www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. x No. x (201x), pp. xx-xx.

www.ui.ac.ir

THE NUMBER OF MAXIMAL SUBGROUPS AND PROBABILISTIC GENERATION OF FINITE GROUPS

ADOLFO BALLESTER-BOLINCHES, RAMÓN ESTEBAN-ROMERO*, PAZ JIMÉNEZ-SERAL AND HANGYANG MENG

Communicated by Patrizia Longobardi

Abstract. In this survey we present some significant bounds for the number of maximal subgroups of a given index of a finite group. As a consequence, new bounds for the number of random generators needed to generate a finite d-generated group with high probability which are significantly tighter than the ones obtained in the paper of Jaikin-Zapirain and Pyber (Random generation of finite and profinite groups and group enumeration, *Ann. Math.*, **183**:769–814, 2011) are obtained. The results of Jaikin-Zapirain and Pyber, as well as other results of Lubotzky, Detomi, and Lucchini, appear as particular cases of our theorems.

1. Introduction

All groups in this survey will be finite. Let G be a d-generated group. The motivating question of this paper is: How many elements one should expect to choose uniformly and randomly to generate G? Let us call this number $\varepsilon(G)$. The first reference we have of this problem is due to Netto [18, page 90], who conjectured that the probability that a randomly chosen pair of elements of $\mathrm{Alt}(n)$ generates $\mathrm{Alt}(n)$ tends to 1 as n tends to infinity, and that the probability that a randomly chosen pair of elements of $\mathrm{Sym}(n)$ generates $\mathrm{Sym}(n)$ tends to $3/4$ as n tends to infinity. Dixon proved in [10] that this conjecture is true, by showing that the proportion of generating pairs for $\mathrm{Alt}(n)$ or $\mathrm{Sym}(n)$ is greater than $1 - 2/(\ln \ln n)^2$ for sufficiently large n, where ln denotes the natural logarithm. He also

MSC(2010): Primary: 20P05; Secondary: 20E07; 20E28
Keywords: Finite group, maximal subgroup, probabilistic generation, primitive group.
Received: 10 Decenber 2018y, Accepted: 24 January 2019.
*Corresponding author.

http://dx.doi.org/10.22108/ijgt.2019.114469.1521

.

conjectured that the proportion of pairs (x, y) of elements of a simple group G that generate G also tends to infinity as $|G|$ tends to infinity. Kantor and Lubotzky [14] and Liebeck and Shalev [15] proved the validity of Dixon's conjecture: if G is an almost simple group with socle S, the probability that a pair of elements of G generates a subgroup containing S tends to 1 as $|G|$ tends to infinity.

Pomerance [20] analysed the expected number $\varepsilon(G)$ of random generators of an abelian group G and proved that $\varepsilon(G) \leq \mathrm{d}(G) + \sigma$, where $\mathrm{d}(G)$ denotes the minimum cardinal of a generating set of G and $\sigma = 2.118456563\cdots$ is obtained from the Riemann zeta function. For abelian groups, this bound is optimal. Pak [19], motivated by potential applications for the product replacement algorithm, useful to obtain random elements in a finitely generated group, studied a related invariant.

Definition 1.1. *Given a group G, $\nu(G)$ denotes the least positive integer k such that the probability of generating G with k random elements is at least $1/\mathrm{e}$.*

Pak proved in [19, Theorem 2.5] that

$$\frac{1}{\mathrm{e}}\varepsilon(G) \leq \nu(G) \leq \frac{\mathrm{e}}{\mathrm{e}-1}\varepsilon(G)$$

and conjectured that $\nu(G) = \mathrm{O}(\mathrm{d}(G)\log\log|G|)$. Here the symbol log is used to denote the logarithm to the base 2, with the convention that $\log 0 = -\infty$.

Let $\{x_1, x_2, \ldots, x_k\} \subseteq G$. Then we have that $\langle x_1, x_2, \ldots, x_k\rangle \neq G$ if, and only if, there exists a maximal subgroup M of G such that $\langle x_1, x_2, \ldots, x_k\rangle \leq M$. This shows the relevance of maximal subgroups in the scope of probabilistic generation of finite groups. Moreover, if the elements x_1, $x_2, \ldots, x_k$ are chosen independently and at random with a uniform distribution and M is a maximal subgroup of G, then

$$\mathrm{Prob}(\langle x_1, x_2, \ldots, x_k\rangle \leq M) = \prod_{i=1}^{k} \mathrm{Prob}(x_i \in M) = \left(\frac{|M|}{|G|}\right)^k = \frac{1}{|G:M|^k}.$$

Therefore the number $\mathrm{m}_n(G)$ of maximal subgroups of G of a given index n is also relevant in this context. Lubotzky [16], by considering the number of chief factors in a given chief series, and Detomi and Lucchini [8], with the help of the number $\lambda(G)$ of non-Frattini chief factors in a given chief series of G and by means of their associated crowns, proved independently that Pak's conjecture is valid.

Theorem 1.2 (Lubotzky, [16, Corollaries 2.6 and 2.8])**.** *If G is a group with r chief factors in a given chief series, then*

$$\mathrm{m}_n(G) \leq r(r + n^{\mathrm{d}(G)})n^2 \leq r^2 n^{\mathrm{d}(G)+2}.$$

Furthermore,

$$\nu(G) \leq \frac{1 + \log\log|G|}{\log \mathrm{i}(G)} + \max\left(\mathrm{d}(G), \frac{\log\log|G|}{\log \mathrm{i}(G)}\right) + 2.02,$$

where $\mathrm{i}(G)$ denotes the smallest index of a proper subgroup of G.

The proof of Theorem 1.2 depends on some results of Mann and Shalev [17] about the number of maximal subgroups of a given index of a group G and on some structural results about primitive groups.

DOI: http://dx.doi.org/10.22108/ijgt.2019.114469.1521

Theorem 1.3 (Detomi and Lucchini [8, Theorem 20]). *There exists a constant c such that, for any group G, $\nu(G) \leq \lfloor \mathrm{d}(G) + c \log \lambda(G) \rfloor$ if $\lambda(G) > 1$, otherwise, $\nu(G) \leq \lfloor \mathrm{d}(G) + c \rfloor$, where $\lambda(G)$ denotes the number of non-Frattini chief factors in a given chief series of G.*

Here the symbol $\lfloor x \rfloor$ is used to denote the defect integer part of x, that is, the largest integer number n with $n \leq x$. The proof of this result depends on some results of Dalla Volta and Lucchini [6] and Dalla Volta, Lucchini, and Morini [7] about the number of generators of powers of certain diagonal-type subgroups of direct products of copies of a monolithic group.

The invariant $\nu(G)$ is related to the following invariant.

Definition 1.4. *Given a group G let*

$$\mathcal{M}(G) = \max_{n \geq 2} \log_n \mathrm{m}_n(G) = \max_{n \geq 2} \frac{\log \mathrm{m}_n(G)}{\log n}.$$

Here $\log_n x$ denotes the loarithm to the base n of x, that is, $\log_n x = \log x / \log n = \ln x / \ln n$. Theorem 1.2 depends on the following interesting result.

Theorem 1.5 (Lubotzky, [16, Proposition 1.2]).

$$\mathcal{M}(G) - 3.5 \leq \nu(G) \leq \mathcal{M}(G) + 2.02.$$

Jaikin-Zapirain and Pyber have obtained in [12] a remarkable result that gives surprisingly explicit lower and upper bounds for $\nu(G)$ with interesting applications (see [5, Theorem 3]). Their bounds depend on the following invariants.

Definition 1.6. *Let G be a group. The symbol $\mathrm{l}(G)$ denotes the least degree of a faithful transitive permutation representation of G, that is, the smallest index of a core-free subgroup of G.*

Definition 1.7. *Let G be a group and let A be a characteristically simple group. The symbol $\mathrm{rk}_A(G)$ denotes the largest number r such that G has a normal section that is the direct product of r non-Frattini chief factors of G that are isomorphic (but not necessarily G-isomorphic) to A.*

This invariant was only defined for non-abelian characteristically simple groups, but there is no problem in extending their definition to elementary abelian groups. The main result of [12] is the following one.

Theorem 1.8 ([12, Theorem 1]). *There exist two absolute constants $0 < \alpha < \beta$ such that for every group G we have*

$$\alpha \left(\mathrm{d}(G) + \max_A \left\{ \frac{\log \mathrm{rk}_A(G)}{\log \mathrm{l}(A)} \right\} \right) < \nu(G) < \beta\, \mathrm{d}(G) + \max_A \left\{ \frac{\log \mathrm{rk}_A(G)}{\log \mathrm{l}(A)} \right\},$$

where A runs through the non-abelian chief factors of G in a given chief series of G.

Here $\mathrm{rk}_n(G)$ denotes the maximum of $\mathrm{rk}_A(G)$ where A runs over the non-abelian characteristically simple groups A with $\mathrm{l}(A) \leq n$. According to private communications with the authors, the maximum on the right-hand side must be understood to be zero when G is soluble, that is, when G has no non-abelian chief factors. This result is a consequence of a stronger result.

DOI: http://dx.doi.org/10.22108/ijgt.2019.114469.1521

Theorem 1.9 ([12, Theorem 9.5]). *Let G be a d-generated group. Then*

$$\max\left\{d, \max_{n\geq 5}\frac{\log \mathrm{rk}_n(G)}{c_7 \log n} - 4\right\} \leq \nu(G) \leq cd + \max_{n\geq 5}\frac{\log \max\{1, \mathrm{rk}_n(G)\}}{\log n} + 3,$$

where c and c_7 are two absolute constants.

The lower bound depends on the following result.

Lemma 1.10 ([12, Corollary 9.3]). *Let G be a group. Then $\mathrm{m}_x(G) \geq \mathrm{rk}_n(G)/n^{c_7}$ for some $x \leq n^{c_7}$.*

There was a misprint in the original version, where the bound $\mathrm{m}_x(G) \geq \mathrm{rk}_n(G)$ was stated instead. This form of this result appears in [13] and so Theorem 1.9 is modified there to the following result.

Theorem 1.11 ([12, Theorem 9.5]). *Let G be a d-generated group. Then*

$$\max\left\{d, \max_{n\geq 5}\frac{\log \mathrm{rk}_n(G)}{c_7 \log n} - 5\right\} \leq \nu(G) \leq cd + \max_{n\geq 5}\frac{\log \max\{1, \mathrm{rk}_n(G)\}}{\log n} + 3,$$

where c and c_7 are two absolute constants.

The aims of this survey are:

(1) to give an interpretation of the invariant $\mathrm{rk}_A(G)$ for a non-abelian characteristically simple group A,
(2) to improve the bounds for $\mathrm{m}_n(G)$ and, hence, for $\nu(G)$, and
(3) to estimate the values of the constants in [12, Theorem 9.5].

These objectives have been developed in depth in [2]. We refer the reader to this paper for more details and proofs.

2. Large characteristically simple sections

Recall that a primitive group is a group with a core-free maximal subgroup. The relevance of primitive groups in the study of the group sturcture is due to the fact that if M is a maximal subgroup of a group G, then M/M_G is a core-free maximal subgroup of G/M_G and so G/M_G is primitive. A detailed study of primitive groups and the crowns associated is presented in [3, Chapter 1]. The following classical result of Baer gives a classification of the primitive groups.

Theorem 2.1 (Baer, [1], see also [3, Theorem 1.1.7]). *Let G be a primitive group and let U be a core-free maximal subgroup of G. Exactly one of the following statements holds:*

(1) $\mathrm{Soc}(G) = S$ *is a self-centralising abelian minimal normal subgroup of* G, $G = US$ *and* $U \cap S = 1$.
(2) $\mathrm{Soc}(G) = S$ *is a non-abelian minimal normal subgroup of* G, $G = US$. *In this case,* $\mathrm{C}_G(S) = 1$.
(3) $\mathrm{Soc}(G) = A \times B$, *where A and B are the two unique minimal normal subgroups of G,* $G = AU = BU$ *and* $A \cap U = B \cap U = A \cap B = 1$. *In this case,* $A = \mathrm{C}_G(B)$, $B = \mathrm{C}_G(A)$, *and* $A \cong B \cong AB \cap U$ *are non-abelian.*

DOI: http://dx.doi.org/10.22108/ijgt.2019.114469.1521

We say that a primitive group is of type 1, of type 2, or of type 3 according to whether G satisfies one of the statements 1, 2, or 3. A maximal subgroup M of G is said to be of type 1, of type 2, or of type 3 when G/M_G is a primitive group of type 1, of type 2, or of type 3, respectively. The theorem of O'Nan and Scott (see [3, Theorem 1.1.52]) gives a complete description of all primitive groups of type 2.

We have the following result.

Theorem 2.2. *Let G be a primitive group of type 2 with socle B. Then G/B has no chief factors isomorphic to B.*

With the help of Theorem 2.2 and the precrown associated to a supplemented chief factor and a maximal subgroup supplementing it, studied in [3, Section 1.2], we can prove the following result.

Theorem A ([2, Theorem B])**.** *Let A be a non-abelian chief factor of a group G and suppose that in a given chief series of G there are k chief factors isomorphic to A. Then there exist two normal subgroups C and R of G such that $R \leq C$ and C/R is isomorphic to a direct product of k minimal normal subgroups of G/R isomorphic to A. In particular, $\mathrm{rk}_A(G)$ is the number of chief factors of G isomorphic to A in a given chief series of G.*

An extension of this result to non-Frattini abelian chief factors is also presented in [2].

3. New bounds for the number of maximal subgroups of a given index

We will improve the lower bound in Theorem 1.9 in two ways: by showing that the constant can be taken to be $c_7 = 2$, and by proving that the result of the original version of the paper is correct and can even be improved. By the O'Nan-Scott theorem, we see that it is enough to prove the result for almost simple groups. We prove the following result.

Theorem 3.1. *Every almost simple group R with socle isomorphic to the simple group S possesses a conjugacy class of core-free maximal subgroups of index $\mathrm{l}(S)$ or a conjugacy class of core-free maximal subgroups with a fixed index $v_s \leq \mathrm{l}(S)^2$, depending only on S.*

Therefore, c_7 can be taken as 2. This result depends on an exhaustive analysis of the maximal subgroups of the smallest index of the simple groups and the associated almost simple groups. As a consequence, we obtain the following bound.

Theorem 3.2. *Let G be a d-generated group. Then*

$$\nu(G) \geq \max\left\{d, \max_A \frac{\log \mathrm{rk}_A(G)}{2\log \mathrm{l}(A)} - 2.63\right\}.$$

These bounds can be improved in groups whose non-abelian composition factors are in a certain class of groups for which every almost simple group with socle isomorphic to S has a conjugacy class of core-free maximal subgroups of index $\mathrm{l}(S)$. For such groups, the constant c_7 in the denominator can be taken to be 1 and the term -2.63 can be replaced by -2.5.

DOI: http://dx.doi.org/10.22108/ijgt.2019.114469.1521

We note that the lower bound is useful only when $\mathrm{rk}_A(G)$ is large for some A. For instance, if $S = \mathrm{Alt}(5)$, which belongs to the abovementioned class, and $G = S^r$, with $r \in \mathbb{N}$, according to a result of Wiegold [22] we need $r = 60^4 = 12\,960\,000$ copies of S to obtain useful bounds, greater than the number of generators $4+2 = 6$. The result of Jaikin-Zapirain and Pyber, even if we accept that $c_7 = 2$, requires 60^{26} copies of S in order to obtain some useful results ($\nu(G) \geq 29$).

Our upper bound for the number of maximal subgroups of index n of a group will depend on the following invariants. The first one concerns the maximal subgroups of type 1.

Definition 3.3. *Let G be a group and let $n \in \mathbb{N}$, $n > 1$. We denote by $\mathrm{cr}_n^{\mathfrak{A}}(G)$ the number of crowns associated to complemented abelian chief factors of order n of G, that is, the number of G-isomorphism classes of complemented abelian chief factors of G of order n.*

It is clear that $\mathrm{cr}_n^{\mathfrak{A}}(G) = 0$ unless n is a power of a prime.

The second invariant is useful to obtain bounds for the number of maximal subgroups of type 2.

Definition 3.4. *Let $n \in \mathbb{N}$. The symbol $\mathrm{rk}\, s_n(G)$ denotes the number of non-abelian chief factors A in a given chief series of G such that the associated primitive group $G/\mathrm{C}_G(A)$ has a core-free maximal subgroup of index n.*

The following invariants concern maximal subgroups of type 3.

Definitions 3.5. *Let $n \in \mathbb{N}$.*

(1) *The symbol $\mathrm{rk}\, o_n(G)$ denotes the number of non-abelian chief factors A in a given chief series of G such that $|A| = n$.*
(2) *The symbol $\mathrm{rk}\, om_n(G)$ denotes the maximum of the numbers $\mathrm{rk}_A(G)$ for A running over the isomorphism types of non-abelian chief factors of G with $|A| = n$.*

We must note that the non-abelian chief factors A of G of order n fall into at most two isomorphism classes. It follows that $\mathrm{rk}\, o_n(G) \leq 2\,\mathrm{rk}\, om_n(G)$. Note also that the invariants $\mathrm{rk}\, s_n(G)$, $\mathrm{rk}\, o_n(G)$, and $\mathrm{rk}\, om_n(G)$ are naturally related to $\mathrm{rk}_n(G)$.

We can use these invariants to give new bounds for $\nu(G)$.

Theorem B ([2, Theorem A])**.** *Let G be a d-generated non-trivial group. Then*

$$\max\left\{d, \max_A \frac{\log \mathrm{rk}_A(G)}{2\log \mathrm{l}(A)} - 2.63\right\} \leq \nu(G) \leq \eta(G),$$

where in the maximum on the left hand side, A runs over the isomorphism classes of non-abelian chief factors in a given chief series of G and $\eta(G)$ is a function bounded by a linear combination of d and the maxima of $\log_n \mathrm{cr}_n^{\mathfrak{A}}(G)$, $\log_n \mathrm{rk}\, s_n(G)$, $\log_n \mathrm{rk}\, o_n(G)$, and $\log_n \mathrm{rk}\, om_n(G)$.

The determination of our function η, presented in Theorem 3.12, follows from estimating the number of maximal subgroups of each type and a given index n. For the maximal subgroups of type 1, we have the following result.

DOI: http://dx.doi.org/10.22108/ijgt.2019.114469.1521

Theorem 3.6. *Let U be a maximal subgroup of type 1 of a d-generated group G and let $n = |G : U|$. Then the number of maximal subgroups M of G such that $\mathrm{Soc}(G/M_G)$ is G-isomorphic to $\mathrm{Soc}(G/U_G)$ is less than or equal to*

$$\frac{n^d - n|H^1(G/C, A)|}{q-1},$$

where $A = C/U_G$ is the unique minimal normal subgroup of G/U_G and $q = |\mathrm{End}_{G/C}(A)|$. In particular, this number is less than n^d.

As a consequence, we obtain the following information about the number of type 1 maximal subgroups.

Corollary 3.7. *The number of type 1 maximal subgroups M of index $n = p^r$ of a d-generated group G is less than or equal to $(n^d - 1)\mathrm{cr}_n^{\mathfrak{A}}(G)$.*

The proof of this result depends on some arguments of Gaschütz [11] and Dalla Volta and Lucchini [6].

The following result is useful to give bounds on the number of maximal subgroups of G of type 2 and index n.

Theorem 3.8 (Pyber [21], see [16, Theorem 1.3] or [9, Theorem 21])**.** *There exists a constant b such that for every group G and every $n \geq 2$, G has at most n^b core-free maximal subgroups of index n. In fact, $b = 2$ will do.*

We can apply it to obtain a bound for the number of maximal subgroups of type 2 and index n.

Theorem 3.9. *Let G be a group and let $n \in \mathbb{N}$. The number of maximal subgroups of G of type 2 and index n is bounded by $\mathrm{rk}\, s_n(G)n^2$.*

Finally, the number of maximal subgroups of type 3 and index n is bounded in the following theorem.

Theorem 3.10. *Let G be a d-generated group and let $n \in \mathbb{N}$ which is a power of the order of a non-abelian simple group. The number of maximal subgroups of G of type 3 and index n is bounded by*

$$n^2 \min\left\{n^d, \frac{\mathrm{rk}\, om_n(G) - 1}{2}\right\} \mathrm{rk}\, o_n(G).$$

This result depends on the study of the crowns associated to non-abelian chief factors, since the minimal normal subgroups of G/M_G are G-connected.

Denote by $\mathbb{T}$ the set of all prime powers and by $\mathbb{S}$ the set of all powers of orders of non-abelian simple groups. Then the elements of $\mathbb{T}$ are possible indices of maximal subgroups of types 1 and 2, the elements of $\mathbb{S}$ are possible indices of maximal subgroups of types 2 and 3, and the elements of $\mathbb{N} \setminus (\mathbb{S} \cup \mathbb{T})$ can appear only as indices of maximal subgroups of type 2. Therefore we have the following result.

Theorem 3.11. (1) *If $n \in \mathbb{T}$ then*

$$\begin{aligned} \mathrm{m}_n(G) &\leq (n^d - 1)\mathrm{cr}_n^{\mathfrak{A}}(G) + n^2 \mathrm{rk}\, s_n(G) \\ &\leq 2 \max\{n^d \mathrm{cr}_n^{\mathfrak{A}}(G), n^2 \mathrm{rk}\, s_n(G)\}. \end{aligned}$$

DOI: http://dx.doi.org/10.22108/ijgt.2019.114469.1521

(2) *If* $n \in \mathbb{S}$, *then*

$$\begin{aligned} \mathrm{m}_n(G) &\leq n^2 \mathrm{rk}\, s_n(G) + n^2 \min\left\{n^d, \frac{\mathrm{rk}\, om(G) - 1}{2}\right\} \mathrm{rk}\, o_n(G). \\ &\leq 2n^2 \max\left\{\mathrm{rk}\, s_n(G), \min\left\{n^d, \frac{\mathrm{rk}\, om_n(G) - 1}{2}\right\} \mathrm{rk}\, o_n(G)\right\}. \end{aligned}$$

(3) *If* $n \notin \mathbb{S} \cup \mathbb{T}$, *then* $\mathrm{m}_n(G) \leq n^2 \mathrm{rk}\, s_n(G)$.

As a consequence, we can precise the function η of Theorem B.

Theorem 3.12. *Let* G *be a* d*-generated non-trivial group. Then, for*

$$\begin{aligned} \eta(G) := \max\Big\{ & d + 2.02 + \max\{\log_n 2 + \log_n \mathrm{cr}_n^{\mathfrak{A}}(G)\}, \\ & 4.02 + \max\{\log_n 2 + \log_n \mathrm{rk}\, s_n(G)\}, \\ & 4.02 + \max\Big\{\min\{d + \log_n 2, \log_n \mathrm{rk}\, om_n(G)\} + \log_n \mathrm{rk}\, o_n(G)\Big\}\Big\}, \end{aligned}$$

we have that

$$\nu(G) \leq \eta(G).$$

4. Discussion

In this section, we will show that Theorem B improves the previously known bounds for $\nu(G)$. First of all, we see that Theorem 1.2 is a consequence of Theorem B. It is a consequence of the facts that in a non-abelian simple group, $|\mathrm{Out}\, S| \leq \sqrt{|S|}$ and in a primitive group of type 3 whose minimal normal subgroups have order n, $\mathrm{i}(G) \leq \sqrt{n}$, both proved in Section 6 of [2]. Theorem 1.3 is also a consequence of Theorem B.

One difficulty we find in order to deduce Theorem 1.9 from Theorem B is the fact that the bounds of Theorem 1.9 depend on some universal constants that are known to exist, but whose values do not appear in the literature. The analysis of the proofs in [12] shows that these constants appear as linear combinations of a set of universal constants, some of them without explicit values known to us. We do this in [2, Section 6].

For instance, Borovik, Pyber, and Shalev [4] presented the following estimation for the number of isomorphism classes of non-abelian simple subgroups of $\mathrm{Sym}(n)$.

Lemma 4.1 ([4])**.** *The number* $g(n)$ *of isomorphism classes of non-abelian simple subgroups of* $\mathrm{Sym}(n)$ *for* $n \geq 5$ *is* $\mathrm{O}(n)$.

We can precise the value $\mathrm{O}(n)$.

Lemma 4.2. *The number* $g(n)$ *of isomorphism classes of non-abelian simple groups of* $\mathrm{Sym}(n)$ *for* $n \geq 5$ *is at most* $4.89n + 1\,141.33$.

We can use this result to precise the known bounds on the number of isomorphism classes of minimal normal subgroups of primitive groups of type 2 with a core-free maximal subgroup of a given index.

DOI: http://dx.doi.org/10.22108/ijgt.2019.114469.1521

Lemma 4.3. *The number $s(n)$ of isomorphism classes of minimal normal subgroups of primitive groups of type 2 with a core-free maximal subgroup of index n satisfies the inequality $s(n) \leq n^{1.266}$.*

Since $\operatorname{rk} s_n(G) \leq s(n)$, we obtain that

$$\log_n \operatorname{rk} s_n(G) \leq 1.266 + \log_n \max\{1, \operatorname{rk}_n(G)\}.$$

Note also that

$$\log_n \operatorname{cr}_n^{\mathfrak{A}}(G) \leq c_6 d + \log_n \max\{1, \operatorname{rk}_n(G)\},$$
$$\log_n \operatorname{rk} o_n(G) \leq \log_2 2 + \log_n \max\{1, \operatorname{rk}_n(G)\}.$$

We can write in a stronger form Corollary 7.3 of [12].

Theorem 4.4 (cf. Jaikin-Zapirain and Pyber [12, Corollary 7.3])**.** *Let G be a d-generated group. Then there exists a constant c_6 such that the number of irreducible G-modules of size n is at most $n^{c_6 d} \max\{1, \operatorname{rk}_n(G)\}$.*

This result and the previous bounds can be used to obtain [12, Theorem 9.5] from our results. We obtain that

$$\nu(G) \leq (c_6 + 1)d + 3.02 + \max \log_n \max\{1, \operatorname{rk}_n(G)\}.$$

We obtain the following result.

Theorem 4.5. *Let G be a d-generated group. Then*

$$\eta(G) \leq cd + \max_{n \geq 5} \frac{\log \max\{1, \operatorname{rk}_n(G)\}}{\log n} + 3,$$

where $c = 375.06$.

This follows from the fact that the constant c_6 in Theorem 4.4 can be taken to be equal to 374.06. A slightly different approach can give a lower bound.

Theorem 4.6. *The number of non-equivalent irreducible G-modules of size $n = p^r$, where p is a prime and $r \in \mathbb{N}$, is at most*

$$n^{\min\{\hat{c}_6 d + k_6 + \log_n \max\{1, \operatorname{rk}_n(G)\}, dr\}},$$

where $\hat{c}_6 = 183.034$ and $k_6 = 74$.

Now we present a construction of d-generated groups with many crowns of abelian chief factors. Let us start with a d-generated primitive group G of type 1. Let Ω be the set of ordered generating d-tuples of G, that we decompose in r orbits under the action of $\operatorname{Aut}(G)$ on Ω, and take for every orbit a representative $(g_{i1}, \ldots, g_{id})$. For $1 \leq i \leq d$, we construct $g_j = \prod_{i=1}^{r} g_{ij} \in G^r$. The group $\hat{G} = \langle g_1, \ldots, g_d \rangle$ is a subdirect product of G^r and $\hat{G}$ has as it socle a direct product of all faithful and irreducible modules for G whose primitive group is isomorphic to G.

DOI: http://dx.doi.org/10.22108/ijgt.2019.114469.1521

This construction can be extended to many d-generated primitive groups with isomorphic socles, by making this construction with each of these groups and by constructing a subdirect product of all of them in a similar way.

For instance, there are three isomorphism classes of 2-generated primitive groups of type 1 with socle of order 8, namely $G_1 = [C_2^3]C_7$, $G_2 = [C_2^3][C_7]C_3$ and $G_3 = [C_2^3]\mathrm{GL}_3(2)$. We can construct a 2-generated group S with all possible crowns of abelian chief factors of order 8. In this group, the non-zero relevant invariants are $\mathrm{cr}_3^{\mathfrak{A}}(S) = 1$, $\mathrm{cr}_7^{\mathfrak{A}}(S) = 9$, $\mathrm{cr}_8^{\mathfrak{A}}(S) = 146$, and $\mathrm{rk}_{\mathrm{GL}_3(2)}(S) = 57$. It follows that the bound of Theorem 1.9 is $\nu(S) \leq 3 + 2c + \log_7 57$, with $3 + 2c + \log_7 57 \geq 5.07 + 2c \approx 755.198$, while the bound of Theorem B is $\nu(S) \leq 6.75$.

Acknowledgments

This work has been supported by the research grants MTM2014-54707-C3-1-P from the *Ministerio de Economía y Competitividad* (Spanish Government) and FEDER (European Union) and PROMETEO/2017/057 from *Generalitat* (Valencian Community, Spain). The fourth author has been supported by the grant number 201606890006 of the China Scholarship Council.

References

[1] R. Baer, Classes of finite groups and their properties, *Illinois J. Math.*, **1** (1957) 115–187.

[2] A. Ballester-Bolinches, R. Esteban-Romero, P. Jiménez-Seral and H. Meng, *Bounds on the number of maximal subgroups with applications to random generation of finite groups*, Preprint.

[3] A. Ballester-Bolinches and L. M. Ezquerro, Classes of finite groups, *Mathematics and Its Applications*, Springer, Dordrecht, **584** (2006).

[4] A. V. Borovik, L. Pyber and A. Shalev, Maximal subgroups in finite and profinite groups, *Trans. Amer. Math. Soc.*, **348** (1996) 3745–3761.

[5] T. C. Burness, M. W. Liebeck and A. Shalev, Generation and random generation: From simple groups to maximal subgroups, *Adv. Math.*, **248** (2013) 59–95.

[6] F. Dalla Volta and A. Lucchini, Finite groups that need more generators than any proper quotient, *J. Austral. Math. Soc. Ser. A*, **64** (1998) 82–91.

[7] F. Dalla Volta, A. Lucchini and F. Morini, On the probability of generating a minimal d-generated group, *J. Aust. Math. Soc.*, **71** (2001) 177–185.

[8] E. Detomi and A. Lucchini, Crowns and factorization of the probabilistic zeta function of a finite group, *J. Algebra*, **265** (2003) 651–668.

[9] E. Detomi and A. Lucchini, Crowns in profinite groups and applications, Noncommutative Algebra and Geometry, *Lect. Notes Pure Appl. Math.*, Chapman & Hall/CRC, **243** (2006) 47–62.

[10] J. D. Dixon, The probability of generating the symmetric group, *Math. Z.*, **110** (1969) 199–205.

[11] W. Gaschütz, Die Eulersche Funktion endlicher auflösbarer Gruppen, *Illinois J. Math.*, **3** (1959) 469–476.

[12] A. Jaikin-Zapirain and L. Pyber, Random generation of finite and profinite groups and group enumeration, *Ann. Math.*, **173** (2011) 769–814.

[13] ———, *Random generation of finite and profinite groups and group enumeration*, http://verso.mat.uam.es/~andrei.jaikin/preprints/pfg.pdf, 2017, Visited 20th February, 2017.

DOI: http://dx.doi.org/10.22108/ijgt.2019.114469.1521

[14] W. M. Kantor and A. Lubotzky, The probability of generating a finite classical group, *Geom. Dedicata*, **36** (1990) 67–87.

[15] M. W. Liebeck and A. Shalev, The probability of generating a finite simple group, *Geom. Dedicata*, **56** (1995) 103–113.

[16] A. Lubotzky, The expected number of random elements to generate a finite group, *J. Algebra*, **257** (2002) 452–459.

[17] A. Mann and A. Shalev, Simple groups, maximal subgroups, and probabilistic aspects of profinite groups, *Israel J. Math.*, **96** (1996) 449–468.

[18] E. Netto, *The theory of substitutions and its applications to algebra*, Second edition. Revised by the author and translated with his permission by F. N. Cole Chelsea Publishing Co., New York, 1964.

[19] I. Pak, *On probability of generating a finite group*, Preprint, http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.43.7319.

[20] C. Pomerance, The expected number of random elements to generate a finite abelian group, *Period. Math. Hungar.*, **43** (2001) 191–198.

[21] L. Pyber, *The number of maximal core-free subgroups of a finite group*, In preparation.

[22] J. Wiegold, Growth sequences of finite groups IV, *J. Austral. Math. Soc. (Ser. A)*, **29** (1980) 14–16.

Adolfo Ballester-Bolinches

Departament de Matemàtiques, Universitat de València, Dr. Moliner, 50, 46100 Burjassot, València, Spain

Email: Adolfo.Ballester@uv.es

Ramón Esteban-Romero

Departament de Matemàtiques, Universitat de València, Dr. Moliner, 50, 46100 Burjassot, València, Spain

Email: Ramon.Esteban@uv.es

Permanent address: Institut Universitari de Matemàtica Pura i Aplicada, Universitat Politècnica de València, Camí de Vera, s/n, 46022 València, Spain

Email: resteban@mat.upv.es

Paz Jiménez-Seral

Departamento de Matemáticas, Universidad de Zaragoza, Pedro Cerbuna, 12, 50009 Zaragoza, Spain

Email: paz@unizar.es

Hangyang Meng

Departament de Matemàtiques, Universitat de València, Dr. Moliner, 50, 46100 Burjassot, València, Spain

Email: hangyangmenges@gmail.com

DOI: http://dx.doi.org/10.22108/ijgt.2019.114469.1521