \emptyset \}$.

From Definition 1.2, it is clear that if $y$ is a mate for the set $S = \{ s_1, s_2, \ldots, s_k \}$, then no maximal subgroup contains both $y$ and $s_i$ for $i = 1, 2, \ldots, k$. We use this property in the following definitions.

**Definition 2.1.** Let $G$ be a finite group and let $M$ be a maximal subgroup of $G$. Let $S$ be a set of elements from a conjugacy class $nX$. When $M \cap nX \neq \emptyset$ we say that $S$ supports $M^G$ if $M' \cap S \neq \emptyset$ for every $M' \in M^G$. In this case we define $\text{support}_{nX}(M)$ for the size of the smallest $S$ that supports $M^G$.

From the Definition 2.1 it is clear that if $M \cap nX \neq \emptyset$ then $\text{support}_{nX}(M) \leq |M^G|$.

Consider a conjugacy class $nY$ of elements in $G$. There exists a conjugacy class $mX \in cl_p(G)$ such that $\text{supp}(mX) \cap \text{supp}(nY) \neq \emptyset$. We use this fact in the following definition.

**Definition 2.2.** Let $G$ be a finite group and $mX$ and $nY$ be two conjugacy classes of elements of $G$ such that $mX \in cl_p(G)$ and $\text{supp}(mX) \cap \text{supp}(nY) \neq \emptyset$. We define,

$$\text{support}(mX, nY) = \min\{ \text{support}_{mX}(M) \mid M \in \mathcal{M} \text{ and } M \in \text{supp}(mX) \cap \text{supp}(nY) \}.$$ 

**Lemma 2.3.** Let $G$ be a finite group and $mX$ and $nY$ be two conjugacy classes of elements of $G$ such that $mX \in cl_p(G)$ and $\text{supp}(mX) \cap \text{supp}(nY) \neq \emptyset$. Consider a set $S$ of elements of the class $mX$. Let $u_{mX,nY}(G)$ be the largest integer such that any set $S$ of this size has a mate from the class $nY$. Then we have,

$$u(G) \leq \max\{ \min\{ u_{mX,nY}(G) \mid mX \in cl_p(G) \text{ and } \text{supp}(mX) \cap \text{supp}(nY) \neq \emptyset \} \mid nY \in cl(G) \}.$$ 

**Proof.** For a given conjugacy class $nY$ consider $u_{mX,nY}(G)$ for each $mX \in cl_p(G)$ where $\text{supp}(mX) \cap \text{supp}(nY) \neq \emptyset$. The minimum of all these is then an upper bound on the size of a set that can have a mate from the class $nY$. The maximum of these values, when $nY$ goes through all classes of $G$, is then an upper bound on $u(G)$.

**Lemma 2.4.** Let $G$ be a finite group. Then,

$$u(G) \leq \max\{ \min\{ \text{support}(mX, nY) \mid mX \in cl_p(G) \text{ and } \text{supp}(mX) \cap \text{supp}(nY) \neq \emptyset \} \mid nY \in cl(G) \} - 1.$$ 

**Proof.** Consider a conjugacy class $nY$ of elements in $G$. There is a conjugacy class $cX \in cl_p(G)$ such that $\text{support}(cX, nY) = \min\{ \text{support}(mX, nY) \mid mX \in cl_p(G) \text{ and } \text{supp}(mX) \cap \text{supp}(nY) \neq \emptyset \}$. From the Definition 2.2 there exists a set $S$ of elements from the class $cX$ of size $\text{support}(cX, nY)$ such that has not any mate from the class $nY$. Therefore $u_{cX,nY}(G) \leq \text{support}(cX, nY) - 1$. Then the result is concluded from Lemma 2.3.
Lemma 2.5. Let $M$ be a maximal subgroup of $G$ and $nX$ be a conjugacy class of elements of $G$ such that $M \cap nX \neq \emptyset$. If each element of the class $nX$ is inside a unique subgroup of class $M^G$ then $\text{support}_{nX}(M) = |M^G|$.

Proof. From the Definition 2.1 we have $\text{support}_{nX}(M) \leq |M^G|$. Let $S$ be a set of elements from the class $nX$ such that supports $M^G$. Let $|S| < |M^G|$. Each element of $S$ is inside a unique subgroup of $M^G$. Therefore $S$ has non-empty intersection with at most $|S|$ subgroups of $M^G$. Then $S$ does not support $M^G$ that is contradiction. $\square$

In Table 1 bounds on the exact spread of thirteen sporadic simple groups are presented. The lower bounds were proved in [7] except for $M_{23}$ whose lower bound was proved in [12]. The upper bounds were proved in [11] except for $Fi_{22}$ and $M_{23}$ whose upper bounds were proved in [4]. According to the fact that $u(G) \leq s(G)$ for any group $G$ in general, upper bounds presented in Table 1 also are upper bounds on the exact uniform spread. The lower bounds given here are lower bounds on $s(G)$ and are therefore not necessarily lower bounds on $u(G)$.

<table>
<thead>
<tr>
<th>$G$</th>
<th>Upper bound</th>
<th>$G$</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Lower bound</td>
<td></td>
<td>Lower bound</td>
</tr>
<tr>
<td>$M_{23}$</td>
<td>8064 8064</td>
<td>$Ly$</td>
<td>1296826874 35049375</td>
</tr>
<tr>
<td>$Ru$</td>
<td>1252799 2880</td>
<td>$Th$</td>
<td>976841774 133997</td>
</tr>
<tr>
<td>$O'N$</td>
<td>2857238 3072</td>
<td>$Fi_{23}$</td>
<td>31670 911</td>
</tr>
<tr>
<td>$Co_2$</td>
<td>1024649 270</td>
<td>$Co_1$</td>
<td>46621574 3671</td>
</tr>
<tr>
<td>$Fi_{22}$</td>
<td>186 13</td>
<td>$J_4$</td>
<td>47766599363 1647124116</td>
</tr>
<tr>
<td>$HN$</td>
<td>74064374 8593</td>
<td>$Fi'_{24}$</td>
<td>7819305288794 269631216855</td>
</tr>
<tr>
<td>$M$</td>
<td>5791748068511982636944259374 3385007637938037777290624</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In our Main Theorem we present upper bounds on the exact uniform spreads for thirteen sporadic simple groups. These results are presented in Table 2.

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Table 2. Upper bounds on the exact uniform spreads of twelve sporadic simple groups.

<table>
<thead>
<tr>
<th>$G$</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{12}$</td>
<td>7</td>
</tr>
<tr>
<td>$J_1$</td>
<td>132</td>
</tr>
<tr>
<td>$M_{22}$</td>
<td>25</td>
</tr>
<tr>
<td>$J_2$</td>
<td>11</td>
</tr>
<tr>
<td>$HS$</td>
<td>32</td>
</tr>
<tr>
<td>$J_3$</td>
<td>458</td>
</tr>
<tr>
<td>$M_{24}$</td>
<td>32</td>
</tr>
<tr>
<td>$McL$</td>
<td>277</td>
</tr>
<tr>
<td>$He$</td>
<td>653</td>
</tr>
<tr>
<td>$Suz$</td>
<td>373</td>
</tr>
<tr>
<td>$Co_3$</td>
<td>1539</td>
</tr>
<tr>
<td>$B$</td>
<td>3843461129719173164826623999999</td>
</tr>
</tbody>
</table>

**Main Theorem.** If $G$ is one of the groups $M_{12}$, $J_1$, $M_{22}$, $J_2$, $HS$, $J_3$, $M_{24}$, $McL$, $He$, $Suz$, $Co_3$ or $B$, then the exact uniform spread $u(G)$ is bounded above by the numbers given in Table 2 and $u(M_{11}) = 3$.

3. Three Algorithms

Let $G$ be a finite group and $M_i$ for $i \in I$ be some maximal subgroups of $G$, such that $(\cup_{i \in I} M_i) \cap nY \neq \emptyset$ which $nY$ goes through all classes of $G$. According to Lemma 2.4 if we can determine $\text{support}_{mX}(M_i)$ for $i \in I$ and each $mX \in cl_p(G)$ that $M_i \in \text{supp}(mX)$, then

$$u(G) \leq \max\{\min\{\text{support}_{mX}(M_i) | mX \in cl_p(G) \text{ and } M_i \in \text{supp}(mX)\} | i \in I\} - 1.$$ 

In this section for a maximal subgroup $M$ and $mX \in cl_p(G)$ which $m \in \text{supp}(mX)$ we present three algorithms to compute upper bounds for $\text{support}_{mX}(M)$. The problem of computing upper bounds of $\text{support}_{mX}(M)$ can be interpreted in terms of intersection graphs.

Let $mX = \{x_1, x_2, \ldots, x_k\}$. Each $x \in mX$ is inside $h$ conjugates of $M$. We denote by $M_{x_i}$ the set of all conjugates of $M$ containing $x_i$ for $i = 1, 2, \ldots, k$. Each two $M_{x_i}$ and $M_{x_j}$ for $1 \leq i, j \leq k$, can be disjoint or have intersection of different size. It is clear that $\cup_{i=1}^{k} M_{x_i} = M^G$. Now we consider graph $\Gamma = (V, E)$ such that $V$, the set of vertices, is $M_{x_i}$ for $i = 1, 2, \ldots, k$. Two vertices $M_{x_i}$ and $M_{x_j}$ are adjacent if $M_{x_i} \cap M_{x_j} \neq \emptyset$. The graph $\Gamma$ is called the intersection graph of $G$ defined by $M$ and $mX$. Note that it is vertex transitive and is therefore regular. If $\text{support}_{mX}(M) = t$ then there exists a set $I$ of size $t$ such that $\cup_{i \in I} M_{x_i} = M^G$. Therefore determining $\text{support}_{mX}(M)$ is equivalent to
finding a set $S$, consisting of vertices of $\Gamma$, of minimum size such that union of its members contains all conjugates of $M$.

In the following algorithms, let $G$ be a finite group and $mX$ and $cY$ be two conjugacy classes of elements of $G$ such that $mX \in cl_p(G)$. Also suppose $M$ is a maximal subgroup of $G$ that has intersection with $mX$ and $n = |M^G|$. With Algorithm-1 we can obtain an upper bound for $support_{mX}(M)$.

Algorithm-1:

**Input**: group $G$, maximal subgroup $M$ and conjugacy class $mX \in cl_p(G)$

**Output**: an upper bound for $support_{mX}(M)$

**Step 1.** Construct graph $\Gamma = (V, E)$ from $G$, $M$ and $mX$. Set $min = n$.

Repeat for each vertex $M_x$ of $V$:

**Step 2.** Find a maximum independent set $S$ of $\Gamma$ containing $M_x$. Then set $L = |S|$ and $N = \cup_{M_y \in S} M_y$. If $|N| = n$ then go to Step 4.

**Step 3.** Select a vertex $M_z$ such that $|N \cap M_z|$ is of minimum size. Then set $N' := N \cup M_z$ and add one to $L$. If $|N'| = n$ then go to Step 4, else replace $N$ with $N'$ and go to Step 3.

**Step 4.** If $L < min$ then $min = L$, else continue.

End repeat.

**Step 5.** Return $min$.

If the maximal subgroup $M$ contains elements of a class $cY$ then $min$, the output of Algorithm-1, is an upper bound of $u_{mX,cY}(G)$. If each element of the class $cY$ is in more than one conjugate of $M$ then by a few changes in Algorithm-1 we can obtain better upper bound for $u_{mX,cY}(G)$.

Algorithm-2:

**Input**: group $G$, maximal subgroup $M$, conjugacy classes $mX \in cl_p(G)$ and $cY$

**Output**: an upper bound for $u_{mX,cY}(G)$

**Step 1.** Construct graph $\Gamma = (V, E)$ from $G$, $M$ and $mX$. Set $min = n$.

Repeat for each vertex $M_x$ of $V$:

**Step 2.** Find a maximum independent set $S$ of $\Gamma$ containing $M_x$. Then set $L = |S|$ and $N = \cup_{M_y \in S} M_y$. Also set $T$ to be the union of elements of the class $cY$ in conjugates of $M$ contained in the set $N$. If $|N| = n$ or $|T| = |cY|$ then go to Step 4.

**Step 3.** Select a vertex $M_z$ such that $|N \cap M_z|$ is of minimum size. Then set $N' := N \cup M_z$ and add one to $L$. Also add to $T$, elements of the class $cY$ in conjugates of $M$ that are in $M_z$. If $|N'| = n$ or $|T| = |cY|$ then go to Step 4, else replace $N$ with $N'$ and go to Step 3.

**Step 4.** If $L < min$ then $min = L$, else continue.

End repeat.

**Step 5.** Return $min$. 

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In the following algorithm let $M_1$ and $M_2$ be two maximal subgroups from different conjugacy classes such that have intersection with $mX$ and $cY$ and $|M_i| = n$. In Algorithm-3 by considering two conjugacy classes of maximal subgroups we can obtain better upper bound for $u_{mX,cY}(G)$.

Algorithm-3:

Input: group $G$, maximal subgroups $M_1$ and $M_2$, conjugacy classes $mX \in cl_p(G)$ and $cY$

Output: an upper bound for $u_{mX,cY}(G)$

Step 1. Construct graph $\Gamma = (V, E)$ from $G$, $M_1$ and $mX$. Set $\text{min} = n$.

Repeat for each vertex $M_1x$ of $V$:

Step 2. Find a maximum independent set $S$ of $\Gamma$ containing $M_1x$. Then set $L = |S|$ and $N = \cup_{M_1y \in S} M_1y$. Also set $T$ to be the union of elements of the class $cY$ in conjugates of $M_1$ contained in $N$ and in conjugates of $M_2$ that contain element $x$. If $|N| = n$ or $|T| = |cY|$ then go to Step 4.

Step 3. Select a vertex $M_1z$ such that $|N \cap M_1z|$ is of minimum size. Then set $N' := N \cup M_1z$ and add one to $L$. Also add to $T$, elements of the class $cY$ in conjugates of $M_1$ that are in $M_1z$ and in conjugates of $M_2$ that have element $z$. If $|N'| = n$ or $|T| = |cY|$ then go to Step 4, else replace $N$ with $N'$ and go to Step 3.

Step 4. If $L < \text{min}$ then $\text{min} = L$, else continue.

End repeat.

Step 5. Return $\text{min}$.

We can extend Algorithm-3 for any number of maximal subgroups from different conjugacy classes that intersect both $mX$ and $cY$. Since Algorithm-2 and Algorithm-3 use large amounts of memory, we often use Algorithm-1 to find upper bounds in the proof of our Main Theorem. Therefore we give an implementation of Algorithm-1 in Appendix A.

4. Proof Of The Main Theorem

In this section we prove our Main Theorem. Therefore we consider the groups presented in Table 2, individually. For some sporadic simple groups for which we use Algorithm-3, the obtained bounds are given in some tables. In these tables upper bound for one of $u_{2A,nY}(G)$ or $u_{2B,nY}(G)$ or $u_{3A,nY}(G)$ for each conjugacy class $nY$, which is minimum, was computed and are presented. We use "-" in tables when corresponding $u_{mX,nY}(G)$ was not computed. In keeping with Atlas notation we will write $5AB$ to indicate the conjugacy classes $5A$ and $5B$ and similarly for other classes.

Our method does not work for sporadic simple groups presented in Table 1. Because for using algorithms of Section 3, we must construct some graphs. For a finite group $G$ size of these graphs are depended on size of conjugacy classes of elements and conjugacy classes of maximal subgroups of $G$. Constructing these graphs for sporadic simple groups presented in Table 1 are not computationally possible yet.
4.1. Mathieu sporadic group $M_{11}$. Bradley and Holmes in [4] proved that $s(M_{11}) = 3$. They really just proved that any set of three elements has a mate from class 11A, in other words $3 \leq u(M_{11})$ which combined with the fact that $u(M_{11}) \leq s(M_{11})$ tells us that $u(M_{11}) = 3$. Thus we have the following proposition.

**Proposition 4.1.** $u(M_{11}) = 3$.

4.2. Mathieu sporadic group $M_{12}$. Group $M_{12}$ has eleven conjugacy classes of maximal subgroups and fifteen conjugacy classes of elements. Group $M_{12}$ has two conjugacy classes of maximal subgroups isomorphic to $M_{11}$. These two conjugacy classes have intersection with all conjugacy classes of elements except 2A, 3B, 6A and 10A. With Algorithm-1 we found $support_{2B}(M_{11}) \leq 3$ for two conjugacy classes of $M_{11}$. For the remaining conjugacy classes 2A, 3B, 6A and 10A we have:
- With Algorithm-3 and conjugacy classes of two maximal subgroups $M_{10}:2$ we found $u_{2B,2A}(M_{12}) < 3$.
- With Algorithm-3 and conjugacy classes of maximal subgroups $L_2(11)$, $2 \times S_5$ and $4^2:D_{12}$ we found $u_{2A,3B}(M_{12}) < 3$.
- With Algorithm-3 and conjugacy classes of maximal subgroups $L_2(11)$, $2 \times S_5$, $4^2:D_{12}$ and $A_4 \times S_3$ we found $u_{2A,6A}(M_{12}) < 8$.
- With Algorithm-3 and conjugacy classes of maximal subgroups $M_{10}:2$, $M_{10}:2$ and $2 \times S_5$ we found $u_{2B,10A}(M_{12}) < 8$.

The results are presented in Table 3.

**Table 3.** An upper bound for $u(M_{12})$.

<table>
<thead>
<tr>
<th>$nY$</th>
<th>2AB, 3A, 4AB, 5A, 6B, 8AB, 11AB</th>
<th>3B</th>
<th>6A</th>
<th>10A</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_{2B,nY}(M_{12}) &lt;$</td>
<td>3</td>
<td>-</td>
<td>-</td>
<td>8</td>
</tr>
<tr>
<td>$u_{2A,nY}(M_{12}) &lt;$</td>
<td>-</td>
<td>3</td>
<td>8</td>
<td>-</td>
</tr>
</tbody>
</table>

For the sake of completeness, we give below some sets of elements that satisfy the conditions of Table 3. Consider the permutation representation of $M_{12}$ on 12 points with standard generators from [1],

$$a = (1, 4)(3, 10)(5, 11)(6, 12), \quad b = (1, 8, 9)(2, 3, 4)(5, 12, 11)(6, 10, 7).$$

In Table 4 we give three elements from conjugacy class 2B that do not have a mate from conjugacy classes 2AB, 3AB, 4AB, 5A, 6B, 8AB and 11AB.

**Table 4.** Three elements of the class 2B.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 2)(3, 7)(5, 8)(6, 12)</td>
</tr>
<tr>
<td>2</td>
<td>(1, 2)(3, 7)(4, 10)(9, 11)</td>
</tr>
<tr>
<td>3</td>
<td>(4, 10)(5, 8)(6, 12)(9, 11)</td>
</tr>
</tbody>
</table>

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In Table 5 we give eight elements from conjugacy class 2B that do not have a mate from conjugacy class 10A.

Table 5. Eight elements of the class 2B.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>1</td>
<td>(1, 3)(2, 7)(5, 8)(9, 11)</td>
</tr>
<tr>
<td>2</td>
<td>(1, 6)(4, 11)(5, 12)(7, 10)</td>
</tr>
<tr>
<td>3</td>
<td>(2, 10)(3, 6)(4, 9)(8, 12)</td>
</tr>
<tr>
<td>4</td>
<td>(2, 4)(3, 12)(5, 6)(10, 11)</td>
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<tr>
<td>5</td>
<td>(1, 4)(3, 11)(5, 10)(7, 8)</td>
</tr>
<tr>
<td>6</td>
<td>(1, 9)(2, 5)(3, 4)(8, 10)</td>
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<tr>
<td>7</td>
<td>(1, 10)(2, 6)(4, 8)(11, 12)</td>
</tr>
<tr>
<td>8</td>
<td>(1, 12)(2, 11)(3, 5)(4, 7)</td>
</tr>
</tbody>
</table>

In Table 6 we give eight elements from conjugacy class 2A that do not have a mate from conjugacy class 6A.

Table 6. Eight elements of the class 2A.

<p>| | |</p>
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<tr>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 2)(3, 12)(4, 6)(5, 10)(7, 8)(9, 11)</td>
</tr>
<tr>
<td>2</td>
<td>(1, 2)(3, 7)(4, 8)(5, 10)(6, 11)(9, 12)</td>
</tr>
<tr>
<td>3</td>
<td>(1, 2)(3, 4)(5, 10)(6, 12)(7, 9)(8, 11)</td>
</tr>
<tr>
<td>4</td>
<td>(1, 2)(3, 11)(4, 9)(5, 10)(6, 7)(8, 12)</td>
</tr>
<tr>
<td>5</td>
<td>(1, 2)(3, 12)(4, 10)(5, 7)(6, 11)(8, 9)</td>
</tr>
<tr>
<td>6</td>
<td>(1, 2)(3, 7)(4, 6)(5, 11)(8, 9)(10, 12)</td>
</tr>
<tr>
<td>7</td>
<td>(1, 2)(3, 11)(4, 5)(6, 12)(7, 10)(8, 9)</td>
</tr>
<tr>
<td>8</td>
<td>(1, 2)(3, 4)(5, 12)(6, 7)(8, 9)(10, 11)</td>
</tr>
</tbody>
</table>

**Proposition 4.2.** $u(M_{12}) \leq 7$.

*Proof.* Applying Lemma 2.3 to the information in Table 3 implies the result. \hfill \Box

The proofs of all subsequent propositions are similar to the above, so we omit them.

4.3. **Janko sporadic group $J_1$.** We know that $J_1$ has seven conjugacy classes of maximal subgroups and fifteen conjugacy classes of elements. Conjugacy classes of maximal subgroups $L_2(11)$, $2 \times A_5$, 19:6 and $D_6 \times D_{10}$ have intersection with all conjugacy classes of elements except 7A. Conjugacy classes of maximal subgroups $2^3:7:3$ and 7:6 have intersection with 7A. We use Algorithm-1 to compute an upper bound for $support_{2A}(M)$ for $M \in \{L_2(11)$, $2 \times A_5$, 19:6, $D_6 \times D_{10}\}$ and Algorithm-3 to compute an upper bound for $u_{3A,7A}(J_1)$ with maximal subgroups $2^3:7:3$ and 7:6. Computed bounds are

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support
\textit{A}(L_2(11)) \leq 33, support
\textit{A}(2 \times A_5) \leq 91, support
\textit{A}(19:6) \leq 127, support
\textit{A}(D_6 \times D_{10}) \leq 133 and u_{3A,7A}(J_1) < 125.

The results are presented in Table 7.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$\nu_Y$ & $2A, 3A, 5AB, 6A, 11A$ & $7A$ & $10AB$ & $15AB$ & $19ABC$ \\
\hline
$\nu_{2A,nY}(J_1) <$ & 33 & - & 91 & 133 & 127 \\
$\nu_{3A,nY}(J_1) <$ & - & 125 & - & - & - \\
\hline
\end{tabular}
\caption{An upper bound for $u(J_1)$.}
\end{table}

**Proposition 4.3.** $u(J_1) \leq 132$.

4.4. **Mathieu sporadic group $M_{22}$.** Group $M_{22}$ has eight conjugacy classes of maximal subgroups and twelve conjugacy classes of elements. Conjugacy classes of maximal subgroups $L_3(4), 2^4 \times A_6$ and $L_2(11)$ have intersection with all conjugacy classes of elements. With Algorithm-1 we found support
\textit{A}($L_3(4)$) \leq 5, support
\textit{A}($2^4 \times A_6$) \leq 7 and support
\textit{A}($L_2(11)$) \leq 26, By Lemma 2.4, an upper bound of $u(M_{22})$ is 25.

**Proposition 4.4.** $u(M_{22}) \leq 25$.

4.5. **Hall-Janko sporadic group $J_2$.** We know that $J_2$ has nine conjugacy classes of maximal subgroups and twenty one conjugacy classes of elements. Conjugacy classes of maximal subgroups $U_3(3)$ and $3.PGL_2(9)$ have intersection with all conjugacy classes of elements except $5CD, 6B$ and $10CD$. With Algorithm-1 we found support
\textit{A}($U_3(3)$) \leq 5 and support
\textit{A}($3.PGL_2(9)$) \leq 9. For the remaining conjugacy classes $5CD, 6B$ and $10CD$ we have:
- With Algorithm-3 and conjugacy classes of maximal subgroups $2^1 + 4: A_5$ and $A_5 \times D_{10}$ we found $u_{3A,5D}(J_2) < 4$ and $u_{3A,5C}(J_2) < 4$.
- With Algorithm-3 and conjugacy classes of maximal subgroups $2^2 + 4:(3 \times S_3), A_4 \times A_5, L_3(2):2$ and $5^2:D_{12}$ we found $u_{2A,6B}(J_2) < 12$.
- With Algorithm-3 and conjugacy classes of maximal subgroups $2^1 + 4: A_5$ and $A_5 \times D_{10}$ we found $u_{3A,10D}(J_2) < 12$ and $u_{3A,10C}(J_2) < 12$.

The results are presented in Table 8.
TABLE 8. An upper bound for $u(J_2)$.

<table>
<thead>
<tr>
<th>$nY$</th>
<th>2A, 3AB, 4A, 6A, 7A, 8A, 12A</th>
<th>2B, 5AB, 10AB, 15AB</th>
<th>5CD</th>
<th>6B</th>
<th>10CD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_{2A,nY}(J_2) &lt;$</td>
<td>5</td>
<td>9</td>
<td>-</td>
<td>12</td>
<td>-</td>
</tr>
<tr>
<td>$u_{3A,nY}(J_2) &lt;$</td>
<td>-</td>
<td>-</td>
<td>4</td>
<td>-</td>
<td>12</td>
</tr>
</tbody>
</table>

Proposition 4.5. $u(J_2) \leq 11$.

Bradley and Holmes in [4] proved that $s(J_2) \leq 24$. Since the upper bounds of $s(J_2)$ and $u(J_2)$ differed by a significant margin we conjecture that $u(J_2) < s(J_2)$. Therefore we give below some sets of elements that satisfy the conditions of Table 8. Consider the permutation representation of $J_2$ on 100 points with standard generators $a$ and $b$ in [1]. Define $x = ab$ and $y = ab^{-1}$.

In Table 9 we give five elements from conjugacy class 2A that do not have a mate from conjugacy classes 2A, 3AB, 4A, 6A, 7A, 8A and 12A.

TABLE 9. Five elements of the class 2A.

| 1 | $by(yx)^4(y^2(xy)^2)$ |
| 2 | $by^2x^2((yx)^2y^2)x(xy)^2(yx)^2a$ |
| 3 | $x(xy)^3(xy^2x)^4yx^2a$ |
| 4 | $b^{-1}x(xy)^2x(xy)^4(x(xy)^2yx^2y$ |
| 5 | $x^{-3}b((yx)^2y)^2yx^2ya$ |

In Table 10 we give nine elements from conjugacy class 2A that do not have a mate from conjugacy classes 2B, 5AB, 10AB and 15AB.

TABLE 10. Nine elements of the class 2A.

| 1 | $((xy)^3x^2y)^2yx(xy)^2a$ |
| 2 | $((ba)^2x^{-1})^2(b^{-1}xa)^2b((yx)^2y)^2x^2y^2xa$ |
| 3 | $b^{-1}xy^2x(xy)^4x^2yx^2$ |
| 4 | $b(x(xy)^2)^2x^2yx^2$ |
| 5 | $b(xy)^2x(yx^2y)^4yx^2$ |
| 6 | $(yx^2y)^2xy((xy)^2x)^2a$ |
| 7 | $b^{-1}(yx)^2(yxy)^2((yx)^2x)^2y(xy^2)^2a$ |
| 8 | $by^2xy(yxy)^2yx^2(yx^2)^2a$ |
| 9 | $b(xy)^2y((xy)^2yx)^2(xy^2)^2yx^2$ |

In Table 11 we give twelve elements from conjugacy class 2A that do not have a mate from conjugacy classes 6B. In Table 12 we give four elements from conjugacy class 3A that do not have
Table 11. Twelve elements of the class 2A.

<table>
<thead>
<tr>
<th></th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((b^{-1}(xy)^2x^2)ya)^2</td>
</tr>
<tr>
<td>2</td>
<td>((bya)^2(ba)^2b^{-1}((yx)^2y^2x^2)xy</td>
</tr>
<tr>
<td>3</td>
<td>((xy(yz)^2)^2y(xy)^4y^2xa</td>
</tr>
<tr>
<td>4</td>
<td>(byxy(xy^2x)^4(yx)^2ya</td>
</tr>
<tr>
<td>5</td>
<td>(b^{-1}(yz)^2xy((yx)^2y^2x^2)yx</td>
</tr>
<tr>
<td>6</td>
<td>(b^{-1}(yz)^2xy((yx)^2x^2)^2x^2</td>
</tr>
<tr>
<td>7</td>
<td>(byx(xy)^2x(yx(xy)^2)^2x(xy)^2a</td>
</tr>
<tr>
<td>8</td>
<td>((ba)^2x^{-2}b(xy^2xy)^2(xy)^3</td>
</tr>
<tr>
<td>9</td>
<td>(((bya)^2ba)^3</td>
</tr>
<tr>
<td>10</td>
<td>(b^{-1}x(xy)^5x^3</td>
</tr>
<tr>
<td>11</td>
<td>(bx(xy^2y)^2xy^2x^2y</td>
</tr>
<tr>
<td>12</td>
<td>(bx(xy^2y)^2xy((xy)^2x^2)^2y^2</td>
</tr>
</tbody>
</table>

Table 12. Four elements of the class 3A.

<table>
<thead>
<tr>
<th></th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((xy)^2yx(xy)^3x^2y^2a</td>
</tr>
<tr>
<td>2</td>
<td>(y(xy)^4xy^2x^2yx</td>
</tr>
<tr>
<td>3</td>
<td>((yxy)^2(xy)^2(yx^2)^2a</td>
</tr>
<tr>
<td>4</td>
<td>(b^{-1}(yz)^2((yx)^2y^2xy)^2</td>
</tr>
</tbody>
</table>

a mate from conjugacy classes 5CD.

In Table 13 we give twelve elements from conjugacy class 3A that do not have a mate from conjugacy classes 10CD.

Table 13. Twelve elements of the class 3A.

<table>
<thead>
<tr>
<th></th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(b^{-1}(yx)^3y(xy^2x)^2x^2</td>
</tr>
<tr>
<td>2</td>
<td>(y(xy)^4xy^2x^2yx</td>
</tr>
<tr>
<td>3</td>
<td>(yxy(xy^2x)^3yxya</td>
</tr>
<tr>
<td>4</td>
<td>((y(xy)^2)^2</td>
</tr>
<tr>
<td>5</td>
<td>(((bya)^2x^{-1})^2</td>
</tr>
<tr>
<td>6</td>
<td>(((ba)^2x^{-1})^2by(yxy)^3xy^2</td>
</tr>
<tr>
<td>7</td>
<td>(x((xy)^2y)^2yx^2y^2xya</td>
</tr>
<tr>
<td>8</td>
<td>(y(xy)^2x^2xy((xy)^2)^2a</td>
</tr>
<tr>
<td>9</td>
<td>(x^{-2}(ba)^2b^{-1}x(yxy)^2x^2yx^2</td>
</tr>
<tr>
<td>10</td>
<td>(y((xy)^2)^2x^2xy^2x^2yx</td>
</tr>
<tr>
<td>11</td>
<td>(x(yx^2y)^2(xy)^3ya</td>
</tr>
<tr>
<td>12</td>
<td>((xy)^2(xyx)^2x(xy)^2a</td>
</tr>
</tbody>
</table>
4.6. **Higman-Sims sporadic group HS**. We know that HS has twelve conjugacy classes of maximal subgroups and twenty four conjugacy classes of elements. Conjugacy classes of maximal subgroups $M_{22}$, $U_3(5):2$ and $S_8$ have intersection with all conjugacy classes of elements. We found $\text{support}_{2A}(S_8)\leq 33$, $\text{support}_{2A}(U_3(5):2)\leq 11$ and $\text{support}_{2A}(M_{22})\leq 5$ with Algorithm-1. By Lemma 2.4, an upper bound of $u(HS)$ is 32. So, we have the following proposition.

**Proposition 4.6.** $u(HS) \leq 32$.

4.7. **Janko sporadic group $J_3$**. Group $J_3$ has nine conjugacy classes of maximal subgroups and twenty one conjugacy classes of elements. Conjugacy classes of maximal subgroups $L_2(16):2$, $L_2(19)$, and $(3 \times A_6):2$ have intersection with all conjugacy classes of elements. With Algorithm-1 we found $\text{support}_{2A}(L_2(16):2)\leq 83$, $\text{support}_{2A}(L_2(19))\leq 375$ and $\text{support}_{2A}(3 \times A_6):2)\leq 459$. By Lemma 2.4, an upper bound of $u(J_3)$ is 458. Therefore the following proposition is concluded.

**Proposition 4.7.** $u(J_3) \leq 458$.

4.8. **Mathieu sporadic group $M_{24}$**. Group $M_{24}$ has nine conjugacy classes of maximal subgroups and twenty six conjugacy classes of elements. Conjugacy classes of maximal subgroups $M_{23}$, $M_{22}:2$, $2^4:A_8$, $M_{12}:2$ and $L_3(4):S_3$ have intersection with all conjugacy classes of elements. By Algorithm-1 $\text{support}_{2A}(M_{23})\leq 3$, $\text{support}_{2A}(M_{22}:2)\leq 13$, $\text{support}_{2A}(2^4:A_8)\leq 16$, $\text{support}_{2A}(M_{12}:2)\leq 27$ and $\text{support}_{2A}(L_3(4):S_3)\leq 33$ were calculated. By Lemma 2.4, an upper bound of $u(M_{24})$ is 32. So, we have the following proposition.

**Proposition 4.8.** $u(M_{24}) \leq 32$.

4.9. **McLaughlin sporadic group $McL$**. We know that $McL$ has twelve conjugacy classes of maximal subgroups and twenty four conjugacy classes of elements. Conjugacy classes of maximal subgroups $U_4(3)$, $M_{22}$, $U_3(5)$, $L_3(4):2$ and $2.A_8$ have intersection with all conjugacy classes of elements. With Algorithm-1 we found $\text{support}_{2A}(U_4(3))\leq 13$, $\text{support}_{2A}(M_{22})\leq 46$, $\text{support}_{2A}(U_3(5))\leq 107$, $\text{support}_{2A}(L_3(4):2)\leq 160$ and $\text{support}_{2A}(2.A_8)\leq 278$. By Lemma 2.4, an upper bound of $u(McL)$ is 277. Therefore the following proposition is concluded.

**Proposition 4.9.** $u(McL) \leq 277$.

4.10. **Held sporadic group $He$**. We know that $He$ has eleven conjugacy classes of maximal subgroups and thirty three conjugacy classes of elements. Conjugacy classes of maximal subgroups $S_4(4):2$, $2^2.L_3(4).S_3$, $2^6:3.S_6$, $2^{1+6}.L_3(2)$ and $3.S_7$ have intersection with all conjugacy classes of elements. With Algorithm-1 we found $\text{support}_{2A}(S_4(4):2)\leq 23$, $\text{support}_{2A}(2^2.L_3(4).S_3)\leq 46$, $\text{support}_{2A}(2^6:3.S_6)\leq 147$, $\text{support}_{2A}(2^{1+6}.L_3(2))\leq 654$ and $\text{support}_{2A}(3.S_7)\leq 642$. By Lemma 2.4, an upper bound of $u(He)$ is 653. Therefore the following proposition is concluded.

**Proposition 4.10.** $u(He) \leq 653$.  

http://dx.doi.org/10.22108/ijgt.2018.111238.1478
4.11. **Suzuki sporadic group** $Suz$. Group $Suz$ has seventeen conjugacy classes of maximal subgroups and forty three conjugacy classes of elements. Conjugacy classes of maximal subgroups $G_2(4)$, $3.U_4(3):2$, $U_5(2)$, $2^{1+6}.U_4(2)$ and $J_2:2$ have intersection with all conjugacy classes of elements of $Suz$. By Algorithm-1 we found $\text{support}_{3A}(G_2(4)) \leq 11$, $\text{support}_{2A}(3.U_4(3):2) \leq 150$, $\text{support}_{2A}(U_5(2)) \leq 69$, $\text{support}_{3A}(2^{1+6}.U_4(2)) \leq 204$ and $\text{support}_{3A}(J_2:2) \leq 374$. By Lemma 2.4, an upper bound of $u(Suz)$ is 373. So, we have the following proposition.

**Proposition 4.11.** $u(Suz) \leq 373$.

4.12. **Conway sporadic group** $Co_3$. We know that $Co_3$ has fourteen conjugacy classes of maximal subgroups and forty two conjugacy classes of elements. Conjugacy classes of maximal subgroups $McL:2$, $HS$, $M_{23}$, $3^5:(2 \times M_{11})$ and $U_3(5):S_3$ have intersection with all conjugacy classes of elements of $Co_3$. By Algorithm-1 we found $\text{support}_{2A}(McL:2) \leq 12$, $\text{support}_{2A}(HS) \leq 79$, $\text{support}_{2A}(M_{23}) \leq 160$, $\text{support}_{2A}(3^5:(2 \times M_{11})) \leq 456$ and $\text{support}_{2A}(U_3(5):S_3) \leq 1540$. By Lemma 2.4, an upper bound of $u(Co_3)$ is 1539. Therefore the following proposition is concluded.

**Proposition 4.12.** $u(Co_3) \leq 1539$.

4.13. **Baby Monster sporadic group** $\mathbb{B}$. Group $\mathbb{B}$ has thirty conjugacy classes of maximal subgroups \cite{20} and one hundred eighty four conjugacy classes of elements. Let $M$ is a maximal subgroup of $\mathbb{B}$ and $nX \in cl_p(\mathbb{B})$. It is clear that if $M \cap nX = \emptyset$ then $\text{support}_{nX}(M) \leq |M^{\mathbb{B}}|$. Therefore by Lemma 2.4, $u(\mathbb{B})$ is less than the index of maximal subgroup of minimum order of $\mathbb{B}$. Maximal subgroup $47:23$ of $\mathbb{B}$ has minimum order among all maximal subgroups. This subgroup has index 38436129719173164826624000000. Then the following proposition is concluded.

**Proposition 4.13.** $u(\mathbb{B}) \leq 38436129719173164826623999999$.

We are now ready to prove our Main Theorem.

**Main Theorem.** If $G$ is one of the groups $M_{12}$, $J_1$, $M_{22}$, $J_2$, $HS$, $J_3$, $M_{24}$, $McL$, $He$, $Suz$, $Co_3$ or $\mathbb{B}$, then the exact uniform spread $u(G)$ is bounded above by the numbers given in Table 2 and $u(M_{11}) = 3$.

**Proof.** The proof follows from Propositions 4.1 to 4.13. \hfill $\square$

**Acknowledgments**

The authors express their deepest gratitude to Professor Ali Reza Ashrafi for introducing them with concept of spread and to the referee whose suggestions and comments led to significant improvement of the presentation of this paper.

**Appendix A**

In the following we present a Magma \cite{8} implementation of Algorithm-1. This code computes an upper bound of $\text{support}_{2A}(3.PGL_2(9))$ in the group $J_2$. By choosing proper values for “f”, “m” and
“size\_mX” and loading groups presented in Table 2 one can check the results of Section 4. For groups $J_3$, $McL$, $He$, $Suz$ and $Co_3$, Magma can not compute maximal subgroups. In these cases we loaded the group and its maximal subgroup $M$ from [1] and omitted the lines “mG:=MaximalSubgroups(G)” and “M:=mG[f]` subgroup” from code.

In the following code “f” is the number corresponding to subgroup $M$ in maximal subgroups of $G$ in Magma ordering. “m” is order of an element of conjugacy class $mX$. “size\_mX” is size of conjugacy class $mX$.

We used Magma version 2.10 [8].

```magma
f:=6;
m:=2;
size\_mX:=315;
load j2;
mG:=MaximalSubgroups(G);
M:=mG[f]` subgroup;
G1:=CosetImage(G,M);
cG1:=ConjugacyClasses(G1);
n:=Index(G,M);
for i in [1..#cG1] do
    if cG1[i,1] eq m and cG1[i,2] eq size\_mX then
        gX:=cG1[i,3];
        break;
    end if;
end for;
v:=Fix(gX)^ G1;
v:=Set(v);
v:=SetToIndexedSet(v);
min:=n;
bound:=#Fix(gX);
for k in [1..#v] do
    best:=bound;
t:=0;
L:=1;
N:={};
range:=[1..#v];
Exclude(~range,k);
N:=N join v[k];
while t ne n do
```

http://dx.doi.org/10.22108/ijgt.2018.111238.1478
for i in range do
    r:=#(v[i] meet N);
    if r le best then
        best :=r;
        temp:=i;
    end if;
end for;
N:=N join v[temp];
L:=L+1;
Exclude(~range,temp);
if L gt min then
    break;
end if;
best:=bound;
t:=#N;
end while;
if L lt min then
    min:=L;
    print "Temporary upper bound is =", min;
end if;
end for;
print "Upper bound is=", min;

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