THE ONE-PRIME POWER HYPOTHESIS FOR CONJUGACY CLASSES RESTRICTED TO NORMAL SUBGROUPS AND QUOTIENT GROUPS

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Communicated by Bijan Taeri

Abstract. We say that a group $G$ satisfies the one-prime power hypothesis for conjugacy classes if the greatest common divisor for all pairs of distinct conjugacy class sizes are prime powers. Insoluble groups which satisfy the one-prime power hypothesis have been classified. However it has remained an open question whether the one-prime power hypothesis is inherited by normal subgroups and quotients groups. In this note we construct examples to show the one-prime power hypothesis is not necessarily inherited by normal subgroups or quotient groups.

For $G$ a finite group let $cs(G) := \{|x|^G| \ x \in G\}$ denote the set of conjugacy class sizes in $G$. A group $G$ satisfies the one-prime power hypothesis for conjugacy classes if for every $m, n \in cs(G)$ with $m \neq n$, then the greatest common divisor of $m$ and $n$ is a prime power. The insoluble groups satisfying the one-prime power hypothesis have been classified in [4] and [1]. This problem is related to the one-prime power hypothesis for character degrees which has been studied for soluble groups in [2] and insoluble groups in [3].

It is well known that for $N \vartriangleleft G$, if $x \in N$ then $|x^N|$ divides $|x^G|$, while if $x \in G$ then $|(xN)^{G/N}|$ divides $|x^G|$. Therefore it is natural to ask whether the one-prime power hypothesis is inherited by normal subgroups and quotient groups. In [1] the authors prove the following lemma.

Lemma. [1, Lemma 3.1] Suppose that $G$ satisfies the one-prime power hypothesis and $r$ is a prime dividing $|G|$. If $N$ is a normal $r$-complement in $G$ then $N$ also satisfies the one-prime power hypothesis.
It is clear that the quotient analogue of [1, Lemma 3.1] holds as $G/N$ being an $r$-group for some prime $r$, trivially implies that it $G/N$ satisfies the one-prime power hypothesis. We first observe a minor generalisation of this result to normal subgroups and a weaker generalisation to quotients.

**Lemma.** Let $N \triangleleft G$ such that $|G/N|$ and $|N|$ are coprime. If $G$ satisfies the one-prime power hypothesis then $N$ also satisfies the one-prime power hypothesis.

**Proof.** This follows from the observation that if $x \in N$ then $|x^G| = |x^N|m$ for some $m$ dividing $|G : N|$.

Note that this proof does not work for quotient groups as it is not clear that $\frac{|x^G|}{|x^N|}$ will divide $|N|$. However when $N$ is taken to be the centre then this does hold.

**Lemma.** Let $G$ be a finite group and $N \leq Z(G)$ such that $|G : N|$ and $|N|$ are coprime. If $G$ satisfies the one-prime power hypothesis then so does $G = G/N$.

**Proof.** Let $\overline{G} := G/N$. The result follows by observing that if $x \in \overline{G}$ then $|x^G| = |\overline{x}|m$ for $m$ dividing $|N|$.

It can be seen that these proofs are a special case, however the authors in [1] claim to not know of any example which shows that the one-prime power hypothesis is not inherited by normal subgroups. We now construct examples to show that in fact the one-prime power hypothesis is not inherited by normal subgroups and quotients Therefore to study soluble one-prime power groups alternative methods are required.

It is easy to construct a group $G$ in which for any $x, y \in G$ the gcd($|x^G|, |y^G|$) is a prime power but this does not hold for a normal subgroup. In particular, let $G = C_p \times C_{p-1}$ such that 4 divides $p - 1$, but $p - 1$ is not a power of 2. Then $cs(G) = \{1, p - 1, p\}$. However in the normal subgroup $N = C_p \times C_{p-1}$ the class of size $p - 1$ splits into two classes of size $\frac{p-1}{2}$. This is the idea we will try to manipulate, however we need to construct a group where we have at least two classes of the same size in $G$ but in $N$ or $G/N$ at least one stays the same size and another splits into two classes.

**A counter example for normal subgroups:**

Let $H$ be the permutation group generated by $g = (1, 2, 3, 4, 5, 6), h_1 = (7, 8, 9, 10)$ and $h_2 = (8, 10)$ so that $H \cong C_6 \times D_8$. There exists an automorphism $x$ of order 2 such that $g^x = g^{-1}h_1, h_1^x = h_1^{-1}$ and $h_2^x = h_1h_2$. Let $G$ be the group $H \rtimes \langle x \rangle$. As $cs(H) = \{1, 2\}$ the class sizes of elements from $H$ in $G$ are either 1, 2 or 4. Thus it remains to consider those elements in $G \setminus H$. By using the relations given it follows that all the remaining classes have size 12. Thus $cs(G) = \{1, 2, 4, 12\}$ and $G$ satisfies the one-prime power hypothesis.

It is clear that $N := \langle g^2, h_1, h_2 \rangle \rtimes \langle x \rangle \triangleleft G$. Moreover, the conjugacy classes of $x$ and $h_2x$ in $G$ must either be the same in $N$ or split into two classes of equal size. As $C_G(h_2x) = C_N(h_2x)$, it follows that the class $(h_2x)^G$ splits into two classes of size 6 in $N$. However $C_G(x) > C_N(x)$ and therefore $x^G = x^N$. Thus $cs(N) = \{1, 2, 4, 6, 12\}$ and $N$ does not satisfy the one-prime power hypothesis.
A counter example for quotient groups:

Let $H$ be the permutation group generated by $g = (1, 2, 3, 4, 5, 6), h_1 = (7, 8, 9, 10)(11, 12, 13, 14)$ and $h_2 = (7, 11, 9, 13)(8, 14, 10, 12)$ so that $H \cong C_6 \times Q_8$. Then there exists an automorphism $x$ of order 4 such that $g^x = g^{-1}, h_1^x = g^3h_1^{-1}$ and $h_2 = h_1h_2$. Set $G = H \rtimes \langle x \rangle$. As before it is clear that all elements in $H$ have class size in $G$ contained in $\{1, 2, 4, 8\}$. Thus we need to find $|(yx)^G|, |(yx^2)^G|$ and $|(yx^3)^G|$ for all $y \in H$. However as $(yx)^{-1} = x^3y^{-1} = y'x^3$ for some $y' \in H$, the class size of $(yx^3)^G$ equals that of $(y'x)^G$. Also, $C_g(yx^2)$ contains $g$ and $h_2^x$ and therefore $|(yx^2)^G| \in \{2, 4, 8\}$. Hence it is enough to compute $|(yx)^G|$ for all $y \in H$. It follows from the relations that $|(yx)^G| = 12$ for all $y \in H$ and thus $cs(G) = \{1, 2, 4, 8, 12\}$ and $G$ satisfies the one-prime power hypothesis.

The action of $x$ fixes $g^2$ and therefore $N := \langle g^2 \rangle$ forms a normal subgroup and the quotient $\overline{G} := G/N \cong (C_3 \times Q_8) \rtimes C_4$. Moreover, $\pi^2$ now acts trivially and thus for $\overline{y} \in \overline{H}$ the class size $|((\pi x^2))^G| = |\overline{\pi^2}| \in \{1, 2, 4\}$. In addition, as before it only remains to compute the class sizes $|((\pi x)^G)|$ with $\overline{y} \in \overline{H}.$ As $N$ is a central subgroup of order 2 it follows that $|x^G|/|\pi^2| = 1$ or 2. Moreover, it can be seen that $|\overline{x}| = 12$ and $|((h_2^2x^2))^G| = 6$. Thus $cs(G) = \{1, 2, 4, 6, 12\}$ and therefore $\overline{G}$ does not satisfy the one-prime power hypothesis.

**Theorem.** The one-prime power hypothesis for conjugacy classes is not inherited by all normal subgroups and quotient groups.

**References**


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DOI: [http://dx.doi.org/10.22108/ijgt.2018.110074.1472](http://dx.doi.org/10.22108/ijgt.2018.110074.1472)