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A PRESENTATION FOR THE SUBGROUP OF COMPRESSED CONJUGATING AUTOMORPHISMS OF A PARTIALLY COMMUTATIVE GROUP

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Abstract. Let G_Γ be a partially commutative group. We find a finite presentation for the subgroup $\mathrm{Conj}_V(G_\Gamma)$ of compressed vertex conjugating automorphisms of the automorphism group $\mathrm{Aut}(G_\Gamma)$ of G. We have written GAP packages which compute presentations for $\mathrm{Aut}(G_\Gamma)$ and its subgroups $\mathrm{Conj}(G_\Gamma)$ and $\mathrm{Conj}_V(G_\Gamma)$.

1. Introduction

The class of partially commutative groups, also known as right-angled Artin groups, is a natural generalisation of both the classes of free and free Abelian groups, has a simple description and its groups are, in many ways algorithmically tractable. They have applications in many areas, for example the theory of 3-manifolds [15] and concurrent processes [6]. Furthermore, groups in the class provide several important examples of pathogenic behaviour in finitely presented group theory. Further details may be found in [3] and [8].

Throughout the paper Γ denotes a finite simplicial graph (i.e. undirected, with no loops or multiple edges). We write $V = V(\Gamma) = \{x_1, \dots, x_n\}$ for the set of vertices of and $E = E(\Gamma)$ for the set of edges, which are 2-subsets of V. The *partially commutative group* with *commutation graph* is the group G with presentation

$$G_\Gamma = \langle V | R \rangle$$

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where

$$R = \{[x_i, x_j] \mid x_i, x_j \in V \text{ and } \{x_i, x_j\} \in E\}$$

and $[x_i\ , x_j] = x_i^{-1}x_j^{-1}x_ix_j$.

We denote by $\text{Aut}(G)$ the automorphism group of a group G. A finite generating set for $\text{Aut}(G_\Gamma)$ was found by Laurence [10], building on the work of Servatius [13]. Several authors have continued the study of this group: see for example [7], [12], [4] and the references therein. In particular, Day [5] constructed a finite presentation for $\text{Aut}(G_)$ and Toinet [14] constructed a finite presentation for the subgroup $\text{Conj}(G_\Gamma) \leq \text{Aut}(G_)$ of all *vertex conjugating automorphisms*: that is the subgroup consisting of automorphisms ϕ having the property that, for all $x \in V$, there exists $g_x \in G$, such that $x\phi = x^{g_x}$.

Here we consider a subgroup $\text{Conj}_V(G_{)\leq \text{Conj}(G_)}$ of *compressed conjugating automorphisms* (see Definition 3.1) studied in [7], for which we find a finite presentation, following the methods of Day and Toinet. The first author, with assistance from George Mitchell and Matthew Fisher, has also developed GAP [9] packages to compute presentations for $\text{Aut}(G_\Gamma)$, $\text{Conj}(G_\Gamma)$ and $\text{Conj}_V(G_\Gamma)$.

2. Automorphisms of partially commutative groups

2.1. **Preliminaries.** Fix a finite simplicial graph and let $G = G_\Gamma$ be the corresponding partially commutative group. Let $L = V \cup V^{-1}$ and, for $x \in V$, define $v(x) = v(x^{-1}) = x$. Define the *link* of $x \in V$ to be $\text{lk}(x) = \{y \in V \mid y \text{ is adjacent to } x\}$ and the *star* of x to be $\text{st}(x) = \text{lk}(x) \cup \{x\}$. For a subset Y of V let $Y^{-1} = \{y^{-1} \mid y \in Y\}$ and extend the definitions of link and star to L by setting, for all $x \in L$,

- $\text{lk}(x) = \text{lk}(v(x))$ and $\text{st}(x) = \text{st}(v(x))$,
- $\text{lk}_L(x) = \text{lk}(v(x)) \cup \text{lk}(v(x))^{-1}$ and $\text{st}_L(x) = \text{st}(v(x)) \cup \text{st}(v(x))^{-1}$.

For $x \in L$ define $_x$ to be the full subgraph on $V \setminus \text{st}(x)$. In the sequel, it is convenient to abuse notation as follows.

- For a connected component $_0$ of $_x$ we shall refer to $V \cap _0$ as a *connected component of* $_x$.

A *word* (over L) is an element of the free monoid L^* generated by L and the length of the word w is denoted $|w|$. The *length of a conjugacy class* C is defined to be the minimum of the lengths of elements of C. A k-tuple $\mathbf{C} = (C_1, \ldots, C_k)$ of conjugacy classes has *length* $|\mathbf{C}| = \sum_{i=1}^{k} |C_i|$. Also, if $\alpha \in \text{Aut}(G_)$ then by $\mathbf{C}\alpha$ we mean $(C_1\alpha, \ldots, C_k\alpha)$.

2.2. **Laurence-Servatius Generators.** In [13] Servatius listed four types of automorphism of G and conjectured that these generated $\text{Aut}(G_)$; showing this to be true in several cases. Subsequently Laurence[10] proved that the Servatius conjecture holds for all G. In this section we describe these four types of automorphism which we call Laurence-Servatius generators. Some initial terminology is required. If $x, y \in L$ then we say that $x \leq y$ if $\text{lk}(x) \subseteq \text{st}(y)$. This gives rise to an equivalence relation $\sim$ on L, given by $x \sim y$ if $x \leq y$ and $y \leq x$. We denote the $\sim$ equivalence class of x by $[x]$.

Definition 2.1. *(i) For all $x, y \in L$ there is an automorphism, denoted $\tau_{x,y}$, sending x to xy and fixing all elements of V except $v(x)$, if and only if $x \le y$ and $v(x) \neq v(y)$ [13].*

For $y, x_1, \ldots, x_r \in L$ such that $x_i \le y$, $v(x_i) \neq v(y)$ and $v(x_i) \neq v(x_j)$ if $i \neq j$, let $D = \{x_1, \ldots, x_r\}$ and define the elementary transvection $\tau_{D,y} = \tau_{x_1,y} \cdots \tau_{x_r,y}$. *Let* $\mathrm{Tr}(G_{)=\mathrm{Tr}}$ *be the set of all such elementary transvections.*

(ii) Given $x \in L$, let C be a connected component of ${}_x$. Then there is an automorphism of G, denoted $\alpha_{C,x}$, sending y to y^x, for all $y \in C$, and fixing all elements of $V \setminus C$ [13].

Let $x \in L$ and let Y be a non-empty union of disjoint connected components $Y_1, \ldots, Y_r$ of ${}_x$. The map $\alpha_{Y,x} = \alpha_{Y_1,x} \cdots \alpha_{Y_r,x}$ is called an elementary vertex conjugating automorphism. *The set of all elementary vertex conjugating automorphisms is denoted* $\mathrm{LI}(G_{)=\mathrm{LI}}$ *(because Servatius called them "locally inner").*

In particular, if $Y = V \setminus \mathrm{st}(x)$ then $\alpha_{Y,x}$, the inner automorphism sending g to g^x for all $g \in G$, is called an elementary inner automorphism *and denoted γ_x.*

(iii) For $x \in V$, there is an automorphism ι_x of G that sends x to x^{-1} and fixes all other elements of V [13]. ι_x is called an inversion *and the set of all inversions is denoted* $\mathrm{Inv}(G_{)=\mathrm{Inv}}$.

(iv) An automorphism π of G is called a graphic automorphism *if π restricted to V determines an automorphism of . Every automorphism of extends to give a graphic automorphism of G [13]. The subgroup of all graphic automorphisms of G_Γ is denoted* $\mathrm{Aut}^{(G_{)=\mathrm{Aut}}}$.

In addition, define $\mathrm{Aut}^{\pm(G_{)}}$ *to be the subgroup of* $\mathrm{Aut}(G_{)}$ *generated by the graphic automorphisms and inversions.*

In the light of the following theorem, we call the set $\mathcal{LS} = \mathrm{Tr} \cup \mathrm{LI} \cup \mathrm{Inv} \cup \mathrm{Aut}$ the *Laurence-Servatius* generators of $\mathrm{Aut}(G_{)}$. (In fact, Servatius and Laurence use a slightly smaller generating set: they consider $\tau_{D,x}$ and $\alpha_{C,x}$ as generators only for $|D| = 1$ and for C a single connected component of ${}_x$.)

Theorem 2.2 ([13], [10])**.** $\mathrm{Aut}(G_{)}$ *is generated by the set $\mathcal{LS}$.*

Day[5] constructs a presentation for $\mathrm{Aut}(G_{)}$ with generating set containing, and closely related to $\mathcal{LS}$. This presentation has relations (R1)–(R10) which are rewritten in [1] to give the following theorem.

Theorem 2.3 ([5], [1])**.** *The group $\mathrm{Aut}(G_\Gamma)$ has finite presentation $\langle \mathcal{LS} \mid \mathcal{R} \rangle$, where $\mathcal{R}$ is a set of relations (R1)–(R10), corresponding to the relations of the presentation in [5].*

2.3. **Vertex conjugating automorphisms.** Of interest here is the following theorem of Laurence.

Theorem 2.4 ([10])**.** *The subgroup* Conj *of vertex conjugating automorphisms is generated by the set* LI.

Toinet[14] gave a presentation for Conj as follows.

Definition 2.5. *Let $\mathcal{C}$ consist of the the relations ($\mathcal{C}$1) to ($\mathcal{C}$4) below, on the set of elementary vertex conjugations* LI. *In these relations, elements of* LI *are written $\alpha_{C,z}$, where $z \in L$ and C is a union of connected components of $_z$.*

($\mathcal{C}$1) $(\alpha_{C,x})^{-1} = \alpha_{C,x^{-1}}$,
($\mathcal{C}$2) $\alpha_{C,x}\alpha_{D,x} = \alpha_{C\cup D,x}$, *if* $C \cap D = \emptyset$,
($\mathcal{C}$3) $\alpha_{C,x}\alpha_{D,y} = \alpha_{D,y}\alpha_{C,x}$, *if* $v(x) \notin D$, $v(y) \notin C$, $x \neq y^{\pm 1}$, *and either* $C \cap D = \emptyset$ *or* $y \in \mathrm{lk}_L(x)$;
($\mathcal{C}$4) $\gamma_y^{-1}\alpha_{C,x}\gamma_y = \alpha_{C,x}$, *if* $v(y) \notin C$ *and* $x \neq y^{\pm 1}$.

Theorem 2.6 (Toinet[14])**.** *The subgroup* $\mathrm{Conj}(G_\Gamma)$ *has a presentation* $\langle \mathrm{LI} \,|\, \mathcal{C}\rangle$.

3. The subgroup of compressed automorphisms

Definition 3.1. *An element $\phi \in \mathrm{Conj}(G_)$ is said to be a* compressed conjugating automorphism *if, for every element $x \in V$, there exists $f_x \in G$ such that $y\phi = y^{f_x}$, for all $y \in [x]$ the equivalence class of the vertex x. The subgroup of all compressed conjugating automorphism is denoted* Conj_V.

(In [7] elements of Conj_V are called vertex conjugating automorphisms. In this paper we have reverted to the earlier use of this terminology, by both Laurence and Toinet, for elements of Conj, as in Sections 1 and 2 above. To maintain consistency with existing GAP packages, [7] and [1], we have nevertheless kept the notation Conj_V for the subgroup of compressed automorphisms.) Here we adapt Toinet's proof of Theorem 2.6 to give a presentation for Conj_V. First we define a candidate generating set for Conj_V.

Definition 3.2. *Define LI_V to be the set of elementary vertex conjugations $\alpha_{C,x}$, where $x \in L$, C is a union of connected components of $_x$ and, for all $z \in V$ either*

(i) $[z] \cap C = \emptyset$; *or*
(ii) $[z] \subseteq C \cup \mathrm{st}(x)$.

Lemma 3.3. $\mathrm{LI}_V = \mathrm{LI} \cap \mathrm{Conj}_V$.

Proof. It follows directly from the definitions that $\mathrm{LI}_V \subseteq \mathrm{LI} \cap \mathrm{Conj}_V$. Suppose then that $\alpha_{C,x} \in \mathrm{LI} \cap \mathrm{Conj}_V$, let $z \in V$, assume that $[z] \cap C \neq \emptyset$ and let $u \in [z] \cap C$. By definition there exists $f_z \in G$ such that $y\alpha_{C,x} = y^{f_z}$, for all $y \in [z]$, so $u^x = u\alpha_{C,x} = u^{f_z}$. Therefore $xf_z^{-1} \in C(u)$, the centraliser of u in G. As $C(u) = \langle \mathrm{st}(u)\rangle$, and $x \notin \mathrm{st}(u)$ (since $u \in C$) it follows that $f_z = g_z \cdot x$, for some $g_z \in C(u)$. If there is an element $v \in [z] \backslash C \cup \mathrm{st}(x)$ then $v = v\alpha_{C,x} = v^{f_z} = v^{g_z x}$. As $x \notin \mathrm{st}(u) \cup \mathrm{st}(v)$, $x \notin \mathrm{supp}(v^{g_z})$, and since $v^{g_z x}$ reduces to v it must be that x commutes with every element of $\mathrm{supp}(v^{g_z})$. In particular x commutes with v, a contradiction. Hence $[z] \subset C \cup \mathrm{st}(x)$ and so $\alpha_{C,x}$ in LI_V. □

We use the following definitions and results from [7] to show that LI_V generates Conj_V.

Definition 3.4 ([7])**.** *An elementary vertex conjugating automorphism $\alpha_{C,u}$, where $u \in L$ and C is a connected component of $_u$, is called a* singular conjugating automorphism *if $|C| = 1$. The set of all singular conjugating automorphisms is denoted $\mathrm{LI}_S = \mathrm{LI}_S(G_)$.*

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To form elements of Conj_V from singular conjugating automorphisms we collect them together, as in the following definition. Note that if $x, u \in V$ such that $u \le x$ and $[u, x] \ne 1$ then no two elements of $[u]$ commute. Indeed, if $z \in [u] \backslash \{u, x\}$, then $[z, x] = 1$ implies $x \in \mathrm{lk}(z) \subseteq \mathrm{st}(u)$, so $[x, u] = 1$, a contradiction. Moreover, if $[z, u] = 1$ then $z \in \mathrm{lk}(u) \subseteq \mathrm{st}(x)$, so $[z, x] = 1$. Hence no element of $[u] \backslash \{u, x\}$ commutes with either u or x. If v and z belong to $[u] \backslash \{x\}$ with $z \ne u$, then $[v, z] = 1$ implies $z \in \mathrm{lk}(v) \subseteq \mathrm{st}(u)$, and so $[u, z] = 1$, a contradiction. Therefore, as claimed, no two elements of $[u]$ commute. In particular, every singleton subset of $[u] \backslash \{x\}$ is a connected component of $_x$.

Definition 3.5 ([7])**.** *For $x, u \in V$ such that $u \le x$ and $[u, x] \ne 1$, define $[u]' = [u] \backslash \{x\}$ and assume that $[u]' = \{v_1, \ldots, v_n\}$. The conjugating automorphism*

$$\alpha_{[u]',x} = \prod_{i=1}^{n} \alpha_{\{v_i\},x}$$

is called a collected conjugating automorphism*. The set of all collected conjugating automorphisms is denoted* $\mathrm{LI}_C = \mathrm{LI}_C(G_)$.

From [7, Lemma 3.40 (ii)], $\mathrm{LI}_C \subseteq \mathrm{Conj}_V$. Next we single out some further elementary conjugating automorphisms which belong to Conj_V.

Definition 3.6 ([7])**.** *The set of* regular conjugating automorphisms *is*

$$\mathrm{LI}_R = \mathrm{LI}_R(G_{)=\mathrm{LI}_V \setminus \mathrm{LI}_S}.$$

Proposition 3.7 ([7])**.** Conj_V *is generated by* $\mathrm{LI}_R \cup \mathrm{LI}_C$.

Theorem 3.8. Conj_V *is generated by* LI_V.

Proof. By definition $\mathrm{LI}_R \subseteq \mathrm{LI}_V$ so, in the light of Lemma 3.3 and Proposition 3.7, it suffices to show that $\mathrm{LI}_C \subseteq \mathrm{LI}_V$. Let $\beta \in \mathrm{LI}_C$, say $\beta = \alpha_{[u]',x}$, where $u, x \in V$, $u \le x$ and $[u, x] \ne 1$, and let $[u]' = [u] \backslash \{x\}$. In this case, for all $z \in V$, either $[z] \cap [u] = \emptyset$ or $[z] = [u]$, in which case $[z] \subseteq [u]' \cup \mathrm{st}(x)$, so one of the conditions (i) or (ii) of Definition 3.2 holds. Hence $\beta \in \mathrm{LI}_V$. Therefore $\mathrm{LI}_R \cup \mathrm{LI}_C \subseteq \mathrm{LI}_V$. □

Lemma 3.9. *Let $\alpha_{C,x}$ and $\alpha_{D,x}$ be elements of LI_V. Then*

(i) $\alpha_{C\cap D,x} \in \mathrm{LI}_V$ *and*
(ii) $\alpha_{C\cup D,x} \in \mathrm{LI}_V$.

Proof. (i) We must check the condition of Definition 3.2 holds when C is replaced by $C \cap D$. Let $z \in V$. If $[z] \cap C = \emptyset$ or $[z] \cap D = \emptyset$ then $[z] \cap (C \cap D) = \emptyset$, so the conditions hold. Thus we may assume $[z] \cap C \ne \emptyset$ and $[z] \cap D \ne \emptyset$, and since $\alpha_{C,x}, \alpha_{D,x} \in \mathrm{LI}_V$ this implies that

$$[z] \subseteq (C \cup \mathrm{st}(x)) \cap (D \cup \mathrm{st}(x)) = (C \cap D) \cup \mathrm{st}(x).$$

Hence, $\alpha_{C\cap D,x} \in \mathrm{LI}_V$.

(ii) We must check the conditions of Definition 3.2 hold when C is replaced by $C \cup D$. Suppose that $[z] \cap (C \cup D) \neq \emptyset$. Then $[z] \cap C \neq \emptyset$ or $[z] \cap D \neq \emptyset$. If $[z] \cap C \neq \emptyset$ then $[z] \subseteq C \cup \mathrm{st}(x) \subseteq (C \cup D) \cup \mathrm{st}(x)$, so $\alpha_{C\cup D,x} \in \mathrm{LI}_V$. A similar argument applies if $[z] \cap D \neq \emptyset$.

□

Given the result of Lemma 3.9 (ii), and the fact that the inner automorphism γ_y, belongs to LI_V, for all $y \in L$, the relations $\mathcal{C}$ may be regarded as relations between words over the set LI_V. We may therefore state our main theorem.

Theorem 3.10. *The subgroup* Conj_V *of* $\mathrm{Aut}(G_\Gamma)$ *has a presentation* $\langle \mathrm{LI}_V \mid \mathcal{C}\rangle$.

Our proof of this theorem relies on the fact that there is a peak lowering theorem for Conj_V.

3.1. **Peak lowering compressed automorphisms.** The peak lowering theorem of [5] holds for elements of Conj_V, but to use it to prove Theorem 3.10 we must show that the automorphisms that arise are all in LI_V. First we prove some preliminary lemmas.

Lemma 3.11. *If* $\alpha_{C,x} \in \mathrm{LI}_V$ *and* $D = V\backslash(C \cup \mathrm{st}(x))$ *then* $\alpha_{D,x^\epsilon} \in \mathrm{LI}_V$ *for* $\epsilon = \pm 1$.

Proof. To prove this it is necessary only to check that the conditions of Definition 3.2 hold when C is replaced by D. First note that, for all $z \in V$, either $[z] \cap C = \emptyset$; or $[z] \subseteq C \cup \mathrm{st}(x)$, by definition of LI_V. If $[z] \subseteq C \cup \mathrm{st}(x)$ then $[z] \cap D = [z] \cap (V\backslash(C \cup \mathrm{st}(x))) = \emptyset$.

On the other hand, if $[z] \cap C = \emptyset$ then $[z] \subseteq V\backslash(C \cup \mathrm{st}(x)) \cup \mathrm{st}(x) = D \cup \mathrm{st}(x)$. □

For $\alpha = \alpha_{C,x} \in \mathrm{Conj}$ define $\bar{\alpha} = \alpha_{D,x^{-1}}$, where $D = V\backslash(C \cup \mathrm{st}(x))$; so $\alpha\bar{\alpha}^{-1} = \bar{\alpha}^{-1}\alpha = \gamma_x$. From Lemma 3.11, we have $\alpha \in \mathrm{LI}_V$ if and only if $\bar{\alpha} \in \mathrm{LI}_V$.

Lemma 3.12. *If* $\pi \in \mathrm{Aut}^{\pm(G_\Gamma)}$ *and* $\alpha_{C,x} \in \mathrm{LI}_V$ *then* $\alpha_{D,y} \in \mathrm{LI}_V$, *where* $D = (C \cup C^{-1})\pi \cap V$ *and* $y = x\pi$.

Proof. It suffices to show that the result holds when π is an inversion or $\pi \in \mathrm{Aut}^{(G_)}$. If $\pi = \iota$ is an inversion and $x\iota = x$ then $\alpha_{D,y} = \alpha_{C,x}$, so there is nothing to prove. If $x\iota = x^{-1}$ then $\alpha_{D,y} = \alpha_{C,x^{-1}}$, which is in LI_V, if $\alpha_{C,x}$ is. Thus we may assume that $\pi \in \mathrm{Aut}^{(G_\Gamma)}$. We must check the conditions of Definition 3.2 hold when C is replaced by $C\pi$ and x by $x\pi$. Let $w, z \in V$ such that $w\pi = z$. Since $\pi\mid_V$ is a graph automorphism we have $[a]\pi = [a\pi]$ and $\mathrm{st}(a)\pi = \mathrm{st}(a\pi)$, for each $a \in V$; in particular $[z] = [w\pi] = [w]\pi$. Suppose that $[z] \cap C\pi \neq \emptyset$; let $u \in [z] \cap C\pi$ and let $v \in V$ such that $v\pi = u$. Then $v\pi \in [z] \cap C\pi = [w]\pi \cap C\pi$ implies $v \in [w] \cap C$, and as $\alpha_{C,x} \in \mathrm{LI}_V$ we have $[w] \subseteq C \cup \mathrm{st}(x)$. Thus $[z] = [w]\pi \subseteq C\pi \cup \mathrm{st}(x)\pi = C\pi \cup \mathrm{st}(x\pi)$; the conditions of Definition 3.2 hold, and $\alpha_{D,y} \in \mathrm{LI}_V$, as required. □

Lemma 3.13 (*cf.* [5])**.** *Suppose* $a, b \in L$, $\alpha_{D,b} \in \mathrm{LI}_V$ *with* $v(a) \notin D$, $v(a)$ *not adjacent to* $v(b)$ *in* Γ *and* $a \neq b$ *(possibly* $a = b^{-1}$*). Then* $\mathrm{lk}_L(a) \cap D = \emptyset$.

Proof. If $a = b^{-1}$ then certainly $\mathrm{lk}_L(a) \cap D = \emptyset$. Assume then that $a \neq b^{\pm 1}$. If $x \in \mathrm{lk}_L(a) \cap D$ then $x \in D$, and $v(a)$ is adjacent to x but not $v(b)$, so $v(a) \in D$, a contradiction. □

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Definition 3.14. *Let* $\mathbf{C}$ *be a* k*-tuple of conjugacy classes of* G_Γ *and let* $\alpha,\ \beta \in \mathrm{LI}_V$*. If*

$$|\mathbf{C}\alpha^{-1}| \geq |\mathbf{C}|,$$
$$|\mathbf{C}\alpha^{-1}| \geq |\mathbf{C}\alpha^{-1}\beta|$$

and one of these inequalities is strict then $a^{-1}\beta$ *is called a* peak *for* $\mathbf{C}$*.*

Lemma 3.15 (*cf.* [5], Sublemma 3.21)**.** *Let* $\alpha = \alpha_{C,a},\ \beta = \beta_{D,b} \in \mathrm{LI}_V$ *and let* $\mathbf{C}$ *be a* k*-tuple of conjugacy classes of* G_Γ*. Assume that* $\alpha^{-1}\beta$ *is a peak for* $\mathbf{C}$ *and that either*

(1) $a \in \mathrm{lk}_L(b)$ *or*
(2) $(C \cap D) \cup (C \cap \{b, b^{-1}\}) \cup (D \cap \{a, a^{-1}\}) \cup (\{a\} \cap \{b\}) = \emptyset$*.*

Then $\mid \mathbf{C}\beta \mid < \mid \mathbf{C}\alpha^{-1} \mid$*.*

This Lemma is proved, in greater generality, in [5].

Assume $\alpha^{-1}\beta$ is a peak for a k-tuple of conjugacy classes $\mathbf{C}$. A factorisation

$$\alpha^{-1}\beta = \delta_1 \cdots \delta_k$$

is called *peak-lowering* if, for all i with $1 \leq i < k$, we have $\delta_i \in \mathrm{LI}_V$ and

$$|\mathbf{C}(\delta_1 \cdots \delta_i)| < |\mathbf{C}\alpha^{-1}|.$$

Lemma 3.16. *Suppose* $\alpha,\ \beta \in \mathrm{LI}_V$ *and* $\mathbf{C}$ *is a* k*-tuple of conjugacy classes of* G_Γ*. If* $\alpha^{-1}\beta$ *is a peak with respect to* $\mathbf{C}$*, then there exists a peak-lowering factorisation of* $\alpha^{-1}\beta$*.*

Proof. Assume that $\alpha = \alpha_{C,a}$ and $\beta = \alpha_{D,b} \in \mathrm{LI}_V$. From the remarks above, $\bar{\alpha}$ and $\bar{\beta}$ belong to LI_V, and $\alpha\bar{\alpha}^{-1} = \gamma_a$ and $\beta\bar{\beta}^{-1} = \gamma_b$, so

$$\mathbf{C}\alpha^{-1} = \mathbf{C}\bar{\alpha}^{-1} \text{ and } \mathbf{C}\alpha^{-1}\beta = \mathbf{C}\alpha^{-1}\bar{\beta}.$$

We claim that if the lemma holds with α or β replaced with $\bar{\alpha}$ or $\bar{\beta}$ respectively, then it holds as originally stated. To see this, suppose $\delta_1 \cdots \delta_k$, with $\delta_i \in \mathrm{LI}_V$, is a peak-lowering factorisation of $\alpha^{-1}\bar{\beta}$. We have

$$\alpha^{-1}\beta = \alpha^{-1}\bar{\beta}\bar{\beta}^{-1}\beta = \delta_1 \cdots \delta_k\gamma_b,$$

and so (since the inner automorphism group is normal) there are elementary inner conjugating automorphisms $\gamma'_1, \ldots, \gamma'_r$, such that

$$\alpha^{-1}\beta = \delta_1 \cdots \delta_k\gamma_b = \gamma'_1 \cdots \gamma'_r\delta_1 \cdots \delta_k.$$

If $|\mathbf{C}\alpha^{-1}\bar{\beta}| < |\mathbf{C}\alpha^{-1}|$ then $\alpha^{-1}\beta = \delta_1 \cdots \delta_k\gamma_b$, is a peak-lowering factorisation of $\alpha^{-1}\beta$, since inner automorphisms do not change the length of a conjugacy class. Similarly, if $|\mathbf{C}| < |\mathbf{C}\alpha^{-1}|$ then $\alpha^{-1}\beta = \gamma'_1 \cdots \gamma'_r\delta_1 \cdots \delta_k$ is a peak-lowering factorisation of $\alpha^{-1}\beta$.

A similar argument holds if $\bar{\alpha}^{-1}\beta$ has a peak lowering factorisation by elements of LI_V, so we may swap $\bar{\alpha}$ for α and $\bar{\beta}$ for β as needs be in the proof of this lemma. Also, by the symmetry in the definition of a peak, we may switch α and β if necessary. We break the proof down into several cases.

Case(1): $a \in \mathrm{lk}_L(b)$. This implies that $a \in \mathrm{st}_L(b)$, $b \in \mathrm{st}_L(a)$, $a \neq b^{\pm 1}$ and, since $C \cap \mathrm{st}(a) = \emptyset = D \cap \mathrm{st}(b)$, that $v(a) \notin D$ and $v(b) \notin C$. Hence, as in relation $(\mathcal{C}3)$, we have a factorisation

$$\alpha^{-1}\beta = \alpha_{C,a^{-1}}\alpha_{D,b} = \alpha_{D,b}\alpha_{C,a^{-1}} = \beta\alpha^{-1}.$$

By Lemma 3.15, we have $|\mathbf{C}\beta| < |\mathbf{C}\alpha^{-1}|$, so this factorisation is peak-lowering.

Case(2): $(C \cap D) \cup (C \cap \{b, b^{-1}\}) \cup (D \cap \{a, a^{-1}\}) \cup (\{a\} \cap \{b\}) = \emptyset$ and $a \notin \mathrm{lk}_L(b)$. The first condition implies that $a \neq b$, so either $a = b^{-1}$ or $a \neq b^{\pm 1}$; and we have the following sub-cases.

Sub-case(2a): $a = b^{-1}$. As in relation $(\mathcal{C}2)$, we have a factorisation:

$$\alpha^{-1}\beta = \alpha_{C,b}\alpha_{D,b} = \alpha_{C \cup D,b},$$

and since from Lemma 3.9, $\alpha_{C \cup D,b} \in \mathrm{LI}_V$, this is trivially peak-lowering (having the form $\alpha^{-1}\beta = \delta_1$).

Sub-case(2b): $a \neq b^{\pm 1}$. In this case the conditions of $(\mathcal{C}3)$ hold and so

$$\alpha^{-1}\beta = \alpha_{C,a^{-1}}\alpha_{D,b} = \alpha_{D,b}\alpha_{C,a^{-1}} = \beta\alpha^{-1}.$$

From Lemma 3.15, we have $|\mathbf{C}\beta| < |\mathbf{C}\alpha^{-1}|$, so this factorisation is peak-lowering.

Case(3): $(C \cap D) \cup (C \cap \{b, b^{-1}\}) \cup (D \cap \{a, a^{-1}\}) \cup (\{a\} \cap \{b\}) \neq \emptyset$ and $a \notin \mathrm{lk}_L(b)$. To begin with we show that we may assume that $a \notin (D \cup D^{-1} \cup \{b\})$ and $b \notin (C \cup C^{-1} \cup \{a\})$. First, by replacing β with $\bar{\beta}$, if necessary, we may assume $a \notin (D \cup D^{-1} \cup \{b\})$. Assume that $b \in (C \cup C^{-1} \cup \{a\})$. If $b = a$ then $a \in (D \cup D^{-1} \cup \{b\})$, a contradiction. Hence $b \neq a$. If $b = a^{-1}$ then $a^{-1} = b \in (C \cup C^{-1})$, again a contradiction. Thus $a \neq b^{\pm 1}$ and swapping α with $\bar{\alpha}$ we have $a \notin (D \cup D^{-1} \cup \{b\})$ and $b \notin (C \cup C^{-1} \cup \{a\})$, as required.

Assume then that $a \notin D \cup D^{-1} \cup \{b\}$ and $b \notin C \cup C^{-1} \cup \{a\}$. From Subcase 4b of Day's proof of Lemma 3.8 in [5], we have in this case either

$$|\mathbf{C}'\alpha_{C \cap D',a}| < |\mathbf{C}'| \ \text{ or } \ |\mathbf{C}'\alpha_{C' \cap D,b}| < |\mathbf{C}'|, \tag{3.1}$$

where $\mathbf{C}' = \mathbf{C}\alpha^{-1}$, $C' = V \backslash (C \cup \mathrm{st}(a))$ and $D' = V \backslash (D \cup \mathrm{st}(b))$. If $a \neq b^{-1}$ then, from Lemma 3.13, we have $C \cap \mathrm{lk}(b) = \emptyset$ and $D \cap \mathrm{lk}(a) = \emptyset$. Thus $\mathrm{st}(a) \cap D = \emptyset$ and $\mathrm{st}(b) \cap C = \emptyset$. By assumption we now have $C \cap D \neq \emptyset$. Let $x \in C \cap D$ and let y be an element of C. There is a path from x to y in C and this path does not meet $\mathrm{st}(b)$; so $x \in D$ implies $y \in D$. It follows that $C = D$. This means that $C \cap D' = \emptyset$, so $\alpha_{C \cap D',a}$ is the identity map. Similarly (if $a \neq b^{-1}$) $\alpha_{C' \cap D,b}$ is the identity map. Thus if $a \neq b^{-1}$ we have a contradiction to (3.1); so we may assume $a = b^{-1}$. In this case, from Lemma 3.11 and Lemma 3.9 (i), $\alpha_{C \cap D,a}$, $\alpha_{C \cap D',a}$ and $\alpha_{C' \cap D,a}$ are in LI_V and we have $\alpha = \alpha_{C \cap D',a}\alpha_{C \cap D,a}$ and $\beta = \alpha_{C \cap D,a^{-1}}\alpha_{C' \cap D,a^{-1}}$.

Suppose $|\mathbf{C}'\alpha_{C \cap D',a}| < |\mathbf{C}'|$. Then $\mathbf{C}'\alpha_{C \cap D',a} = \mathbf{C}\alpha^{-1}_{C \cap D,a}$ and so $\alpha^{-1}_{C \cap D',a}\beta$ is a peak for $\mathbf{C}\alpha^{-1}_{C \cap D,a}$. We may apply Case 2a to obtain a peak-lowering factorisation $\alpha^{-1}_{C \cap D',a}\beta = \alpha_{(C \cap D') \cup D,b} = \alpha^{-1}_{C \cup D,a}$

of this peak, and then $\alpha^{-1}_{C\cap D,a}\alpha^{-1}_{C\cup D,a}$ is a peak-lowering factorisation of $\alpha^{-1}\beta$. A similar argument applies if $|\mathbf{C}'| < |\mathbf{C}'\alpha_{C'\cap D,b}|$. □

3.2. Proof of Theorem 3.10.

Theorem 3.10. Our proof is based on Toinet's [14, Theorem 3.1], which is an adaption of arguments of McCool [11], using Day's peak-lowering results for long-range automorphisms of partially commutative groups [5]. As the proof may be read off from Toinet's, after suitable substitutions and adjustments have been made, we do not give full details; which can be found in the proof of Theorem 4.3.15 of [1]. To obtain a proof of the current theorem from Toinet's, replace H (which we call Conj) by Conj_V, replace the set of elementary vertex automorphisms S by LI_V, and use $\mathcal{C}$ in place of R. Instead of Day's peak lowering lemma[5, Lemma 3.18] use our Lemma 3.16. Toinet's proof involves an analysis of the relators (R1)–(R10) in Day's presentation, to determine which give rise to relations on words over LI. The analysis in our version proceeds in the same way, and results in the same list of relations, but in our case it is necessary to ensure that all the elementary vertex automorphisms which arise belong to LI_V. Once this has been verified there is nothing further to do but apply the arguments of Toinet's proof, with the adaptions above.

The relation Toinet initially finds, between elements of Conj_V, are $(\mathcal{C}1)$ – $(\mathcal{C}4)$ above and

$$\pi^{-1}(\alpha_{C,a})\pi = \alpha_{D,b}, \tag{3.2}$$

for $\alpha_{C,a} \in \mathrm{LI}_V$ and π in the finite subgroup $\mathrm{Aut}^{\pm(G_\Gamma)}$, where $D = (C \cup C^{-1})\pi \cap V$ and $b = a\pi$. From Lemma 3.12, $\alpha_{D,b} \in \mathrm{LI}_V$, and all other elements of LI arising in the proof then belong to LI_V. Finally the conclusion of this version of the proof is that Conj_V has presentation $\langle \mathrm{LI}_V \mid \mathcal{C}\rangle$. □

4. GAP Packages

We have written a GAP package *AutParCommGrp* [2] for the computation of finite presentations for the automorphism group of a partially commutative group and some subgroups of the automorphism group. The main functions of the package *AutParCommGrp* are

- `FinitePresentationOfAutParCommGrp`(V, E) which computes Day's presentation of the automorphism group;
- `FinitePresentationOfSubgroupConj`(V, E) which computes Toinet's presentation of the subgroup Conj and
- `FinitePresentationOfSubgroupConjv`(V, E) which computes our presentation of the subgroup Conj_V.

All of these have input (V, E), the vertices and edges of the commutation graph of a partially commutative group. Output consists of two sets, *gens* and *rels*, the sets of generators and relations of the presentations computed.

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