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CLASSIFYING FAMILIES OF CHARACTER DEGREE GRAPHS OF SOLVABLE GROUPS

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ABSTRACT. We investigate prime character degree graphs of solvable groups. In particular, we consider a family of graphs $\Gamma_{k,t}$ constructed by adjoining edges between two complete graphs in a one-to-one fashion. In this paper we determine completely which graphs $\Gamma_{k,t}$ occur as the prime character degree graph of a solvable group.

1. Introduction

We note that throughout this paper, G will be a finite solvable group. We denote $\text{Irr}(G)$ for the set of irreducible characters of G , and $\text{cd}(G) = \{\chi(1) \mid \chi \in \text{Irr}(G)\}$. Write $\rho(G)$ to be the set of primes that divide degrees in $\text{cd}(G)$. When working solely with graphs Γ (and not necessarily degree graphs of a finite solvable group G), we will also use the notation $\rho(\Gamma)$ to signify the vertex set. The degree graph of G , written as $\Delta(G)$, is the graph whose vertex set is $\rho(G)$. Two vertices p and q of $\rho(G)$ are adjacent in $\Delta(G)$ if there exists $a \in \text{cd}(G)$ where pq divides a . We identify each vertex of a graph with a prime in $\rho(G)$. Throughout this paper, for simplicity, when labeling a vertex we associate that label also with a prime $p \in \rho(G)$. Character degree graphs have been studied in a variety of places; for example, see [2], [5], [6], [7], [8], and [9]. We expand upon the work done in [1].

Families of graphs have been considered and studied by way of direct products for some time. However, showing that a family of graphs cannot occur as the prime character degree graph of any solvable group was done recently in [1]. One of the main tools in that paper was to classify vertices as

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admissible. In particular, it was shown that the graph cannot occur when every vertex is admissible. Many arguments in this paper rely on this result.

In this paper, we construct a family of a graphs, denoted $\Gamma_{k,t}$ (for natural numbers $k \geq t \geq 1$), and inquire for which values of k and t does $\Gamma_{k,t}$ occur as the prime character degree graph of some solvable group. We construct $\Gamma_{k,t}$ by taking two complete graphs (uniquely determined by the number of vertices, of size k and t , respectively), and we place an edge between the two graphs injectively. That is, we attach edges uniquely from one complete graph to the other in a one-to-one fashion. The construction clearly gives a graph that satisfies Pálffy's condition and is at most diameter two. However, when determining if the graph can or cannot occur as the prime character degree graph of a solvable group, many of our arguments rely on facts about graphs that are diameter three. This has been studied more extensively in [5] and [9], and their techniques and results will play an important role.

For $k \geq t \geq 1$, we handle the cases separately for $t = 1$ (Proposition 3.1), $t = 2$ (Proposition 3), $t = 3$ (Proposition 3.2), and $t \geq 4$ (Proposition 3.3). We can sum up their parts in the result below:

Main Theorem. *The graph $\Gamma_{k,t}$ occurs as the prime character degree graph of a solvable group precisely when $t = 1$ or $k = t = 2$. Otherwise $\Gamma_{k,t}$ does not occur as the prime character degree graph of any solvable group.*

2. Preliminaries

Here we present some classic and more recent results that are immediately related to prime character degree graphs of solvable groups. Our aim is to keep this paper as self-contained as possible, and one can see [3] or [7] for more reading. Most of the results from this section, however, can be found in [1], [8], or [9].

Lemma 2.1 (Pálffy's condition). ([8]) *Let G be a solvable group and let π be a set of primes contained in $\Delta(G)$. If $|\pi| = 3$, then there exists an irreducible character of G with degree divisible by at least two primes from π . (In other words, any three vertices of the prime character degree graph of a solvable group span at least one edge.)*

Definition 2.2. ([1]) *A vertex p of a graph Γ is **admissible** if:*

- (i) *the subgraph of Γ obtained by removing p and all edges incident to p does not occur as the prime character degree graph of any solvable group, and*
- (ii) *none of the subgraphs of Γ obtained by removing one or more of the edges incident to p occur as the prime character degree graph of any solvable group.*

Lemma 2.3. ([1]) *If Γ is a graph in which every vertex is admissible, then Γ is not the prime character degree graph of any solvable group.*

Taking direct products is a method to generate graphs from smaller ones. We present the detailed construction below.

Definition 2.4. A graph Γ is a **direct product** if it can be constructed in the following way:

We start with graphs A and B . We have that $\rho(\Gamma) = \rho(A) \cup \rho(B)$, where $\rho(A)$ and $\rho(B)$ are disjoint sets. There is an edge between vertices p and q in Γ if any of the following are satisfied:

- (i) $p, q \in \rho(A)$ and there is an edge between p and q in A ,
- (ii) $p, q \in \rho(B)$ and there is an edge between p and q in B , or
- (iii) $p \in \rho(A)$ and $q \in \rho(B)$.

Lemma 2.5. Complete graphs occur as $\Delta(G)$ for some solvable group G .

The following two results are due to Lewis. They will aid in constructions throughout this paper.

Lemma 2.6. ([7]) For every positive integer N , there is a solvable group G so that $\Delta(G)$ has two complete, connected components: one having an isolated vertex, and the other having N vertices.

Lemma 2.7. ([5]) Let G be a solvable group and let $p \in \rho(G)$. If P is a normal Sylow p -subgroup of G , then $\rho(G/P') = \rho(G) \setminus \{p\}$.

The above Lemma 2.7 implies the subgraph $\Delta(G/P')$ is a subgraph of $\Delta(G)$ without the vertex p , all edges incident to p , and possibly without edges between vertices adjacent to p .

The following was presented in [9]. Suppose Γ is a graph of diameter three. We can partition $\rho(\Gamma)$ into four nonempty disjoint sets: $\rho(\Gamma) = \rho_1 \cup \rho_2 \cup \rho_3 \cup \rho_4$. One can do this in the following way: find vertices p and q where the distance between them is three. Let ρ_4 be the set of all vertices that are distance three from the vertex p , which will include the vertex q . Let ρ_3 be the set of all vertices that are distance two from the vertex p . Let ρ_2 be the set of all vertices that are adjacent to the vertex p and some vertex in ρ_3 . Finally, let ρ_1 consist of p and all vertices adjacent to p that are not adjacent to anything in ρ_3 . This notation is not unique, and depends on the vertices p and q . Using this notation, we have the following result.

Proposition 2.8. ([9]) Let G be a solvable group where $\Delta(G)$ has diameter three. One then has the following:

- (i) $|\rho_3| \geq 3$,
- (ii) $|\rho_1 \cup \rho_2| \leq |\rho_3 \cup \rho_4|$,
- (iii) if $|\rho_1 \cup \rho_2| = n$, then $|\rho_3 \cup \rho_4| \geq 2^n$, and
- (iv) G has a normal Sylow p -subgroup for exactly one prime $p \in \rho_3$.

Next we state the main theorem from [1]. It gives rise to two families of graphs that satisfy Pálffy's condition. We will concern ourselves with one of them, which is our motivation of this paper.

Theorem 2.9. ([1]) Let Γ be a graph satisfying Pálffy's condition with $k \geq 5$ vertices. Assume that there exists two vertices p_1 and p_2 in Γ such that

- (i) both p_1 and p_2 are of degree two,
- (ii) p_1 is adjacent to p_2 , and

(iii) p_1 and p_2 share no common neighbor.

Then Γ is not the prime character degree graph of any solvable group.

3. Main Results

Similar to a direct product, here we construct the graph $\Gamma_{k,t}$. This will yield a family of graphs that we investigate and classify in our Main Theorem. We say that a graph Γ is in the family $\Gamma_{k,t}$ if it can be constructed in the following way:

We start with complete graphs A and B . Let A have k vertices (arbitrarily labeled a_1, a_2, \dots, a_k), and let B have t vertices (arbitrarily labeled b_1, b_2, \dots, b_t). Without loss of generality, we fix $k \geq t$.

Letting $\Gamma_{k,t} := \Gamma$, we have that $\rho(\Gamma_{k,t}) = \rho(A) \cup \rho(B)$, where $\rho(A)$ and $\rho(B)$ are disjoint sets. There is an edge between vertices p and q in $\Gamma_{k,t}$ if any of the following are satisfied:

- (i) $p, q \in \rho(A)$,
- (ii) $p, q \in \rho(B)$, or
- (iii) $p = a_i \in \rho(A)$ and $q = b_i \in \rho(B)$ for some $1 \leq i \leq t$.

Notice how this construction generates a graph that satisfies Pálffy’s condition and is at most diameter two. We now proceed towards proving the propositions needed for the Main Theorem.

Proposition 3.1. *Let $k \geq 1$. The graph $\Gamma_{k,1}$ occurs as $\Delta(G)$ for some solvable group G .*

Proof. Observe that $\Gamma_{1,1}$ occurs by Lemma 2.5. For $k \geq 2$, we consider the following: let A be a graph consisting of two complete, connected components: one having an isolated singleton, and the other having $k - 1$ vertices. Let B be a singleton. Notice that A occurs as $\Delta(G_A)$ for some solvable group G_A by Lemma 2.6, and B occurs as $\Delta(G_B)$ for some solvable group G_B by Lemma 2.5. Therefore, the direct product of A and B occurs as $\Delta(G)$ for some solvable group G . Finally, observe that $\Gamma_{k,1}$ is precisely the direct product of A and B . □

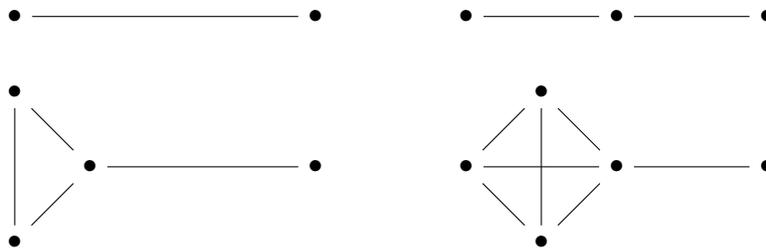


FIGURE 1. Graphs in Proposition 3.1

It was shown in [2] that $\Gamma_{2,2}$, represented in Figure 2, occurs as $\Delta(G)$ for some solvable group G . It is in fact a direct product. The rest of the cases for $t = 2$ can be extrapolated from Theorem 2.9.



FIGURE 2. The graph $\Gamma_{2,2}$

beginprop([1]) Let $k \geq 3$. The graph $\Gamma_{k,2}$ is not the prime character degree graph of any solvable group.

Proof. Observe that Theorem 2.9 yields two families, one of which is $\Gamma_{k,2}$. □

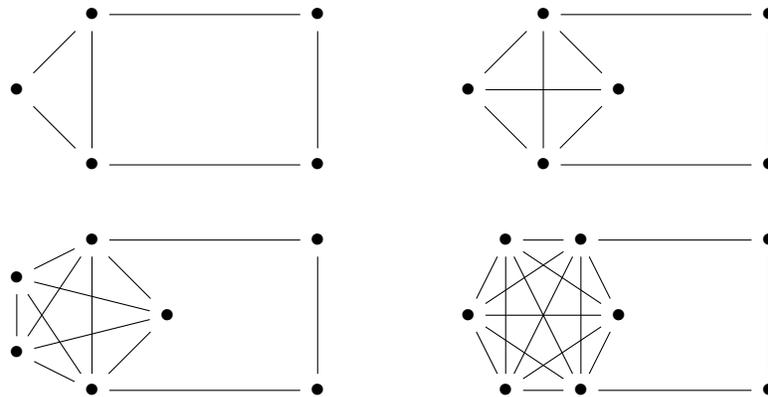


FIGURE 3. Graphs in Proposition 3

Proposition 3.2. Let $k \geq 3$. The graph $\Gamma_{k,3}$ is not the prime character degree graph of any solvable group.

Proof. We proceed by induction on k . As our base case, we will show that $\Gamma_{3,3}$ does not occur as the prime character degree graph of any solvable group. We label the vertices a_i and b_i for $1 \leq i \leq 3$, where the a 's and b 's each form a complete graph, with edges between a_i and b_i . For the sake of contradiction, suppose that the solvable group G is a counterexample with $|G|$ minimal such that $\Delta(G) = \Gamma_{3,3}$.

We will show that a_1 is admissible. Removing the edge between a_1 and a_i (with $1 \leq i \leq 3$ and $i \neq 1$) violates Pálffy's condition with a_1 , a_i , and b_j , where $1 \leq j \leq 3$, $j \neq 1$, and $j \neq i$. Removing the edge between a_1 and b_1 reduces to a graph of diameter three. This graph cannot occur since $|\rho_3| = 2$, which contradicts Proposition 2.8(i). Removing the vertex a_1 with all incident edges yields the graph $\Gamma_{3,2}$, which does not occur by Proposition 3. Thus, a_1 is admissible. By symmetry, we note that every remaining vertex is also admissible. Hence, $\Gamma_{3,3}$ cannot occur by Lemma 2.3.

Next we consider $k \geq 4$. As the inductive hypothesis, we suppose that $\Gamma_{k-1,3}$ does not occur as the prime character degree graph of any solvable group. We now proceed to the inductive step. In order to see that $\Gamma_{k,3}$ also does not occur as the prime character degree graph of any solvable group, one again shows that every vertex is admissible. We start by labeling the vertices a_i and b_j for $1 \leq i \leq k$ and $1 \leq j \leq 3$, where the a 's and b 's each form a complete graph, with edges between a_j and b_j . For the sake of contradiction, suppose that the solvable group G is a counterexample with $|G|$ minimal such that $\Delta(G) = \Gamma_{k,3}$.

First we consider the vertex a_1 . Removing a_1 and all incident edges yields a graph of diameter three such that $|\rho_3| = 2$, which contradicts Proposition 2.8(i). Removing the edge between a_1 and a_i , where $2 \leq i \leq k$, violates Pálffy's condition with a_1, a_i , and b_j , where $1 \leq j \leq 3, j \neq 1$, and $j \neq i$. Removing the edge between a_1 and b_1 again gives a graph of diameter three, which once again contradicts Proposition 2.8(i). Thus, a_1 is admissible, and by symmetry so too are a_2 and a_3 .

Next we consider the vertex a_k . Removing a_k and all incident edges yields the graph $\Gamma_{k-1,3}$, which we are assuming does not occur as the prime character degree graph of any solvable group by our inductive hypothesis. Removing the edge between a_k and a_i for $1 \leq i \leq k - 1$ again violates Pálffy's condition with a_k, a_i , and b_j , where $1 \leq j \leq 3$ and $j \neq i$. Thus, a_k is admissible. By symmetry, we also get that a_i is admissible for $4 \leq i \leq k - 1$.

Finally we consider the vertex b_1 . Removing b_1 and all incident edges gives us the graph $\Gamma_{k,2}$, which does not occur as the prime character degree graph of any solvable group by way of Proposition 3. Removing the edge between b_1 and b_j for $2 \leq j \leq 3$ violates Pálffy's condition with b_1, b_j , and a_k . Removing the edge between b_1 and a_1 was already considered above and shown to result in a graph which cannot occur. Thus, b_1 is admissible, and by symmetry we now have that b_2 and b_3 are admissible as well.

Since all vertices are admissible, we then invoke Lemma 2.3 to get our desired result. □

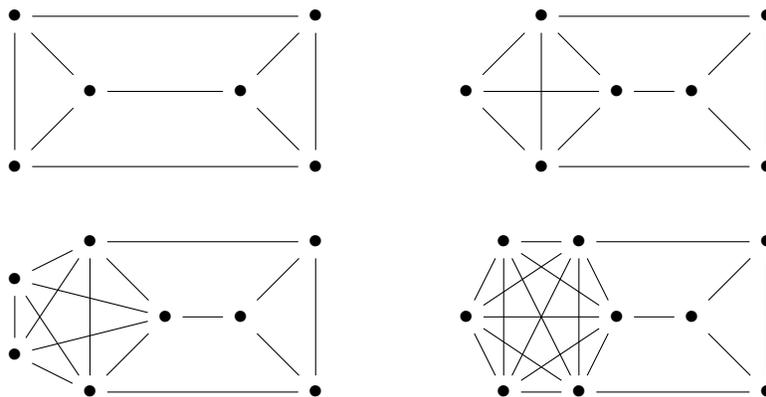


FIGURE 4. Graphs in Proposition 3.2

Proposition 3.3. *Let $k \geq t \geq 4$. The graph $\Gamma_{k,t}$ is not the prime character degree graph of any solvable group.*

Proof. We proceed by induction on both k and t . For our base case, we will show that $\Gamma_{k,4}$ does not occur as the prime character degree graph of any solvable group. To do this, one must first consider $\Gamma_{4,4}$ and then induct on k . To see that $\Gamma_{4,4}$ does not occur, we start by labeling the vertices a_i and b_i for $1 \leq i \leq 4$, where the a 's and b 's each form a complete graph, with edges between a_i and b_i . For the sake of contradiction, suppose that the solvable group G is a counterexample with $|G|$ minimal such that $\Delta(G) = \Gamma_{4,4}$.

We will show that a_1 is admissible. Removing the edge between a_1 and a_i (with $1 \leq i \leq 4$ and $i \neq 1$) violates Pálffy's condition with a_1, a_i , and b_j , where $1 \leq j \leq 4, j \neq 1$, and $j \neq i$. Removing the edge between a_1 and b_1 yields a graph of diameter three. Notice that $|\rho_1 \cup \rho_2| = 4$ and $|\rho_3 \cup \rho_4| = 4$, which contradicts Proposition 2.8(iii). Removing the vertex a_1 and all incident edges yields the graph $\Gamma_{4,3}$, which does not occur by Proposition 3.2. Thus a_1 is admissible. By symmetry, notice that every remaining vertex is also admissible. Hence $\Gamma_{4,4}$ cannot occur by Lemma 2.3.

Next we consider $k \geq 5$. As the inductive hypothesis, we assume that $\Gamma_{k-1,4}$ does not occur as the prime character degree graph of any solvable group. To see that $\Gamma_{k,4}$ does not occur as the prime character degree graph of any solvable group, we start by labeling the vertices a_i and b_j for $1 \leq i \leq k$ and $1 \leq j \leq 4$, where the a 's and b 's each form a complete graph, with edges between a_j and b_j . For the sake of contradiction, suppose that the solvable group G is a counterexample with $|G|$ minimal such that $\Delta(G) = \Gamma_{k,4}$.

First we show that a_1 is admissible. Removing the edge between a_1 and a_i (where $1 \leq i \leq k$ and $i \neq 1$) violates Pálffy's condition with a_1, a_i , and b_j , where $1 \leq j \leq 4, j \neq 1$, and $j \neq i$. Removing the edge between a_1 and b_1 yields a graph of diameter three (which we denote by $\Delta(H)$ for some solvable group H). Notice that Proposition 2.8(ii) forces $\rho_3 = \{a_2, a_3, a_4\}$, and by Proposition 2.8(iv), we know that H has a normal Sylow p -subgroup for exactly one prime $p \in \rho_3$. Without loss of generality, let $p = a_2$, and call the normal subgroup P . By Lemma 2.7, we know that $\rho(H/P') = \rho(H) \setminus \{p\}$. Observe that $\Delta(H/P')$ is a connected subgraph obtained from $\Delta(H)$ by removing the vertex a_2 and all incident edges, and possibly edges between vertices adjacent to a_2 . However, in $\Delta(H/P')$, we have that $|\rho_3| = 2$, which contradicts Proposition 2.8(i). Thus the edge between a_1 and b_1 cannot be lost. Removing the vertex a_1 and all incident edges yields a graph of diameter three. By following an argument similar to the above, we can again get that $|\rho_3| = 2$, which cannot happen. Thus a_1 is admissible. By symmetry, a_2, a_3 , and a_4 are admissible as well.

Following similar arguments, one can show that b_j is admissible for $1 \leq j \leq 4$ (relying on $\Gamma_{k,3}$ from Proposition 3.2), as well as a_i for $5 \leq i \leq k$ (relying on $\Gamma_{k-1,4}$ from the above inductive hypothesis). Thus, all vertices are admissible, and we get our result from Lemma 2.3.

Finally, for $k \geq t \geq 5$, we verify that $\Gamma_{k,t}$ does not occur. The inductive hypothesis requires that $\Gamma_{k,t-1}$ does not occur as the prime character degree graph of any solvable group. We then need to

show $\Gamma_{k,t}$ does not occur, which requires induction. First we will verify that $\Gamma_{t,t}$ does not occur; we will follow a similar argument as above, mimicking the proof for $\Gamma_{4,4}$. Label the vertices a_i and b_i for $1 \leq i \leq t$, where the a 's and b 's each form a complete graph, with edges between a_i and b_i . For the sake of contradiction, suppose that the solvable group G is a counterexample with $|G|$ minimal such that $\Delta(G) = \Gamma_{t,t}$.

We start by considering the vertex a_1 . Removing the edge between a_1 and a_i (with $1 \leq i \leq t$ and $i \neq 1$) violates Pálffy's condition with a_1 , a_i , and b_j , where $1 \leq j \leq t$, $j \neq 1$, and $j \neq i$. Removing the edge between a_1 and b_1 yields a graph of diameter three. Notice that $|\rho_1 \cup \rho_2| = t$ and $|\rho_3 \cup \rho_4| = t$, which contradicts Proposition 2.8(iii). Removing the vertex a_1 and all incident edges yields the graph $\Gamma_{t,t-1}$, which does not occur by the inductive hypothesis. Thus a_1 is admissible. By symmetry, notice that every remaining vertex is also admissible. Hence $\Gamma_{t,t}$ cannot occur by Lemma 2.3.

Then we suppose that for $k > t \geq 5$, we have that $\Gamma_{k-1,t}$ does not occur as the prime character degree graph of any solvable group. Showing $\Gamma_{k,t}$ does not occur employs the same argument as for $\Gamma_{k,4}$, using Proposition 2.8(iv) and Lemma 2.7 multiple times before getting a contradiction with Proposition 2.8(i). To start, we label the vertices a_i and b_j for $1 \leq i \leq k$ and $1 \leq j \leq t$, where the a 's and b 's each form a complete graph, with edges between a_j and b_j . For the sake of contradiction, suppose that the solvable group G is a counterexample with $|G|$ minimal such that $\Delta(G) = \Gamma_{k,t}$.

We show that a_1 is admissible. Removing the edge between a_1 and a_i (where $1 \leq i \leq k$ and $i \neq 1$) violates Pálffy's condition with a_1 , a_i , and b_j , where $1 \leq j \leq t$, $j \neq 1$, and $j \neq i$. Removing the edge between a_1 and b_1 yields a graph of diameter three (which we denote by $\Delta(H)$ for some solvable group H). Notice that Proposition 2.8(ii) forces $\rho_3 = \{a_2, a_3, \dots, a_t\}$, and by Proposition 2.8(iv), we know that H has a normal Sylow p_1 -subgroup for exactly one prime $p_1 \in \rho_3$. Without loss of generality, let $p_1 = a_2$, and call the normal subgroup P_1 . By Lemma 2.7, we know that $\rho(H/P'_1) = \rho(H) \setminus \{p_1\}$. Observe that $\Delta(H/P'_1)$ is a connected subgraph obtained from $\Delta(H)$ by removing the vertex a_2 and all incident edges, and possibly edges between vertices adjacent to a_2 . However, notice that $\Delta(H/P'_1)$ is also of diameter three. For this graph, Proposition 2.8(ii) forces $\rho_3 = \{a_3, \dots, a_t\}$, and by Proposition 2.8(iv) we know that H/P'_1 has a normal Sylow p_2 -subgroup for exactly one prime $p_2 \in \rho_3$. Without loss of generality, let $p_2 = a_3$, and call the normal subgroup P_2 . Observe one can show that $P_1P_2 = P_1 \times P_2$ is normal in H . Notice this implies that P_2 is also a normal Sylow p -subgroup of H , where $p \in \rho_3$. This contradicts Proposition 2.8(iv), since we now have two such subgroups: P_1 and P_2 . Finally, removing the vertex a_1 and all incident edges yields a graph of diameter three. By following an argument similar to the above, we can again get that $|\rho_3| = 2$, which cannot happen. Thus a_1 is admissible. By symmetry, a_j is admissible for all $1 \leq j \leq t$.

One can show that b_j is admissible, which relies on the graph $\Gamma_{k,t-1}$ from the inductive hypothesis. Furthermore, showing a_i is admissible ($t+1 \leq i \leq k$) relies on the graph $\Gamma_{k-1,t}$, which we are supposing does not occur. Since all vertices are admissible, our result follows by way of Lemma 2.3. \square

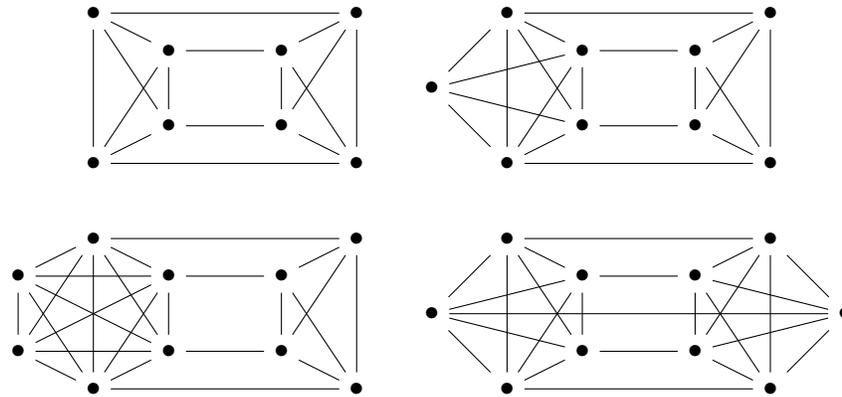


FIGURE 5. Graphs in Proposition 3.3

As consequence of Propositions 3.1, 3, 3.2, and 3.3, our Main Theorem is proved. We state it again below:

Theorem 3.4. *The graph $\Gamma_{k,t}$ occurs as the prime character degree graph of a solvable group precisely when $t = 1$ or $k = t = 2$. Otherwise $\Gamma_{k,t}$ does not occur as the prime character degree graph of any solvable group.*

Corollary 3.5. *Let $k \geq t \geq 2$. No connected proper subgraph of $\Gamma_{k,t}$ with the same vertex set is the prime character degree graph of any solvable group.*

The case for the disconnected subgraph can also be considered. The group structure of two connected components is fully outlined in [4].

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