www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. x No. x (201x), pp. xx-xx.

www.ui.ac.ir

4-QUASINORMAL SUBGROUPS OF PRIME ORDER

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Communicated by Patrizia Longobardi

Abstract. Generalizing the concept of quasinormality, a subgroup H of a group G is said to be 4-quasinormal in G if, for all cyclic subgroups K of G, $\langle H, K\rangle = HKHK$. An intermediate concept would be 3-quasinormality, but in finite p-groups - our main concern - this is equivalent to quasinormality. Quasinormal subgroups have many interesting properties and it has been shown that some of them can be extended to 4-quasinormal subgroups, particularly in finite p-groups. However, even in the smallest case, when H is a 4-quasinormal subgroup of order p in a finite p-group G, precisely how H is embedded in G is not immediately obvious. Here we consider one of these questions regarding the commutator subgroup $[H, G]$.

1. Introduction

A subgroup H of a group G is said to be 4-*quasinormal in* G if $\langle H, K\rangle = HKHK$ for every cyclic subgroup K of G and we write

$$H\ qn_4\ G.$$

In [1] several properties of cyclic 4-quasinormal subgroups of finite p-groups were discovered. In particular if H is such a subgroup of G and K is also a cyclic subgroup of G, then

$$\langle H, K\rangle \textit{ is a regular group provided } p \geqslant 5.$$

Using results from [1], the precise structure of $\langle H, K\rangle$ was described in [6] as follows:-

MSC(2010): Primary: 20E07, 20E15.
Keywords: Finite group, Sylow subgroup, abnormal subgroup, seminormal subgroup.
Received: 17 October 2018, Accepted: 09 December 2018.
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http://dx.doi.org/10.22108/ijgt.2018.113482.1510

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Lemma 1.1. *Let* $G=\langle h,k\rangle$ *be a finite* p*-group* $(p\geqslant 5)$ *with* $H=\langle h\rangle$ *cyclic of order* p^m *and* $K=\langle k\rangle$ *cyclic of order* p^n *and suppose that* $H\ qn_4\ G$*. Let* $C=\langle[h,k]\rangle$ *of order* p^ℓ*. Then* (i) $G=HCK$; (ii) $m\geqslant \ell$ *and* $n\geqslant \ell$; *and* (iii) *both* H^{p^ℓ} *and* K^{p^ℓ} *lie in* $\zeta_1(G)$*, the centre of* G*.*

Of course when $m=n=1$, then G is either elementary abelian or E_p, *the extraspecial group of order* p^3 *and exponent* p. Also when $m=1$, then

$$[H,K^p]=1 \tag{1.1}$$

and

$$[H,K,H] \text{ and } [H,K,K] \text{ lie in } K^{p^{n-1}}. \tag{1.2}$$

We shall use these facts repeatedly throughout the rest of the paper. Working with groups of exponent p, the situation is even simpler:-

Lemma 1.2. *Let* $G=\langle H,K\rangle$ *be a finite* p*-group with* $H\ qn_4\ G$, $|H|=p$ *and* K *a subgroup of exponent* p*. Then* H^G *is elementary abelian.*

Let $H=\langle h\rangle$ and $k\in K$. Then $\langle h,k\rangle$ is elementary abelian, E_p or D_8 (the dihedral group of order 8). Therefore h commutes with $[h,k]$ and hence also with h^k. Thus $H^G=H^K$ is elementary abelian.

When p is odd, then we even have G of exponent p. However, it is easy to see that Lemma 1.2 can fail when K has exponent $\geqslant p^2$.

Example 1.3. *Let* $C=\langle c\rangle$ *be a cyclic group of prime order* p $(\geqslant 3)$*,* $K=\langle k\rangle$ *cyclic of order* p^2 *and form the direct product* $CK=\langle c\rangle\times\langle k\rangle$*. Then define* $G=(CK)\rtimes\langle h\rangle$ *where* $H=\langle h\rangle$ *is cyclic of order* p *and* $c^h=ck^p$*,* $k^h=ck$*. Clearly* G *has order* p^4 *and class 3 and* $H\ qn_4\ G$*. But*

$$H^G=\langle h,c,k^p\rangle\cong E_p.$$

However, for an arbitrary subgroup K*, the situation doesn't become more complicated provided* $p\geqslant 5$*, as we now show.*

Lemma 1.4. *Let* G *be a finite* p*-group* $(p\geqslant 5)$ *with a subgroup* H *of order* p *such that* $H\ qn_4\ G$*. Then* H^G *has class* $\leqslant 2$ *and exponent* p*.*

Let $H=\langle h\rangle$. By Lemma 1.1, for each $g\in G$, $[h,g]$ has order at most p and g^p centralizes H. Thus G^p centralizes H and hence also H^G. Moreover by Lemma 1.2, H^G is elementary abelian modulo G^p. Therefore $H^G\cap G^p\leqslant\zeta_1(H^G)$ and so H^G has class $\leqslant 2$. Then H^G is regular and hence has exponent p (see [3, 10.2 Satz and 10.5 Hauptsatz]). In the light of the Example above and Lemma 1.4, a natural question is the following:-

(1.3) *With the hypotheses of Lemma* 1.4*, is* $[H,G]$ *always abelian?*

By Lemma 1.2, the answer is 'yes' when G has exponent p. We shall show throughout the rest of the paper that the answer remains 'yes' in many cases. Of course when $G=\langle H,K\rangle$ with H of order p, then $[H,G]=[H,K]$.

http://dx.doi.org/10.22108/ijgt.2018.113482.1510

Our notation is standard. Cyclic groups of order p are denoted by C_p and E_p denotes the extraspecial group of order p^3 and exponent p. For each integer n, $\gamma_n(G)$ is the n-th term of the lower central series of G and $\zeta_n(G)$ is the n-th term of the upper central series of G. Also the subgroup of G generated by the elements of order p is denoted by $\Omega_1(G)$.

2. Cases where $[H,G]$ is abelian

We begin by considering the case $G = \langle H, K\rangle$ where $K = C_p \times C_p$.

Proposition 2.1. *Let $G{=}\langle H, K\rangle$ be a finite p-group (where $p \geqslant 5$), $H \cong C_p$, $K \cong C_p \times C_p$ and $H\ qn_4\ G$. Then G has class $\leqslant 2$.*

Clearly $|G| \leqslant p^5$ and so G has class $\leqslant 4$. Therefore G is regular and hence has exponent p. We may assume that $H = \langle h\rangle \nleqslant G'$ and that $|G/G'| = p^3$, otherwise $G = \langle h, k\rangle$ with $|k| = p$ and then G has class at most 2.

Thus G has class at most 3. If G has class 3, then $\gamma_2(G) \cong C_p \times C_p$ and $\gamma_3(G) \cong C_p$. Then it follows that $G/\gamma_3(G) \cong E_p \times C_p$ (see [5, p. 196]) and we may assume that $K = \langle k_1\rangle \times \langle k_2\rangle$ with

$$\gamma_2(G) = \langle [h, k_1]\rangle \times \langle [h, k_2]\rangle \text{ and } \gamma_3(G) = \langle [h, k_2]\rangle.$$

But then by the Hall-Witt identity (see [4, Section 2.1, (3)])

$$[k_1, h, k_2] = 1.$$

Therefore k_2 centralizes $\gamma_2(G)$ and so G has class $\leqslant 2$. Considering the case where $K \cong E_p$ is surely the obvious move from here.

Proposition 2.2. *Let $G{=}\langle H, K\rangle$ be a finite p-group (where $p \geqslant 5$), $H \cong C_p$, $K \cong E_p$ and $H\ qn_4\ G$. Then G has class $\leqslant 3$.*

Let $K = \langle k_1, k_2\rangle$, $H = \langle h\rangle$ and $c = [k_1, k_2]$. For any $k \in K$, $\langle h, c, k\rangle$ has class $\leqslant 2$, by Proposition 2.1. Therefore $[h, c] \in \zeta_1(G)$. So $c \in \zeta_2(G)$ and thus

$$c \text{ commutes with } \gamma_2(G) \tag{2.1}$$

(see [4, Lemma 2.21]). Also $\langle [h, c], c\rangle \lhd G$ and

$$G/\langle [h, c], c\rangle \text{ has class} \leqslant 2, \tag{2.2}$$

again by Proposition 2.1. Thus G has class at most 4. Moreover by Lemma 1.2

$$H^G \text{ is elementary abelian;} \tag{2.3}$$

and by (5)

$$\gamma_3(G) \text{ commutes with } K. \tag{2.4}$$

Since $\langle h\rangle\ qn_4\ \langle h, k_1k_2\rangle$, using (4), (6) and (7) we have

$$1 = [h, k_1k_2, k_1k_2] = [[h, k_2][h, k_1]^{k_2}, k_1k_2] = [h, k_2, k_1k_2][h, k_1, k_1k_2].$$

Also $[h, k_2, k_1k_2] = [h, k_2, k_1]$ by (7) and $[h, k_1, k_1k_2] = [h, k_1, k_2]$. Therefore

$$[h, k_2, k_1] = [h, k_1, k_2]^{-1}. \tag{2.5}$$

Now by the Hall-Witt identity

$$[h, k_1, k_2][k_1^{-1}, k_2^{-1}, h][k_2, h^{-1}, k_1^{-1}] = 1$$

using (6). Thus $[h, k_1, k_2][c, h] = [h, k_2, k_1]$ (using (7)) and substituting in (8) we have

$$[h, k_1, k_2]^2 = [h, c]. \tag{2.6}$$

Therefore G has class $\leqslant 3$. Of course, by Lemma 1.2, with the hypotheses of Proposition 2.1 or Proposition 2.2, $[H, G]$ *is always abelian.* Also when $G = \langle H, K\rangle$ is a finite p-group ($p \geqslant 5$), with $|H| = p$, H qn_4 G and K is abelian, then Lemma 1.1 shows that $K^p \leqslant \zeta_1(G)$; and G/K^p has class $\leqslant 2$, by Proposition 2.1. Therefore G has class $\leqslant 3$ and again

$$[H, G] \text{ is abelian.} \tag{2.7}$$

We proceed now to show that the answer to Question 3 is 'yes' for significantly larger classes of subgroups K. We shall need the following consequence of Proposition 2.2.

Lemma 2.3. *With the hypotheses of Proposition 2.2, suppose $[h, k_1] = 1$. Then G has class $\leqslant 2$.*

By (9), $[h, c] = 1$. Also by (7), $[h, k_2] \in \zeta_1(G)$. Since c now belongs to $\zeta_1(G)$ and $G/\langle c, [h, k_2]\rangle$ is abelian, the Lemma follows. Our first main result is the following.

Theorem 2.4. *Let $H = \langle h\rangle$ qn_4 $G = \langle h, k_1, k_2\rangle$, a finite p-group ($p \geqslant 5$) with $|h| = |k_1| = p$. Then $[[h, k_1], [h, k_2]] = 1$.*

Assume, for a contradiction, that the result is false and let G be a counterexample of minimal order. Then there is a unique central subgroup $\langle c\rangle$ of order p in G and

$$[[h, k_1], [h, k_2]] = c.$$

Let $|k_2| = p^n$ and $x = k_1[h, k_2]$. Since $\langle h^{k_2}, k_1\rangle$ is either elementary abelian or E_p, it follows from Propositions 2.1 and 2.2 that $\langle h, h^{k_2}, k_1\rangle$ has class $\leqslant 3$. Therefore $\langle h, h^{k_2}, k_1\rangle$ is regular and so

$$x^p = 1. \tag{2.8}$$

Let $k_3 = [h, [h, k_2]]$. If $n = 1$, then $k_3 = 1$, and if $n \geqslant 2$, then $k_3 = k_2^{\alpha p^{n-1}}$ where $0 \leqslant \alpha < p$ (by (2)). Also

$$[h, x] = [h, k_1[h, k_2]] = k_3[h, k_1]c.$$

Therefore by (11)

$$1 = [h, x, x] = [k_3[h, k_1]c, k_1[h, k_2]] = [k_3[h, k_1], k_1[h, k_2]]$$
$$= [k_3[h, k_1], [h, k_2]][k_3[h, k_1], k_1]^{[h,k_2]} = c[k_3, k_1]$$

using 1.1. Thus

$$[k_1, k_3] = c \tag{2.9}$$

and so

$$\langle k_3 \rangle = \Omega_1 \langle k_2 \rangle \text{ with } |k_2| \geqslant p^2. \tag{2.10}$$

Now let $y = k_2[h, k_1]$. Then

$$[h, y] = [h, k_2[h, k_1]] = [h, k_2]^{[h,k_1]} = [h, k_2]c^{-1}.$$

Therefore

$$[h, y, y] = [h, k_2, y] = [h, k_2, k_2[h, k_1]] = c^{-1}[h, k_2, k_2] \tag{2.11}$$

by 1.1 and 1.2. Since $[h, h^{k_1}] = 1$, we also have, by 1.1 and Lemma 2.3,

$$\langle h, h^{k_1}, k_2 \rangle / \langle k_2^p \rangle \text{ has class } \leqslant 2.$$

Also $y \in \langle h, h^{k_1}, k_2 \rangle$ and so by Lemma 1.4

$$[h, y, y] \in \Omega_1 \langle k_2 \rangle.$$

Therefore (14) implies that $\langle c \rangle = \Omega_1 \langle k_2 \rangle = \langle k_3 \rangle$ (by (13)), contradicting (12). The Theorem follows.
Another case where we get a positive result is when $K = \langle k_1, k_2 \rangle$ is *powerful*, i.e. K/K^p is abelian.

Theorem 2.5. *Let $H = \langle h \rangle$ qn_4 $G = \langle h, k_1, k_2 \rangle$, a finite p-group ($p \geqslant 5$) with $|H| = p$ and suppose that $K = \langle k_1, k_2 \rangle$ is powerful. Then $[H, K]$ is abelian.*

By [2, 2.8 Corollary], $K = \langle k_1 \rangle \langle k_2 \rangle$. Then by [3, 11.5 Satz], K is metacyclic. Therefore we may assume that $\langle k_1 \rangle \lhd K$. We shall prove first that

$$[[h, k_1], [h, k_2]] = 1. \tag{2.12}$$

By Theorem 2.4, we may also assume that $|k_1| = m \geqslant 2$. Then since $\Omega_1 \langle k_1 \rangle \leqslant \zeta_1(G)$ (by 1.1), arguing by induction on $|G|$, we may assume, for a contradiction, that

$$[[h, k_1], [h, k_2]] = z,$$

where $\langle z \rangle = \Omega_1 \langle k_1 \rangle$. Then

$$[h, k_1]^{k_2} = [h[h, k_2], k_1[k_1, k_2]] = [h[h, k_2], k_1] = [h, k_1]z[h, k_2, k_1], \tag{2.13}$$

using (2). Also by (2)

$$[h, k_1, h] = k_1^{\lambda p^{m-1}}, \quad 0 \leqslant \lambda < p. \tag{2.14}$$

Therefore $[h, k_1, h] = [h, k_1, h]^{k_2} = [[h, k_1]^{k_2}, h[h, k_2]]$

$$= [[h, k_1]z[h, k_2, k_1], h[h, k_2]] \text{ (by (16))}$$

$$= [[h, k_1], h[h, k_2]] \text{ (by 1.1 and Proposition 2.1).}$$

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Thus, by 1.1 and 2.14, $[h, k_1, h] = z[h, k_1, h]$, i.e. $z = 1$, a contradiction, proving 2.12.

Finally to prove the Theorem, let $k_3 = k_1^\alpha k_2^\beta$ and $k_4 = k_1^\gamma k_2^\delta$. Then

$$[[h, k_3], [h, k_4]] = [[h, k_1^\alpha k_2^\beta], [h, k_1^\gamma k_2^\delta]] =$$
$$[[h, k_2^\beta][h, k_1^\alpha]^{k_2^\beta}, [h, k_2^\delta][h, k_1^\gamma]^{k_2^\delta}] = [([h, k_2^\beta][h, k_1^\alpha])^{k_2^\beta}, ([h, k_2^\delta][h, k_1^\gamma])^{k_2^\delta}]$$

using 1.1 and 1.2. Therefore $[[h, k_3], [h, k_4]] =$

$$[[h, k_2^\beta][h, k_1^\alpha][[h, k_2^\beta][h, k_1^\alpha], k_2^\beta], [h, k_2^\delta][h, k_1^\gamma][[h, k_2^\delta][h, k_1^\gamma], k_2^\delta]].$$

Therefore, using 1.1 and Proposition 2.1,

$$[[h, k_3], [h, k_4]] = [[h, k_2^\beta][h, k_1^\alpha], [h, k_2^\delta][h, k_1^\gamma]] = 1,$$

since $[h, k_1^\alpha] \equiv [h, k_1]^\alpha \bmod \langle k_1^p \rangle$ etc. Thus the Theorem follows. In order to find an answer to (3), the next case to consider should be where $G = \langle H, K\rangle$ with $K = \langle k_1, k_2\rangle$ of class 2. We conclude by showing that the answer is still 'yes' in this case if, in addition, $|[k_1, k_2]| = p$.

Theorem 2.6. *Let $H = \langle h\rangle$ qn_4 $G = \langle h, k_1, k_2\rangle$, a finite p-group $(p \geqslant 5)$ with $|H| = p$, $K = \langle k_1, k_2\rangle$ and $|K'| = p$. Then $[H, K]$ is abelian.*

By (10) it is sufficient to show that $[[h, k_1], [h, k_2]] = 1$. Let $|k_1| = p^m$ and $|k_2| = p^n$. It follows from Theorem 2.4 that we may assume $m \geqslant n \geqslant 2$. Suppose, for a contradiction, that the Theorem is false and let G be a counterexample with $m + n$ minimal. Since k_1^p and k_2^p are in $\zeta_1(G)$, we must have $\Omega_1\langle k_1\rangle = \Omega_1\langle k_2\rangle$. Then it is easy to see that we may assume

$$k_1^{p^{m-1}} = k_2^{p^{n-1}} \text{ and } \langle[[h, k_1], [h, k_2]]\rangle = \Omega_1\langle k_1\rangle. \tag{2.15}$$

Cleartly K is regular and by [3, 10.8 Satz],

$$(k_1^{-p^{m-n}} k_2)^{p^{n-1}} = k_1^{-p^{m-1}} k_2^{p^{n-1}} = 1.$$

Thus by choice of G

$$[[h, k_1^{-p^{m-n}} k_2], [h, k_1]] = 1. \tag{2.16}$$

However, if $m > n$, then $[h, k_1^{p^{m-n}}] = 1$ and 2.16 contradicts 2.15. Therefore we must have $m = n$ and so

$$1 = [[h, k_1^{-1}k_2], [h, k_1]] = [[h, k_2][h, k_1^{-1}]^{k_2}, [h, k_1]]$$
$$= [[h, k_2][h, k_1^{-1}][h, k_1^{-1}, k_2], [h, k_1]].$$

But G/K^p has class at most 3 (by Proposition 2.1) and $K^p \leqslant \zeta_1(G)$. Therefore G has class $\leqslant 4$. Also $[h, k_1^{-1}, k_2] \in \gamma_3(G)$ and so commutes with $\gamma_2(G)$ (see [4, Lemma 2.21 (ii)]). Thus

$$1 = [[h, k_2][h, k_1^{-1}], [h, k_1]] = [[h, k_2], [h, k_1]]$$

by 1.1, contradicting 2.15. The Theorem follows.

http://dx.doi.org/10.22108/ijgt.2018.113482.1510

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http://dx.doi.org/10.22108/ijgt.2018.113482.1510